R-GROUPS AND ELLIPTIC REPRESENTATIONS
FOR UNITARY GROUPS

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ABSTRACT. We determine the reducibility and number of components of any representation of a quasi-split unitary group which is parabolically induced from a discrete series representation. The R-groups are computed explicitly, in terms of reducibility for maximal parabolics. This gives a description of the elliptic representations.

INTRODUCTION

Let G be a reductive p-adic group. That is, $G = \mathbf{G}(F)$, where $\mathbf{G}$ is a reductive algebraic group, defined over a p-adic field F of characteristic zero. Let $P = MN$ be a parabolic subgroup of G. Suppose σ is a discrete series representation of M. Consider the unitarily induced representation $i_{G,M}(\sigma) = \mathrm{Ind}_P^G(\sigma \otimes 1_N)$. We are interested in determining when $i_{G,M}(\sigma)$ is reducible, and if so, what can be said about the constituents. A complete classification of the irreducible constituents of the representations $i_{G,M}(\sigma)$ gives rise to a classification of the tempered spectrum of G.

The theory of R-groups tells one exactly when $i_{G,M}(\sigma)$ is reducible, and describes the number of irreducible components and their multiplicity. In [4] we explicitly described all the R-groups for the split classical groups $Sp(2n)$ and $SO(n)$. In [3] we described the R-groups for $SL(n)$. We now explicitly describe the structure of the R-groups when G is either of the quasi-split unitary groups, $U(n, n)$ or $U(n, n+1)$. These results are remarkably similar to those for the groups $Sp(2n)$ and $SO(2n+1)$. Every R-group we consider is of the form $\mathbb{Z}_2^d$, and the integer d can be described explicitly in terms of σ.

Let G^e be the set of regular elliptic elements of G. Suppose π is an irreducible tempered representation of G with character Θ_π. Let Θ_π^e denote the restriction of Θ_π to G^e. Then π is said to be elliptic if Θ_π^e is nonzero. In [1], Arthur gives a characterization of the elliptic constituents of $i_{G,M}(\sigma)$ in terms of R-groups. In [7], Herb classifies the elliptic representations of $Sp(2n)$ and $SO(n)$ by using Arthur's description. We describe the elliptic representations of quasi-split unitary groups by following [7]. Again the results are similar to those for the split groups.

In §1 we will review the theory of intertwining operators and R-groups. In §2 we will describe the structure of the parabolic subgroups of $U(n, n)$ and $U(n, n+1)$. In §3 we compute the R-group attached to any discrete series representation of any Levi component of G. In §4 we describe the commuting algebra of $i_{G,M}(\sigma)$ and use this to describe the elliptic representations. As a result of the description of the commuting algebra, we find that every discrete series representation of every Levi component induces with multiplicity 1. Moreover, every irreducible tempered representation is either elliptic or is irreducibly induced from a tempered elliptic representation of a proper parabolic subgroup. This is not the case for every group G, as is pointed out in [7]. Further counterexamples can be found in [3].

1. PRELIMINARIES

Let F be a locally compact, nondiscrete, nonarchimedean local field of characteristic zero and residual characteristic q_F. Let G be the F-rational points of a connected reductive quasi-split algebraic group defined over F. Let G' be the set of regular elements of G [6]. We say an element x of G is elliptic if its centralizer is compact, modulo the center of G. We denote by G^e the set of regular elliptic elements of G.

We let $\mathscr{E}_2(G)$ denote the collection of equivalence classes of irreducible discrete series representations of G. We write $\mathscr{E}_t(G)$ for the collection of equivalence classes of irreducible tempered representations of G. Then $\mathscr{E}_2(G) \subset \mathscr{E}_t(G)$. If $\pi \in \mathscr{E}_t(G)$, then we write Θ_π for the character of π and Θ_π^e for its restriction to G^e. We say that π is elliptic if $\Theta_\pi^e \neq 0$.

We say that $M \subseteq G$ is a Levi subgroup of G if there is a parabolic subgroup P of G, with M the Levi component of P. Let A_0 be a maximal F-split torus of G, and let $\Phi(G, A_0)$ be the set of restricted roots of G with respect to A_0. Let Δ be a collection of simple roots in Φ. Then the conjugacy classes of parabolic subgroups are in one-to-one correspondence with the subsets of Δ [16]. If $\theta \subset \Delta$, then we let $A_\theta \subset A_0$ be the subtorus corresponding to θ. Then $M_\theta = Z_G(A_\theta)$ is the the Levi subgroup corresponding to θ. We denote by $B = TU$ the Borel subgroup associated to $A_\varnothing = A_0$.

Let M be a Levi subgroup of G. Let (σ, V) be a smooth representation of M. Suppose $P = MN$ is a parabolic subgroup with Levi component M. We let $\text{Ind}_P^G(\sigma)$ be the representation unitarily induced by σ. Since the class of $\text{Ind}_P^G(\sigma)$ depends only on M, and not P, we also write $i_{G,M}(\sigma)$ for $\text{Ind}_P^G(\sigma)$.

Let $A = A_\theta$ and $M = M_\theta$. We denote by $W(A)$ or $W(G/A)$, the Weyl group $N_G(A)/M$ of G with respect to A. For $w \in W(A)$, we make no distinction between w and a representative for w in $N_G(A)$. If σ is an irreducible smooth representation of M and $w \in W(A)$, then we let $w\sigma$ be the representation defined by $w\sigma(m) = \sigma(w^{-1}mw)$. We say that σ is ramified if there is some $w \in W(A)$, $w \neq 1$, so that $\sigma \simeq w\sigma$.

We denote by $X(M)_F$ the collection of F-rational characters of M. Then $\mathfrak{a} = \text{Hom}(X(M)_F, \mathbb{R})$ is the real Lie algebra of A, and we let $\mathfrak{a}_{\mathbb{C}}^*$ be the complexified dual of $\mathfrak{a}$ [6]. There is a homomorphism $H_P: M \to \mathfrak{a}$ satisfying

$$q_F^{\langle \chi, H_P(m) \rangle} = |\chi(m)|_F, \quad \forall \chi \in X(M)_F, \, m \in M.$$

Let $\nu \in \mathfrak{a}_{\mathbb{C}}^*$ and $\sigma \in \mathscr{E}_2(M)$. We denote the representation $\text{Ind}_P^G\left(\sigma \otimes q_F^{\langle \nu, H_P(-) \rangle}\right)$

by $I(\nu, \sigma)$. Let $w \in W(A)$, and let $N_w = U \cap w^{-1}\bar{N}w$, where $\bar{N}$ is the unipotent radical opposed to N. Suppose f is a function in the space of $I(\nu, \sigma)$. We formally define an operator by

$$(1.1) \qquad A(\nu, \sigma, w)f(g) = \int_{N_w} f(w^{-1}ng)\, dn.$$

We say $A(\nu, \sigma, w)$ converges if (1.1) converges for each choice of f and g. Normalizing by a meromorphic scalar factor we obtain a family of intertwining operators between $I(\nu, \sigma)$ and $I(w\nu, w\sigma)$, which are holomorphic on the unitary axis $i\mathfrak{a}^*$ [14, 16]. We denote these operators by $\mathscr{A}(\nu, \sigma, w)$ and write $\mathscr{A}(\sigma, w) = \mathscr{A}(0, \sigma, w)$. Then these operators satisfy the cocycle condition

$$(1.2) \qquad \mathscr{A}(\sigma, w_1 w_2) = \mathscr{A}(w_2\sigma, w_1)\mathscr{A}(w_2, \sigma).$$

Theorem 1.1 (Harish-Chandra). *Let $w \in W(A)$ and $\sigma \in \mathscr{E}_2(M)$. Then there is a complex number $\mu(\nu, \sigma, w)$ so that, if Haar measure is chosen appropriately,*

$$A(\nu, \sigma, w)A(w\nu, w\sigma, w^{-1}) = \mu(\nu, \sigma, w)^{-1}\gamma_w^2(G/P),$$

where $\gamma_w(G/P)$ is defined in [6]. Moreover, $\nu \mapsto \mu(\nu, \sigma, w)$ is meromorphic on $\mathfrak{a}_{\mathbb{C}}^*$, and holomorphic and nonnegative on the unitary axis $i\mathfrak{a}^*$. $\quad\square$

The scalar $\mu(\nu, \sigma, w)$ is called the Plancherel measure attached to ν, σ, and w. If w is the longest element of the Weyl group, then we write $\mu(\nu, \sigma)$ for $\mu(\nu, \sigma, w)$. We also write $\mu(\sigma)$ for $\mu(0, \sigma)$. If M is maximal and proper, then $i_{G,M}(\sigma)$ is reducible if and only if σ is ramified and $\mu(\sigma) \neq 0$ [16, Corollary 5.4.2.3].

Let $W(\sigma) = \{w \in W(A) \mid w\sigma \simeq \sigma\}$. Choose an intertwining operator $T(w)$ satisfying $T(w)(w\sigma) = \sigma T(w)$. Then $\mathscr{A}'(\sigma, w) = T(w)\mathscr{A}(\sigma, w)$ is a self-intertwining operator for $i_{G,M}(\sigma)$. Let $C(\sigma)$ be the commuting algebra of $i_{G,M}(\sigma)$.

Theorem 1.2 (Harish-Chandra, [16, Theorem 5.5.3.2]). *The collection*

$$\{\mathscr{A}'(\sigma, w) \mid w \in W(\sigma)\}$$

spans the commuting algebra $C(\sigma)$. $\quad\square$

Therefore, we would like to be able to determine a basis for $C(\sigma)$ from among the operators $\mathscr{A}'(\sigma, w)$. This leads to the construction of the Knapp-Stein *R*-group. Let $\Phi(P, A)$ be the set of reduced roots of P with respect to A. Let $\beta \in \Phi(P, A)$, and let A_β be the torus $(\ker \beta \cap A)^0$. Suppose M_β is the centralizer of A_β in G. Note that M is a maximal proper Levi subgroup of M_β. Let $\mu_\beta(\sigma)$ denote the Plancherel measure attached to $i_{M_\beta,M}(\sigma)$. Let $\Delta' = \{\beta \in \Phi(P, A) \mid \mu_\beta(\sigma) = 0\}$. We let $R = \{w \in W(\sigma) \mid w\beta > 0, \forall \beta \in \Delta'\}$. Since $\mu_{w\beta}(\sigma) = \mu_\beta(w^{-1}\sigma)$ [6, p. 183], $R = \{w \in W(\sigma) \mid w\Delta' = \Delta'\}$. Let W' be the subgroup of $W(\sigma)$ generated by the reflections in the roots of Δ'.

Theorem 1.3 (Knapp-Stein, Silberger [11, 15]). *For any $\sigma \in \mathscr{E}_2(M)$, $W(\sigma) = R \ltimes W'$. Furthermore, $W' = \{w \in W(\sigma) \mid \mathscr{A}'(\sigma, w)$ is scalar$\}$.* $\quad\square$

Therefore, $\{\mathscr{A}'(\sigma, w) \mid w \in R\}$ is a basis for the commuting algebra $C(\sigma)$. However, we can say even more. If $w_1, w_2 \in R$, then

$$\mathscr{A}'(\sigma, w_1 w_2) = \eta(w_1, w_2)\mathscr{A}'(\sigma, w_1)\mathscr{A}'(\sigma, w_2),$$

where the 2-cocycle $\eta : R \times R \to \mathbb{C}^*$ satisfies

$$T(w_1 w_2) = \eta(w_1, w_2) T(w_1) T(w_2).$$

Theorem 1.4 (See [10]). (a) $C(\sigma) \simeq \mathbb{C}[R]_\eta$, *where* $\mathbb{C}[R]_\eta$ *is the group algebra twisted by the cocycle* η.

(b) *Suppose* R *is abelian and* η *splits, i.e.,* $C(\sigma) \simeq \mathbb{C}[R]$. *Then* $i_{G,M}(\sigma)$ *decomposes with multiplicity one.* □

The number of inequivalent constituents is equal to the number of irreducible representations of R, and the irreducible component corresponding to $\rho \in \hat{R}$ appears with multiplicity dim ρ. Let $\mathcal{H}$ be the space of $\text{Ind}_P^G(\sigma)$. Let π_ρ be the irreducible constituent corresponding to ρ, and let θ_ρ be the character of ρ. We define an operator $\mathcal{A}_\rho$ by

$$\mathcal{A}_\rho = |R|^{-1} \dim \rho \sum_{w \in R} \bar{\theta}_\rho(w) \mathcal{A}'(\sigma, w).$$

Then $\{\mathcal{A}_\rho \mid \rho \in \hat{R}\}$ is a collection of nonzero [15] orthogonal projections, and $\mathcal{H}_\rho = \mathcal{A}_\rho \mathcal{H}$ is the π_ρ isotypic subspace of $\mathcal{H}$. Note that if R is abelian and η splits, then $\mathcal{H}_\rho = \{v \in \mathcal{H} \mid \mathcal{A}'(w, \sigma)v = \rho(w)v \text{ for all } w \in R\}$ and $(\pi_\rho, \mathcal{H}_\rho)$ is irreducible [10].

We now recall some results of Arthur [1] and Herb [7] on elliptic representations. For each $w \in R$ define $\mathfrak{a}_w = \{H \in \mathfrak{a} \mid w \cdot H = H\}$. Let Z be the split component of G, i.e., the maximal F-split torus in the center of G; and let $\mathfrak{z}$ be the real Lie algebra of Z. Then $\mathfrak{z} \subset \mathfrak{a}_w$ for each $w \in R$. Let $\mathfrak{a}_R = \bigcap_{w \in R} \mathfrak{a}_w$.
We state a weak version of Arthur's theorem. The hypotheses will apply in all of our examples.

Theorem 1.5 (Arthur [1, Proposition 2.1]). *Suppose* R *is abelian and* $C(\sigma) \simeq \mathbb{C}[R]$. *Then the following are equivalent*:
  (a) $i_{G,M}(\sigma)$ *has an elliptic constituent*,
  (b) *all the constituents of* $i_{G,M}(\sigma)$ *are elliptic*,
  (c) *there is a* $w \in R$ *so that* $\mathfrak{a}_w = \mathfrak{z}$. □

For the remainder of this section we suppose R is abelian and $C(\sigma) \simeq \mathbb{C}[R]$. Suppose $M' \supset M$ is a Levi subgroup of G satisfying Arthur's compatibility condition with respect to Δ' [1, §2]. Then $R' = R \cap W(M'/A)$ is the R-group attached to $i_{M',M}(\sigma)$. If $\kappa' \in \hat{R}'$, then we let $\tau_{\kappa'}$ to be the irreducible component of $i_{M',M}(\sigma)$ corresponding to κ'. Let $\hat{R}(\kappa') = \{\kappa \in \hat{R} \mid \kappa(w) = \kappa'(w), \forall w \in R'\}$. We let π_κ be the irreducible constituent of $i_{G,M}(\sigma)$ corresponding to κ.

Theorem 1.6 (Arthur [1, §2]). *For any* $\kappa' \in \hat{R}'$, $i_{G,M'}(\tau_{\kappa'}) = \bigoplus_{\kappa \in \hat{R}(\kappa')} \pi_\kappa$. □

Proposition 1.7 (Herb [7]). *Suppose* R *is abelian and* $C(\sigma) \simeq \mathbb{C}[R]$. *Let* π *be an irreducible constituent of* $i_{G,M}(\sigma)$. *Then* $\pi = i_{G,M'}(\tau)$ *for a proper Levi subgroup* M' *and some* $\tau \in \mathcal{E}_t(M')$, *if and only if* $\mathfrak{a}_R \neq \mathfrak{z}$. *Moreover,* M' *and* τ *can be chosen so that* τ *is elliptic if and only if there is a* $w_0 \in R$ *with* $\mathfrak{a}_R = \mathfrak{a}_{w_0}$.

2. Parabolic subgroups of $U(n, n)$ and $U(n, n+1)$

Let E/F be a quadratic extension of locally compact, nondiscrete, nonar-chimedean fields of characteristic zero. Let $x \mapsto \bar{x}$ be the nontrivial element of $\mathrm{Gal}(E/F)$. Choose an element $\beta \in E$ such that $E = F(\beta)$ and $\bar{\beta} = -\beta$. Let $G = U(n, n)$ or $U(n, n+1)$. We define J_n, J_n' by

$$J_n = \begin{pmatrix} & \beta I_n \\ -\beta I_n & \end{pmatrix} \quad \text{and} \quad J_n' = \begin{pmatrix} & & \beta I_n \\ & 1 & \\ -\beta I_n & & \end{pmatrix}.$$

Then

$$U(n, n) = \{g \in GL(2n, E) \mid {}^t\bar{g}J_n g = J_n\},$$
$$U(n, n+1) = \{g \in GL(2n+1, E) \mid {}^t\bar{g}J_n' g = J_n'\}.$$

Each of these groups is quasi-split, but not split. Let T be the maximal torus of diagonal elements in G. For notational convenience, we denote a diagonal element of $GL(k, E)$ by $\mathrm{diag}\{x_1, \dots, x_k\}$. Then

$$T = \left\{\mathrm{diag}\left\{\lambda_1, \lambda_2, \dots, \lambda_n, \bar{\lambda}_1^{-1}, \dots, \bar{\lambda}_n^{-1}\right\} \mid \lambda_i \in E^*\right\}$$

if $G = U(n, n)$, and

$$T = \left\{\mathrm{diag}\left\{\lambda_1, \dots, \lambda_n, y, \bar{\lambda}_1^{-1}, \dots, \bar{\lambda}_n^{-1}\right\} \mid \lambda_i \in E^*, y\bar{y} = 1\right\}$$

if $G = U(n, n+1)$. Let A_0 be the maximal F-split subtorus of T. Then A_0 is the collection of elements in T with each $\lambda_i \in F^*$, and $y = 1$ if $G = U(n, n+1)$. Notice that $\mathfrak{z}$, the real Lie algebra of the split component Z of G, is $\{0\}$.

Let $\Phi(G, A_0)$ be the restricted roots of G with respect to A_0. We choose the ordering on the roots so that the Borel subgroup is the collection of upper triangular matrices in G. If $G = U(n, n)$, then $\Phi(G, A_0)$ is of type C_n, while if $G = U(n, n+1)$, then $\Phi(G, A_0)$ is of type BC_n. Let Δ be the simple roots given by $\Delta = \{\alpha_i\}_{i=1}^n$, with $\alpha_i = e_i - e_{i+1}$, $1 \leq i \leq n-1$, $\alpha_n = 2e_n$ if $G = U(n, n)$, and $\alpha_n = e_n$ if $G = U(n, n+1)$.

The Weyl group $W(G/A_0)$ is isomorphic to $S_n \ltimes \mathbb{Z}_2^n$. Here S_n acts by permutations on the matrix entries λ_i, $1 \leq i \leq n$. We use standard cycle notation for the elements of S_n. Thus, (ij) interchanges λ_i and λ_j. If c_i is the non-trivial element of the ith copy of $\mathbb{Z}_2$, then c_i interchanges λ_i and λ_i^{-1}. The element c_i is called a sign change, because the action of c_i on $\Phi(G, A_0)$ takes e_i to $-e_i$. Any element which is a product of sign changes is also called a sign change.

Let $\theta \subset \Delta$. Suppose $\theta_1 \cup \theta_2 \cdots \cup \theta_k$ is the decomposition of θ as a disjoint union of connected components of the Dynkin diagram. We assume that if $\alpha_n \in \theta$, then $\alpha_n \in \theta_k$. Let $n_i = |\theta_i| + 1$, unless $\alpha_n \in \theta$ and $i = k$, in which case we let $n_k = |\theta_k|$. Let A_θ be the subtorus of A_0 associated to θ. Let $G = U(n, n)$. We denote a diagonal matrix which is scalar in blocks by $\mathrm{diag}\{x_1 I_{m_1}, \dots, x_r I_{m_r}\}$, where the m_i are the appropriate dimensions. Then, if $\alpha_n \notin \theta$,

$$A_\theta = \left\{\mathrm{diag}\left\{\lambda_1 I_{n_1}, \dots, \lambda_k I_{n_k}, \lambda_1^{-1} I_{n_1}, \dots, \lambda_k^{-1} I_{n_k}\right\} \mid \lambda_i \in F^*\right\}.$$

If $\alpha_n \in \theta$, then

$$A_\theta = \left\{ \mathrm{diag}\left\{ \lambda_1 I_{n_1}, \ldots, \lambda_{k-1} I_{n_{k-1}}, I_{2n_k}, \lambda_1^{-1} I_{n_1}, \ldots, \lambda_{k-1}^{-1} I_{n_{k-1}} \right\} \mid \lambda_i \in F^* \right\}.$$

If $G = U(n, n+1)$, then

$$A_\theta = \left\{ \mathrm{diag}\left\{ \lambda_1 I_{n_1}, \ldots, \lambda_k I_{n_k}, 1, \lambda_1^{-1} I_{n_1}, \ldots, \lambda_k^{-1} I_{n_k} \right\} \mid \lambda_i \in F^* \right\},$$

if $\alpha_n \notin \theta$ and

$$A_\theta = \left\{ \mathrm{diag}\left\{ \lambda_1 I_{n_1}, \ldots, \lambda_{k-1} I_{n_{k-1}}, I_{2n_k+1}, \lambda_1^{-1} I_{n_1}, \ldots, \lambda_{k-1}^{-1} I_{n_{k-1}} \right\} \mid \lambda_i \in F^* \right\}$$

if $\alpha_n \in \theta$. Let M_θ be the centralizer of A_θ. A straightforward matrix computation gives a proof of the following lemma.

Lemma 2.1. *Let $\theta \subset \Delta$. Then there are integers $n_1, \ldots, n_r > 0$ and $m \geq 0$, so that $n_1 + \cdots + n_r + m = n$, and*

$$M_\theta \simeq GL(n_1, E) \times \cdots \times GL(n_r, E) \times G(m),$$

where

$$G(m) = \begin{cases} U(m, m) & \text{if } G = U(n, n), \\ U(m, m+1) & \text{if } G = U(n, n+1). \end{cases}$$

Note that we adopt the convention that

$$G(0) = \begin{cases} 1 & \text{if } G = U(n, n), \\ U(1) & \text{if } G = U(n, n+1). \end{cases} \qquad \square$$

Suppose we write $a \in A_\theta$ as $a = (\lambda_1, \ldots, \lambda_r)$, with each $\lambda_i \in F^x$. The Weyl group $W(G/A_\theta)$ is a subgroup of $S_r \ltimes \mathbb{Z}_2^r$. Namely, for $(ij) \in S_r$, $(ij) \in W(G/A_\theta)$ if and only if $n_i = n_j$. In this case

$$(ij) \cdot (\lambda_1, \ldots, \lambda_r) = (\lambda_1, \ldots, \lambda_{i-1}, \lambda_j, \ldots, \lambda_{j-1}, \lambda_i, \ldots, \lambda_r).$$

For each $1 \leq i \leq r$, there is a block sign change $C_i \in W(G/A_\theta)$, such that $C_i \cdot (\lambda_1, \ldots, \lambda_r) = (\lambda_1, \ldots, \lambda_i^{-1}, \ldots, \lambda_r)$. We use the term "block sign change" because C_i is the product of the n_i sign changes $c_{b_i+1}, \ldots, c_{b_i+n_i} \in W(G/A_0)$, where $b_i = \sum_{j<i} n_j$.

If $g \in M_\theta$, we write $g = (g_1, \ldots, g_r, g')$, with $g_i \in GL(n_i, E)$ and $g' \in G(m)$. If $(ij) \in W(G/A_\theta)$, then

$$(ij) \cdot g = (g_1, \ldots, g_{i-1}, g_j, \ldots, g_{j-1}, g_i, \ldots, g_r, g').$$

For $1 \leq i \leq r$ we have $C_i \cdot g = (g_1, \ldots, g_{i-1}, \varepsilon(g_i), \ldots, g_r, g')$, where ε is the automorphism of $GL(n_i, E)$ given by $\varepsilon(g_i) = {}^t \bar{g}_i^{-1}$. We fix a subset $\theta \subset \Delta$, and write $A = A_\theta$ and $M = M_\theta$. Let $\sigma \in \mathscr{E}_2(M)$. Then $\sigma \simeq \sigma_1 \otimes \cdots \otimes \sigma_r \otimes \rho$, where $\sigma_i \in \mathscr{E}_2(GL(n_i, E))$ and $\rho \in \mathscr{E}_2(G(m))$. By the above calculations, we see that $C_i \sigma \simeq \sigma_1 \otimes \cdots \otimes \sigma_i^\varepsilon \otimes \cdots \otimes \sigma_r \otimes \rho$, where $\sigma_i^\varepsilon(g_i) = \sigma_i(\varepsilon(g_i))$. If $s \in S_r \cap W(G/A)$, then $s\sigma = \sigma_{s(1)} \otimes \cdots \otimes \sigma_{s(r)} \otimes \rho$. Note we can write the conditions for σ to be ramified as follows:

$$C_i \sigma \simeq \sigma \Leftrightarrow \sigma_i \simeq \sigma_i^\varepsilon,$$
$$(ij)\sigma \simeq \sigma \Leftrightarrow \sigma_i \simeq \sigma_j,$$
$$(ij)C_i C_j \sigma \simeq \sigma \Leftrightarrow \sigma_i \simeq \sigma_j^\varepsilon.$$

That is, if σ is ramified, then at least one of these conditions holds. If $1 \leq i \leq j \leq r - 1$, we denote the reduced root $e_{b_i} - e_{b_j+1}$ by α_{ij}. We write $\beta_{ij} = e_{b_i} + e_{b_j+1}$. If $1 \leq i \leq n$, then we let

$$\gamma_i = \begin{cases} 2e_{b_i} & \text{if } G = U(n, n), \\ e_{b_i} & \text{if } G = U(n, n+1). \end{cases}$$

Then $\{\alpha_{ij}, \beta_{ij}\}_{i \leq j} \cup \{a\gamma_i \mid a\gamma_i \in \Phi^+\}_{1 \leq i \leq r}$ constitutes a collection of positive reduced roots $\Phi(P, A)$.

3. R-GROUPS FOR $U(n, n)$ AND $U(n, n+1)$

For $\beta \in \Phi(P, A)$, the Levi subgroup M_β is the centralizer of $(A \cap \ker \beta)^0$, and $\mu_\beta(\sigma)$ is the Plancherel measure associated to $i_{M_\beta, M}(\sigma)$. Thus, $\mu_\beta(\sigma) = 0$ if and only if there is some $w \neq 1$ in $W(M_\beta/A)$ such that $w\sigma \simeq \sigma$, and $i_{M_\beta, M}(\sigma)$ is irreducible. We denote by Δ' the collection of $\beta \in \Phi(P, A)$ such that $\mu_\beta(\sigma) = 0$. Then

$$R = \{w \in W(\sigma) | w\beta > 0, \forall \beta \in \Delta'\}.$$

Lemma 3.1. *Let M be a standard Levi subgroup of G. For any $\sigma \in \mathscr{E}_2(M)$, the reduced root $\alpha_{ij} \in \Delta'$ if and only if $\sigma_i \simeq \sigma_{j+1}$. Similarly, $\beta_{ij} \in \Delta'$ if and only if $\sigma_i \simeq \sigma_{j+1}^e$.*

Proof. Note that

$$M_{\alpha_{ij}} \simeq \prod_{k \neq i, j+1} GL(n_k, E) \times GL(n_i + n_{j+1}, E) \times G(m)$$

and

$$W(M_\beta/A) = \begin{cases} \langle\langle (i(j+1)) \rangle\rangle & \text{if } n_i = n_{j+1}, \\ \{1\} & \text{if } n_i \neq n_{j+1}. \end{cases}$$

Thus, the result for α_{ij} follows from the work of Ol'šanskii [12], Bernstein and Zelevinski [2], and Jacquet [8] (see [4, Lemma 3.4]). Note that $\beta_{ij} = C_{j+1}(\alpha_{ij})$. Therefore, $\mu_{\beta_{ij}}(\sigma) = \mu_{\alpha_{ij}}(C_{j+1} \cdot \sigma)$. Thus, $\mu_{\beta_{ij}}(\sigma) = 0$ if and only if $\sigma_i \simeq \sigma_{j+1}^e$. $\square$

The following lemma is critical to our argument. Its proof is identical to a lemma of Keys [9].

Lemma 3.2. *Let $w = sc \in R$, with $s \in S_r$ and $c \in \mathbb{Z}_2^r$. Then $s = 1$.*

Proof. By conjugating by a sign change we may assume that c changes the sign of at most one e_{b_i} in each orbit of s. Suppose that s has a nontrivial cycle, which we may assume is of the form $(1, \ldots, j+1)$. Suppose c changes no signs among $\{e_{b_1}, \ldots, e_{b_{j+1}}\}$. Since $w \in R$, we have $\sigma_1 \simeq \sigma_2 \simeq \cdots \simeq \sigma_{j+1}$. Therefore, by Lemma 3.1, $\alpha_{1j} \in \Delta'$. However, $w\alpha_{1j} < 0$, contradicting the assumption that $w \in R$. Now suppose c changes the sign of $e_{b_{j+1}}$. Then $w\sigma \simeq \sigma$ implies $\sigma_1 \simeq \cdots \simeq \sigma_{j+1} \simeq \sigma_1^e$. Therefore, by Lemma 3.2, $\beta_{1j} \in \Delta'$ and $w\beta_{1j} = s \cdot \alpha_{1j} < 0$. This contradicts our assumption that $w \in R$. $\square$

Definition 3.3. Let $\sigma \in \mathscr{E}_2(GL(k, E))$ and $\rho \in \mathscr{E}_2(G(m))$. We say that the condition $\mathscr{H}_{m,k,G}(\sigma \otimes \rho)$ holds if $i_{G(m+k), GL(k, E) \times G(m)}(\sigma \otimes \rho)$ is reducible.

Note that if $G = G(n) = G(m + k)$ and $M = GL(k, E) \times G(m)$, then M is the maximal proper Levi subgroup of G determined by omitting the simple root

$$\alpha = \begin{cases} e_k - e_{k+1} & \text{if } m > 0; \\ 2e_n & \text{if } m = 0, G = U(n, n); \\ e_n & \text{if } m = 0, G = U(n, n + 1). \end{cases}$$

Thus, $\mathcal{X}_{m,k,G}(\sigma \otimes \rho)$ holds if and only if $\sigma \simeq \sigma^\varepsilon$ and $\mu_\alpha(\sigma) \neq 0$.

Remark. If $m = 0$, then [5] determines this condition in terms of lifting from $U(k)$. Using the methods of [13], we hope to determine the condition $\mathcal{X}_{G,m,k}(\sigma \otimes \rho)$ explicitly for all m. This will be the subject of joint work with Shahidi.

Theorem 3.4. *Let $G = U(n, n)$ or $U(n, n + 1)$. Let $P = MN$ be a parabolic subgroup of G. Suppose that $M \simeq GL(n_1, E) \times \cdots \times GL(n_r, E) \times G(m)$. Let $\sigma \in \mathscr{E}_2(M)$, with $\sigma \simeq \sigma_1 \otimes \cdots \otimes \sigma_r \otimes \rho$. Let d be the number of inequivalent σ_i such that $\mathcal{X}_{m,n_i,G}(\sigma_i \otimes \rho)$ holds. Then $R \simeq \mathbb{Z}_2^d$.*

Proof. By Lemma 3.2, $R \subset \mathbb{Z}_2^r$, the subgroup of sign changes in $W(G/A)$. For $w \in W(G/A)$, we let $R(w) = \{\alpha \in \Phi(P, A) \mid w\alpha < 0\}$. Suppose $c = C_1 \cdots C_j \in R$. Since $c \in W(\sigma)$, $\sigma_i \simeq \sigma_i^\varepsilon$ for $1 \leq i \leq j$. Therefore, $C_i \in W(\sigma)$ for $1 \leq i \leq j$. Note that $R(C_i) \subseteq R(c)$. Since $c \in R$, $R(c) \cap \Delta' = \varnothing$. Thus, for $1 \leq i \leq j$, $R(C_i) \cap \Delta' = \varnothing$, and hence $C_i \in R$. Therefore, R is generated by single block sign changes C_i. Note that

$$R(C_i) = \{\alpha_{ik}, \beta_{ik}\}_{i \leq k \leq r-1} \cup (\{a\gamma_i\}_{a=1,2} \cap \Phi^+).$$

Suppose $\sigma_i \simeq \sigma_i^\varepsilon$. By Lemma 3.1, $\alpha_{ik}, \beta_{ik} \in \Delta'$ if and only if $\sigma_i \simeq \sigma_{k+1}$. By inspection, $a\gamma_i \in \Delta'$ if and only if $\mathcal{X}_{m,n_i,G}(\sigma_i \otimes \rho)$ does not hold. Therefore, $C_i \in R$ if and only if $\sigma_i \not\simeq \sigma_k$ for all $k > i$, and $\mathcal{X}_{m,n_i,G}(\sigma_i \otimes \rho)$ holds. Thus, there is one generator of R for each equivalence class among the σ_i such that $\mathcal{X}_{m,n_i,G}(\sigma_i \otimes \rho)$ holds. $\square$

4. Elliptic representations of $U(n, n)$ and $U(n, n + 1)$.

In this section we use the arguments of [7] to describe the elliptic representations of $U(n, n)$ and $U(n, n + 1)$. Notice that it is the manner in which the structure of the parabolics and R-groups for $U(n, n)$ (respectively $U(n, n+1)$) parallels those for $Sp(2n)$ (respectively $SO(2n + 1)$) which allows us to use these arguments.

Proposition 4.1. *Let $G = U(n, n)$ or $U(n, n + 1)$. Suppose M is any Levi subgroup of G, $\sigma \in \mathscr{E}_2(M)$, and R is the reducibility group associated to $i_{G,M}(\sigma)$. Then the commuting algebra $C(\sigma) \simeq \mathbb{C}[R]$.*

Remark. By Theorems 1.4(b) and 3.4, this proves that every $i_{G,M}(\sigma)$ decomposes with multiplicity one.

Proof. We need to show that we can choose intertwining operators T_w, $w \in R$, so that $T_w(\sigma) = w\sigma T_w$ and $T_{w_1 w_2} = T_{w_1} T_{w_2}$. Suppose $\sigma = \sigma_1 \otimes \cdots \otimes \sigma_r \otimes \rho$ and σ acts on $V = V_1 \otimes \cdots \otimes V_r \otimes V'$.

If $C_i \in W(\sigma)$, then $\sigma_i \simeq \sigma_i^\varepsilon$. Let $T_i : V_i \to V_i$ satisfy $T_i\sigma_i = \sigma_i^\varepsilon T_i$. Since $\varepsilon^2 = 1$, we must have $T_i^2 = \lambda I_{V_i}$, with $\lambda \in \mathbb{C}^*$. Therefore, without loss of generality, we can assume that $T_i^2 = I_{V_i}$. Now extend T_i to V by setting

$$T_i^V(v_1 \otimes \cdots \otimes v_r \otimes v') = v_1 \otimes \cdots \otimes T_iv_i \otimes \cdots \otimes v_r \otimes v'.$$

Then $T_i^V\sigma = (C_i\sigma)T_i^V$ and $(T_i^V)^2 = 1$. Suppose $i < j$ and $C_i, C_j \in W(\sigma)$. Then $T_i^V T_j^V = T_j^V T_i^V$. Let $T_{C_i} = T_i^V$. If $c \in R$ and $c = C_{i_1}C_{i_2}\cdots C_{i_j}$, then $T_c = T_{C_{i_1}}\cdots T_{C_{i_j}}$ is an intertwining operator between σ and $c\sigma$. Moreover, the commutativity of the operators T_{C_i}, and our normalization, guarantees that $T_{cc'} = T_cT_{c'}$ for all $c, c' \in R$. $\square$

Lemma 4.2. *For any* $\sigma \in \mathscr{E}_2(M)$, *there is a* $w_0 \in R$ *with* $\mathfrak{a}_{w_0} = \mathfrak{a}_R$. *Further,* $\mathfrak{a}_R = \{0\}$ *if and only if* $R \simeq \mathbb{Z}_2^r$.

Proof. By Theorem 3.4, we know $R \simeq \mathbb{Z}_2^d$ and $R \subset \langle C_1, \ldots, C_r \rangle$. We can assume that $R = \langle C_1, \ldots, C_d \rangle$. Let $w_0 = C_1 \cdots C_d$. Then, for all $w \in R$, $\mathfrak{a}_{w_0} \subseteq \mathfrak{a}_w$, and thus, $\mathfrak{a}_{w_0} = \mathfrak{a}_R$. If $w \in \langle C_1, \ldots, C_r \rangle$ and $\mathfrak{a}_w = \{0\}$, then $w = C_1 \cdots C_r$. Therefore, $\mathfrak{a}_R = \{0\}$ if and only $R \simeq \mathbb{Z}_2^r$. $\square$

Theorem 4.3. *Let* $G = U(n, n)$ *or* $U(n, n+1)$. *Let* M *be a Levi component of* G *with* $M = GL(n_1, E) \times \cdots \times GL(n_r, E) \times G(m)$. *Then the following are equivalent*:
  (a) $i_{G,M}(\sigma)$ *has an elliptic constituent,*
  (b) *all the constituents of* $i_{G,M}(\sigma)$ *are elliptic,*
  (c) $R \simeq \mathbb{Z}_2^r$.

Proof. By Theorem 3.4, $R \simeq \mathbb{Z}_2^d$, and, by Proposition 4.1, $C(\sigma) \simeq \mathbb{C}[R]$. Therefore, by Theorem 1.5, (a) and (b) are equivalent, and both are equivalent to the existence of a $w_0 \in R$ with $\mathfrak{a}_{w_0} = \{0\}$. Thus, by Lemma 4.2, (a) and (c) are equivalent. $\square$

The next result is now a direct consequence of Proposition 1.7.

Proposition 4.4. *Let* $G = U(n, n)$ *or* $U(n, n+1)$. *Let* $\pi \in \mathscr{E}_t(G)$. *Then* π *is either elliptic or* $\pi = i_{G,M}(\tau)$ *for some proper Levi subgroup* M *of* G *and some elliptic* $\tau \in \mathscr{E}_t(M)$. $\square$

Suppose $R \simeq \mathbb{Z}_2^r$. For every $\kappa \in \hat{R}$, we let $\varepsilon(\kappa) = \kappa(C_1 \cdots C_r) = \pm 1$. Let $1 \in \hat{R}$ denote the trivial character.

Proposition 4.5. *Suppose* $R \simeq \mathbb{Z}_2^r$ *and* $\kappa \in \hat{R}$. *Let* π_κ *be the corresponding irreducible constituent of* $i_{G,M}(\sigma)$ *and* Θ_κ *its character. Then* $\Theta_\kappa^e = \varepsilon(\kappa)\Theta_1^e$.

Proof. For $1 \leq i \leq r$, we let M_i be the maximal Levi subgroup of G containing M, such that $M \simeq GL(n_i) \times G(n-n_i)$. Let R_i denote the R-group attached to $i_{M_i,M}(\sigma)$. Then, by Theorem 3.4, $R_i \simeq \mathbb{Z}_2^{r-1}$ and is generated by $\{C_j\}_{j\neq i}$. Thus, by Theorem 4.3, every irreducible constituent τ of $i_{M_i,M}(\sigma)$ is elliptic. Let $\kappa_i \in \hat{R}_i$, and let $\hat{R}(\kappa_i) = \{\kappa_i^+, \kappa_i^-\}$, where $\kappa_i^+(C_i) = 1$ and $\kappa_i^-(C_i) = -1$. Now, by Theorem 1.6, $i_{G,M_i}(\tau_{\kappa_i}) = \pi_{\kappa_i^+} \oplus \pi_{\kappa_i^-}$. Therefore, $\Theta_{\kappa_i^+}^e = -\Theta_{\kappa_i^-}^e$.

Now we proceed by induction. Let $s(\kappa)$ be the number of indices, $1 \leq i \leq r$, such that $\kappa(C_i) = -1$. If $s(\kappa) = 0$, then $\kappa = 1$, $\varepsilon(\kappa) = 1$, and the

proposition is trivially true. Suppose it is true for $s(\kappa) = s$. Suppose $s(\kappa) = s + 1$. Let i be an index with $\kappa(C_i) = -1$, and let $\kappa_i = \kappa|_{R_i}$. Then $\kappa = \kappa_i^-$ and $s(\kappa_i^+) = s$. Therefore,

$$\Theta_\kappa^e = \Theta_{\kappa_i^-}^e = -\Theta_{\kappa_i^+}^e = -\varepsilon(\kappa_i^+)\Theta_1^e = (-1)^{s+1}\Theta_1^e = \varepsilon(\kappa)\Theta_1^e. \quad \square$$

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