TWO RESULTS ON THE 2-LOCAL EHP SPECTRAL SEQUENCE

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Abstract. The $E_2$-term of the 2-local EHP spectral sequence is shown to be a $\mathbb{Z}/2$ module. 4 is the order of the identity map on the double loop space of the fiber $W(n)$ of the double suspension $E^2: S^{2n-1} \to \Omega^2 S^{2n+1}$.

1. Introduction

Restrict attention to the category of 2-local spaces. The EHP fibrations
\[ \Omega^2 S^{2q+1} \xrightarrow{p} S^q \xrightarrow{E} \Omega S^{q+1} \xrightarrow{H} \Omega^2 S^{2q+1} \]
give a tower of fibrations converging to $Q(S^0)$, whose homotopy spectral sequence is the EHP spectral sequence \[ E_1^{p,q} = \pi_{p+q}(S^{2q-1}) \Rightarrow \pi_p \], with differentials $d_r: E_r^{p,q} \to E_r^{p-r,q-1}$. James [6] proved that $2^{2n} \pi_*(S^{2n+1}) = 0$, by showing that $2\pi_*(S^{2n+1}) \subset \text{Im}(E)$ and $E(2\pi_*(S^{2n})) \subset \text{Im}(E^2)$. Thus the $E_\infty$-term of the EHP spectral sequence is a $\mathbb{Z}/2$ module. James's work was translated to spaces [1, §5]:

Lemma 1.1. (1) $\Omega^2 S^{2n+1} \xrightarrow{\Omega H} \Omega^2 S^{4n+1} \xrightarrow{2} \Omega^2 S^{4n+1}$ is nullhomotopic.

(2) There exists a map $\phi$ making the following diagram homotopy commutative.

\[ \Omega^2 S^{2n} \xrightarrow{2} \Omega^2 S^{2n} \xrightarrow{\Omega^2 E} \Omega^3 S^{2n+1} \]

\[ \phi \]

\[ \Omega S^{2n-1} \]

Mahowald [8] made the following conjecture, which will follow from Lemma 1.1 and an extensive amount of diagram chasing.
Theorem 1.2. The $E_2$-term of the EHP spectral sequence is a $\mathbb{Z}/2$ module. The $4^{th}$ power map of $\Omega^2W(n)$ is nullhomotopic, and $\pi_*(W(n))$ has exponent 4.

Let $d_1^+: \Omega^3S^{4n+1} \xrightarrow{\Omega p} \Omega^2S^{2n} \xrightarrow{H} \Omega^2S^{4n-1}$ and $d_1^-: \Omega^3S^{4n+1} \xrightarrow{\Omega p} \Omega^2S^{2n+1} \xrightarrow{H} \Omega^2S^{4n+1}$ denote the composites which realize the first EHP differential. Selick [11] improved the James exponent to $2^{2n-[n/2]}$. Cohen [1, §6] reformulated this as a compression of the $H$-space squaring map on $\Omega^4S^{4n+1}$ through $\Omega^2S^{4n-1}$. Theorem 1.2 is implied by the following compression result, which extends their work.

Theorem 1.3. There exist maps $\mathcal{F}^+: \Omega^2S^{4n+1} \to \Omega^4S^{4n+1}$ and $\mathcal{F}^+: \Omega^2S^{4n-3} \to \Omega^2W(n)$ making the following diagrams homotopy commutative.

\[
\begin{array}{ccc}
\Omega^4S^{4n+1} & \xrightarrow{\Omega d_1^+} & \Omega^2S^{4n-1} \\
& \downarrow & \downarrow \\
\Omega^4S^{4n+1} & \xrightarrow{\Omega d_1^-} & \Omega^2S^{4n-3} \\
& \downarrow & \downarrow \\
\Omega^4S^{4n+1} & \xrightarrow{\Omega^2\partial} & \Omega^2W(n)
\end{array}
\]

2. Proofs

For any space $X$, we will denote by $2: \Omega X \to \Omega X$ the $H$-space squaring map. We will often use the following fact. For any map $f: \Omega X \to \Omega Y$, the composites $\Omega^2X \xrightarrow{\Omega f} \Omega^2X \xrightarrow{2} \Omega^2Y$ and $\Omega^2X \xrightarrow{2} \Omega^2Y \xrightarrow{\Omega f} \Omega^2Y$ are homotopic. We will use the following result about coliftings, which we state without proof.

Lemma 2.1. Let $F \xrightarrow{p} E \xrightarrow{\partial} B$ be a fibration, and let $f: B \to X$ be a map such that $f \cdot p: E \to X$ is nullhomotopic. Then $\Omega f$ factors through $\partial$, by a colifting $\mathcal{B}: F \to \Omega X$, which makes the following diagram commutes up to homotopy.

\[
\begin{array}{ccc}
\Omega B & \xrightarrow{\partial} & F \\
\downarrow & & \downarrow \\
\Omega f & \downarrow & \mathcal{B} \\
& \Omega X & \\
\end{array}
\]

Proof of Theorem 1.3. The EHP fibration $\Omega S^{2n} \xrightarrow{\Omega E} \Omega^2S^{2n+1} \xrightarrow{\Omega H} \Omega^2S^{4n+1}$ and Lemmas 1.1(1) and 2.1 give a colifting $\mathcal{B}: \Omega S^{2n} \to \Omega^3S^{4n+1}$ making the diagram

\[
\begin{array}{ccc}
\Omega^3S^{4n+1} & \xrightarrow{\Omega p} & \Omega^3S^{2n} \\
& \downarrow & \downarrow \\
\Omega^3S^{4n+1} & \xrightarrow{\mathcal{B}} & \Omega^3S^{2n}
\end{array}
\]

homotopy commutative. But $\mathcal{B} \cdot E: S^{2n-1} \to \Omega^3S^{4n+1}$ is nullhomotopic. By Lemma 2.1 and the EHP fibration $\Omega^2S^{4n-1} \xrightarrow{p} S^{2n-1} \xrightarrow{E} \Omega^2S^{2n}$, there exists
a colifting $\mathcal{F}^+: \Omega^2 S^{4n-1} \to \Omega^4 S^{4n+1}$ making the diagram

\[\begin{array}{ccc}
\Omega^2 S^{2n} & \xrightarrow{\Omega E} & \Omega^2 S^{4n-1} \\
\Omega \mathcal{F} & \downarrow & \downarrow \\
\Omega^4 S^{4n+1} & \xrightarrow{\mathcal{F}^+} & \Omega^4 S^{4n+1} \\
\end{array}\]

homotopy commutative. This proves the first part of Theorem 1.3.

By Lemma 1.1(2), the composite $\Omega^2 S^{2n} \xrightarrow{2-\Omega E \cdot \phi} \Omega^2 S^{2n} \xrightarrow{\Omega^2 E} \Omega^3 S^{2n+1}$ is nullhomotopic. Hence there exists a map $\psi: \Omega^2 S^{2n} \to \Omega^4 S^{4n+1}$ making the diagram

\[\begin{array}{ccc}
\Omega^2 S^{2n} & \xrightarrow{\Omega E} & \Omega^2 S^{4n-1} \\
| & & | \\
\Omega^2 S^{4n+1} & \xrightarrow{\Omega^2 \mathcal{P} E} & \Omega^2 S^{2n} & \xrightarrow{\Omega H} & \Omega^2 S^{4n-1} \\
\end{array}\]

commute up to homotopy, since (cf. [1, Proof of Lemma 4.1]) $\Omega H$ is linear.

We have an induced map of fibers $\beta: \Omega S^{2n-1} \to \Omega W(n)$, obtained by pulling back the outer trapezoid to the left, making the following diagram homotopy commutative.

\[\begin{array}{ccc}
\Omega^3 S^{4n-1} & \xrightarrow{\Omega \mathcal{P}} & \Omega^2 S^{2n-1} & \xrightarrow{\Omega E} & \Omega^2 S^{2n} & \xrightarrow{\Omega H} & \Omega^2 S^{4n-1} \\
\downarrow 2 & & \downarrow \beta & & \downarrow \psi & & \downarrow 2 \\
\Omega^3 S^{4n-3} & \xrightarrow{\Omega \mathcal{P} \mathcal{E}} & \Omega W(n) & \xrightarrow{\Omega \mathcal{E} \cdot \phi} & \Omega^4 S^{4n+1} & \xrightarrow{\Omega \mathcal{E}^+ \cdot \phi} & \Omega^2 S^{4n-1} \\
\end{array}\]

But $\beta \cdot \mathcal{F}: S^{2n-2} \to \Omega W(n)$ is nullhomotopic. The EHP fibration $\Omega^2 S^{4n-3} \xrightarrow{\mathcal{E} \cdot \phi} S^{2n-2} \xrightarrow{\mathcal{E} \cdot \phi} \Omega S^{2n-1}$ and Lemma 2.1 then yield the colifting $\mathcal{F}^-: \Omega^2 S^{4n-3} \to \Omega^2 W(n)$ making the following diagram homotopy commutative.

\[\begin{array}{ccc}
\Omega^2 S^{2n-1} & \xrightarrow{\Omega H} & \Omega^2 S^{4n-3} \\
\Omega \beta & & \mathcal{F}^- \\
\Omega^2 W(n) & & \\
\end{array}\]

Proof of Theorem 1.2. By Theorem 1.3, \(\text{Ker}(d^+_{1}) \subset \pi_{\ast}(S^{4n+1})\) is a $\mathbb{Z}/2$ module. Thus each $E^*_{2, 2n+1}$ is a $\mathbb{Z}/2$ module. By Theorem 1.3, any cycle $\alpha \in \text{Ker}(d^+_{1}) \subset \pi_{\ast}(S^{4n-1})$ satisfies $2\alpha \in \text{Im}(d^+_{1})$. Hence each $E^*_{2, 2n}$ is a $\mathbb{Z}/2$ module.

We have the fibration sequence $\Omega^2 S^{4n-1} \xrightarrow{\mathcal{E} \cdot \phi} W(n) \xrightarrow{\mathcal{E} \cdot \phi} \Omega^3 S^{4n+1} \xrightarrow{d^+_{1}} \Omega S^{4n-1}$.

By Theorem 1.3, the composite $\Omega W(n) \xrightarrow{\mathcal{E} \cdot \phi} \Omega^4 S^{4n+1} \xrightarrow{2} \Omega^4 S^{4n+1}$ is nullhomotopic. As indicated by the following homotopy commutative diagram, there exists a lifting $f: \Omega W(n) \to \Omega^3 S^{4n-1}$ of the $H$ space squaring map of $\Omega W(n)$.
We have the following homotopy commutative diagrams.

\[ \begin{array}{ccc}
\Omega^2 S^{4n-1} & \xrightarrow{p} & S^{2n-1} \\
\downarrow \vartheta & & \downarrow \iota \\
\Omega W(n) & & W(n) \\
\end{array} \]

\[ \begin{array}{ccc}
\Omega^3 S^{4n-1} & \xrightarrow{d^{-}_1} & \Omega S^{4n-3} \\
\downarrow \vartheta & & \downarrow \iota \\
\Omega W(n) & & \Omega S^{2n-1} \\
\end{array} \]

By looping the above parallelogram and applying Lemma 1.1(1), we see that the composite $\Omega^2 W(n) \xrightarrow{\Omega \vartheta} \Omega^4 S^{4n-1} \xrightarrow{\Omega d^{-}_1} \Omega^2 S^{4n-3}$ is nullhomotopic. The composite $\Omega^2 W(n) \xrightarrow{\Omega \vartheta} \Omega^4 S^{4n-1} \xrightarrow{2} \Omega^4 S^{4n-1} \xrightarrow{\Omega^2 \vartheta} \Omega^2 W(n)$ is nullhomotopic, by Theorem 1.3. Hence $4: \Omega^2 W(n) \to \Omega^2 W(n)$, the 4\text{th} power map, is nullhomotopic. \( \square \)

### 3. Remarks

James [6] also showed that $2E(x) = 0$ for all $x \in \text{Ker}(E^2) \subset \pi_*(S^q)$. When $q = 2n - 1$, this gives evidence for Theorem 1.2, as it is implied by $4\pi_*(W(n)) = 0$. We used the case $q = 2n$ of James's result in an earlier version of our paper.

Richter [10] strengthened Theorem 1.3, showing that $2 \simeq -\Omega E^2 \cdot d^+_1$ on $\Omega^3 S^{4n+1}$ and $2 - \Omega^3(2i) \simeq -\Omega E^2 \cdot d^-_1$ on $\Omega^3 S^{4n-1}$, solving a conjecture of Gray [3] and Mahowald, which Harper [5] proved at odd primes. At an odd prime $p$, Cohen, Moore, and Neisendorfer [2] showed that the $p$\text{th} power map on $\Omega W(n)$ is nullhomotopic. Gray [4] showed that $W(n)$ deloops, essentially by delooping the map $d^+_1$. It was already known that $\pi_*(W(2))$ had exponent 4, by Cohen's [1, Theorem 19.1] splitting $\Omega^2 S^5(2) \simeq W(2) \times \Omega^2 S^3(3)$.

Mahowald [8] further conjectured that $(d^-_1)_* : \pi_*(S^{4n-1}) \to \pi_*(S^{4n-3})$. Note that James shows that $\text{Ker}(E) \subset \pi_*(S^{2n+1})$ is a $\mathbb{Z}/4$ module.
The conjecture implies that \( \text{Ker}(E) \) is a \( \mathbb{Z}/2 \) module. By [10], the conjecture also implies

**Conjecture C2.** For any element \( \alpha \in \pi_\ast(S^{4n-1}) \), \( (2i) \cdot \alpha = 2\alpha \in \pi_\ast(S^{4n-1}) \).

One might wonder whether \( 2 \simeq \Omega^k(2i) \) on \( \Omega^kS^{4n-1} \) for some \( k \). Note [1, §§11 and 12] that away from Arf invariant one or Hopf invariant one dimensions, \( k \) must be at least 3.

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**REFERENCES**


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