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A COUNTER-EXAMPLE ON A QUASI-VARIATIONAL INEQUALITY WITHOUT LOWER SEMICONTINUITY ASSUMPTION

PAOLO CUBIOTTI

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ABSTRACT. In this note we show that a recent result on quasi-variational inequalities, which seems to improve deeply some well-known results, is not correct.

In a very recent paper, N. H. Dien stated the following assertion.

Assertion A (Theorem 3 in [2]). *Let C be a nonempty compact and convex set in $\mathbb{R}^n$, $K: C \to 2^C$ a multifunction, and $\varphi: C \to \mathbb{R} \cup \{\pm\infty\}$, $f: C \to \mathbb{R}^n$, and $\eta: C \times C \to \mathbb{R}^n$ three (single-valued) functions. Assume that:*

(1) *K is upper semicontinuous with nonempty compact convex values;*
(2) *φ is lower semicontinuous (in the sense of single-valued maps) and convex;*
(3) *η and f are continuous and $\langle f(x), \eta(x, x)\rangle \geq 0$ for all $x \in C$;*
(4) *for each $x \in C$, the function $\langle f(x), \eta(\cdot, x)\rangle$ is convex on C.*

Then there exists $\tilde{x} \in C$ such that

$$(1) \quad \tilde{x} \in K(\tilde{x}) \quad \text{and} \quad \langle f(\tilde{x}), \eta(u, \tilde{x})\rangle + \varphi(u) - \varphi(\tilde{x}) \geq 0 \quad \text{for all } u \in K(\tilde{x}).$$

The aim of the present note is to show that Assertion A, in general, is false. Notice that the above statement, if true, would imply (taking $\eta(y, x) = y - x$ and $\varphi \equiv 0$) that the classical and celebrated Chan and Pang's result on quasi-variational inequalities (Corollary 3.1 in [1]) is still true (for single-valued f) without supposing the multifunction K to be lower semicontinuous on C. The following simple example shows at once that Assertion A is incorrect and that the mentioned improvement of Chan and Pang's theorem is not possible.

Example. Take $C = [0, 1]$, $\varphi \equiv 0$, $\eta(y, x) = y - x$, $f(x) = 1$, and let the multifunction K be defined by

$$K(x) = \begin{cases} [\frac{1}{2}, 1] & \text{if } x \in [0, \frac{1}{2}[, \\ [0, 1] & \text{if } x = \frac{1}{2}, \\ [0, \frac{1}{2}] & \text{if } x \in]\frac{1}{2}, 1]. \end{cases}$$

The reader can easily verify that the graph of the multifunction K is closed and then, since $[0, 1]$ is compact, K is upper semicontinuous. Moreover, the other

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assumptions of Assertion A are trivially satisfied. However, the multifunction K has just one fixed point which is not a solution to the problem (1). Notice that such K is not lower semicontinuous.

Remark 1. The mistake in the proof of Theorem 3 in [2] resides in the assertion that the marginal function S is upper semicontinuous. To see this, let us consider the space C, the multifunction K, and the maps f, η, φ as in the example. In this case, the multifunction S becomes

$$S(x) = \begin{cases} \{\frac{1}{2}\} & \text{if } x \in [0, \frac{1}{2}[, \\ \{0\} & \text{if } x \in [\frac{1}{2}, 1], \end{cases}$$

which is not upper semicontinuous.

Remark 2. Of course, Theorem 4 of [2] also fails.

REFERENCES

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MESSINA, 98166 SANT'AGATA-MESSINA, ITALY
E-mail address: `pcub@imeuniv.bitnet`