Abstract. Multiplication by $x$ determines an automorphism of the compact dual group of $\Lambda_g = \mathbb{Z}[x, x^{-1}]/(g)$ for $g \in \mathbb{Z}[x]$. We determine the $K$-groups of the $C^*$-algebra associated with this dynamical system if $g$ is irreducible and has degree one or two. Partial results are included if the degree of $g$ is three.

To each $g \in \mathbb{Z}[x]$ associate a $C^*$-algebra $B_g$, the crossed product $C^*$-algebra associated to the dynamical system $(\hat{\Lambda}_g, \alpha, Z)$. Here $\hat{\Lambda}_g$ is the dual group of the discrete abelian group $\Lambda_g = \mathbb{Z}[x, x^{-1}]/(g)$ where $(g)$ is the principal ideal generated by $g$ in the ring $\mathbb{Z}[x, x^{-1}]$. The automorphism $\alpha$ of $\hat{\Lambda}_g$ is that defined by multiplication by $x$ on $\Lambda_g$. The dynamical systems $(\hat{\Lambda}_g, \alpha)$ are examples of (abelian) Markov groups ([8]), in particular they are generalized solenoids ([2]).

In this note the $K$-groups of $B_g$ are computed for $g$ nonconstant, irreducible, and of degree one or two. Partial results for $g$ degree three are also obtained. This is accomplished by a straightforward (though involved) application of the Pimsner-Voiculescu six-term exact sequence ([7]). In addition, the range of any state on $K_0(B_g)$ arising from a tracial state on $B_g$ is shown to be $\mathbb{Z}$ for any (nonconstant, irreducible) $g$.

Although of little interest in their own right, these calculations do allow a comparison of the computed $K$-groups with the known (for degree $g$ equal to one or two) $*$ (or anti-$*$)-isomorphism classes of these algebras [1]. This yields many examples of non-$*$ (or anti-$*$)-isomorphic algebras with isomorphic $K$-groups and, since both the tracial states on $K_0$ and (at least in degree one) the possible order structures on $K_0$ provide no additional information, one is left with the interesting (especially in light of the questions raised in [3]) possibility that the $K$-groups are of limited value in determining the isomorphism classes of this family of amenable algebras. The isomorphism classification arrived at in [1] (for degree $g$ one or two) used a sequence of invariants related to the entropy of the underlying dynamical system. One can contrast this with the

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family of rotation algebras, for example, where $K$-theoretic means provide a classification and the entropy of each underlying dynamical system is zero.

Admittedly the algebras $B_g$ are not simple, however, they do possess a separating family of finite-dimensional quotients. Although the algebras are not AF-algebras ($K_1$ is nonzero), they are finite and embeddable in AF-algebras. This follows from [5], since the finite periodic points in $\Lambda_g$ are dense ([2], [4]) and the dynamical system $(\widehat{\Lambda}_g, \alpha)$ is thus chain recurrent.

In the following, if $g = \sum_{i=0}^d a_i x^i \in \mathbb{Z}[x]$ with $g(0) \neq 0$, then $g^0$ denotes the polynomial $\sum_{i=0}^d a_{d-i} x^i$ and $\deg(g)$ denotes the degree of $g$. The content of $g$ is written $\text{cont}(g)$. For $n, m \in \mathbb{Z}$, $(n, m)$ is the greatest common divisor of $n$ and $m$ and $(n, m)$ is the least common multiple of $n$ and $m$. If $\phi$ is a $\mathbb{Z}$-module map, $\text{im} \phi$ denotes the image of $\phi$, $\ker \phi$ denotes the kernel of $\phi$ and $\text{coker} \phi$ denotes the cokernel of $\phi$.

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We compute the $K$-groups of the abelian $C^*$-algebra $C(\widehat{\Lambda}_g)$ for irreducible nonconstant $g \in \mathbb{Z}[x]$ and prove some preliminary algebraic results.

For $g \in \mathbb{Z}[x]$ nonconstant, irreducible let $a \in \mathbb{C} \setminus \{0\}$ denote a root of $g$, so $\Lambda_g \simeq \mathbb{Z}[a, a^{-1}]$. The automorphism $\alpha$ of $\widehat{\Lambda}_g$ is dual to the $\mathbb{Z}$-module map $M_a$ (multiplication by $a$) in the ring $\Lambda_g$. Note that $\Lambda_g$ is torsion free if and only if $\text{cont}(g) = 1$. Thus, $\Lambda_g$ is a discrete, torsion free, abelian, rank $d$ group where $d = \deg(g)$. Although the group $\Lambda_g$ is not finitely generated (unless both $g$ and $g^0$ are monic), $\Lambda_g$ is a direct limit of its finitely generated submodules. Since any finitely generated submodule of $\Lambda_g$ is torsion free (and thus free), write $\Lambda_g$ as a direct limit of submodules each isomorphic to $\mathbb{Z}^d$. For example, let $\mathcal{M}_n$ be the submodule of $\Lambda_g$ generated by \{\{a^{-n}, \ldots, 1, a, \ldots, a^{d+n}\}\} (identifying $\Lambda_g$ with $\mathbb{Z}[a, a^{-1}]$). Then $\mathcal{M}_n \simeq \mathbb{Z}^d$ and $\Lambda_g = \lim_{n \to \infty} (\mathcal{M}_n, i_n)$ where $i_n : \mathcal{M}_n \to \mathcal{M}_{n+1}$ is the natural inclusion.

The group $K_*(C^*(\mathcal{M}_n))$ is isomorphic to $\bigoplus_{j=0}^d \wedge^j \mathbb{Z}^d$ (denoted by $\wedge \mathbb{Z}^d$) with $K_0$ corresponding to the even indices and $K_1$ to the odd indices. The induced map $(i_n)_* : \bigoplus_{j=0}^d \wedge^j \mathbb{Z}^d \to \bigoplus_{j=0}^d \wedge^j \mathbb{Z}^d$ (denoted $\wedge i_n$). Since $\Lambda_g$ is discrete and abelian, it is straightforward to see that $C^*(\Lambda_g) \simeq \lim_n (C^*(\mathcal{M}_n))$. Since $K_*$ commutes with lim and the wedge product of $\mathbb{Z}$-modules commutes with lim, it follows that $K_*(C^*(\Lambda_g)) \simeq \bigoplus_{j=1}^d \wedge^j \Lambda_g \simeq \bigoplus_{j=1}^d \wedge^j \mathbb{Z}[a, a^{-1}]$.

**Proposition 1.1.** The automorphism $\alpha$ of $C(\widehat{\Lambda}_g)$ induces the map $\bigoplus_{j=0}^d \wedge^j M_a$ on $K_*(C^*(\Lambda_g))$.

**Proof.** Let $\mathcal{M}_n$ be the submodules of $\Lambda_g$ defined above and note that $M_a(\mathcal{M}_n) \subseteq \mathcal{M}_{n+1}$. The maps $M_n = M_a|_{\mathcal{M}_n}$ define a homomorphism of the directed system $(\mathcal{M}_n, i_n)$ to $(\mathcal{M}_{n+1}, i_{n+1})$ and the homomorphism $\lim M_n$ of $\lim \mathcal{M}_n = \Lambda_g$ to $\lim \mathcal{M}_{n+1} = \Lambda_g$ is the map $M_a$. The corresponding maps $\tilde{M}_n : C^*(\mathcal{M}_n) \to C^*(\mathcal{M}_{n+1})$ on the $C^*$-algebra level form a map of the directed system $(C^*(\mathcal{M}_n), i_n)$ to itself. The homomorphism $\lim \tilde{M}_n$ of $\lim C^*(\mathcal{M}_n) = C^*(\Lambda_g)$ to itself is the map $\alpha$.

The functor $K_*$ commutes with direct limits, so the endomorphism $\alpha_*$ of the group $K_*(C^*(\Lambda_g))$ is the homomorphism $(\tilde{M}_n)_*$ of the directed system.


\( (K_n(\mathfrak{M}_n)), (i_{n+i}) \) to \( (K_n(\mathfrak{M}_{n+1})), (i_{n+1+i}) \). However, \( (\widehat{M}_n)_n \) is the map 
\( \bigoplus_{j=0}^d \wedge^j M_n \). Finally, note that \( \lim \) commutes with direct sums and with wedge products. □

Since \( \Lambda_g \) is a limit of \( \mathbb{Z} \)-submodules isomorphic to \( \mathbb{Z}^d \), \( \wedge^d \Lambda_g \) is isomorphic to a limit of \( \mathbb{Z} \)-submodules isomorphic to \( \mathbb{Z} \). In particular, the elements \( a^{j_0} \wedge \cdots \wedge a^{j_{d-1}} \) (with \( j_k \in \mathbb{Z} \) and \( j_0 < \cdots < j_{d-1} \)) generate \( \wedge^d \Lambda_g \) as a \( \mathbb{Z} \)-module.

Note also that \( \wedge^d \Lambda_g \) is a limit of torsion free modules, so is torsion free.

**Lemma 1.2.** Let \( g = \sum_{i=0}^d a_i x^i \) (nonconstant, irreducible) and \( l = \{a_0, a_d\} \). If \( e \) is the element \( 1 \wedge a \wedge \cdots \wedge a^{d-1} \) of \( \wedge^d \Lambda_g \), then \( l^{-1}e \in \wedge^d \Lambda_g \).

**Proof.** Let \( c_j = 1 \wedge \cdots \wedge \hat{a}_l \wedge \cdots \wedge a^d \) and \( b_j = a^{-1} \wedge \cdots \wedge \hat{a}_j \wedge \cdots \wedge a^{d-1} \) for \( 0 \leq j \leq d - 1 \). Since \( g(a) = 0 \), it follows that \( a_d c_j = (-1)^{d-j} a_j e \) and \( a_0 b_j = (-1)^{j+1} a_{j+1} e \) for \( 0 \leq j \leq d - 1 \). If \( r = (a_0, a_d) \), then \( l = a_0 a_d \); so \( a_j a_0 r^{-1} e = (-1)^{d-j} l c_j \) and \( a_{j+1} a_d r^{-1} e = (-1)^{j+1} l b_j \) for \( 0 \leq j \leq d - 1 \). Since \( a_d a_0 r^{-1} e = l e \in l \wedge^d \Lambda_g \) also, it follows that \( a_j a_0 r^{-1} e \) and \( a_{j+n} a_d r^{-1} e \in l \wedge^d \Lambda_g \) for \( 0 \leq j \leq d \). Since \( (a_d a_0 r^{-1}, a_d r^{-1}) = 1 \), we have \( a_j e \in l \wedge^d \Lambda_g \) for \( 0 \leq j \leq d \). However, \( \text{cont}(g) = 1 \), so \( (a_0, a_1, \ldots, a_d) = 1 \) and \( e \in l \wedge^d \Lambda_g \). The result follows since \( \wedge^d \Lambda_g \) is torsion free. □

The elements \( a^{j_0} \wedge \cdots \wedge a^{j_{d-1}} \) of \( \wedge^d \Lambda_g \) are all contained in the \( \mathbb{Z} \)-module \( \mathbb{Z}[a_0^{-1}, a_d^{-1}]e = \mathbb{Z}[l^{-1}]e \), and since they generate \( \wedge^d \Lambda_g \) as a \( \mathbb{Z} \)-module, \( \wedge^d \Lambda_g \subseteq \mathbb{Z}[l^{-1}]e \).

**Proposition 1.3.** If \( g, e, l \) are as in the preceding lemma, then \( \mathbb{Z}[l^{-1}]e = \wedge^d \Lambda_g \).

**Proof.** It is enough to show \( \mathbb{Z}[l^{-1}]e \subseteq \wedge^d \Lambda_g \). Since \( l^{-1}e \in \wedge^d \Lambda_g \), the result will follow if \( \wedge^d \Lambda_g \) is a commutative ring with unit \( e \).

First, define a multiplication on the generators \( a^{j_0} \wedge \cdots \wedge a^{j_{d-1}}, j_0 < \cdots < j_{d-1} \), of \( \wedge^d \Lambda_g \). Using the alternating \( d \)-multilinear map \( \Delta \) of \( \mathbb{Q}^d \) (identified with \( \mathbb{Q}[a] \)) into \( \mathbb{Q} \) taking the value 1 on the basis \( \{1, a, \ldots, a^{d-1}\} \) of \( \mathbb{Q}[a] \), we identify \( \wedge^d \mathbb{Q}^d \) with \( \mathbb{Q} \) (\( e \) corresponding to 1). Let \( \varphi_{j_0 \cdots j_{d-1}} = \varphi \) denote the \( \mathbb{Q} \)-linear map of \( \mathbb{Q}[a] \) determined by mapping \( a^k \) to \( a^{j_k} \), \( 0 \leq k \leq d - 1 \). This map is also \( \mathbb{Z} \)-linear and maps \( \Lambda_g \) to itself. We have \( a^{j_0} \wedge \cdots \wedge a^{j_{d-1}} = \wedge^d \varphi(e) = (\det \varphi)e \). Define the product of \( a^{j_0} \wedge \cdots \wedge a^{j_{d-1}} \) with \( a^{i_0} \wedge \cdots \wedge a^{i_{d-1}} \) as \( \wedge^d (\varphi_{j_0 \cdots j_{d-1}} \circ \varphi_{i_0 \cdots i_{d-1}})(e) = \det(\varphi_{j_0 \cdots j_{d-1}} \circ \varphi_{i_0 \cdots i_{d-1}}) \cdot \det(\varphi_{i_0 \cdots i_{d-1}})e \) and extend this linearly to a product on \( \wedge^d \Lambda_g \) (\( \wedge^d \Lambda_g \) is a subring of \( \mathbb{Q} \)). □

The proof of the above proposition shows \( \wedge^d \Lambda_g \) is a ring isomorphic to \( \mathbb{Z}[l^{-1}] \). The endomorphism \( \wedge^d \Lambda_g \) of \( \wedge^d \Lambda_g \) is multiplication by \( \det(M_a) = (-1)^d a_0 a_d e \) in \( \mathbb{Z}[l^{-1}] \).

**Proposition 1.4.** Let \( g = \sum_{i=0}^d a_i x^i \in \mathbb{Z}[x] \) be nonconstant, irreducible. Then \( \Lambda_g / (1 - M_a) \Lambda_g \simeq \mathbb{Z} / (\sum_{i=0}^d a_i) \mathbb{Z} \).

**Proof.** Let \( R = \mathbb{Z}[x] / (1 - x) \mathbb{Z}[x] \). Then \( R \simeq \mathbb{Z} \) and \( \Lambda_g / (1 - M_a) \Lambda_g \simeq \mathbb{Z} / \overline{g} \mathbb{Z} \) where \( \overline{g} = \sum_{i=0}^d a_i \) is the class of \( g \) in \( R \) (cf. [2]). □

**Definition.** If \( n \neq 0 \), \( l \in \mathbb{N} \), define \( n : l = n(n, l^{-m_0})^{-1} \) where \( m_0 \) is the maximum multiplicity of any prime dividing \( n \). Thus, \( n : l \) is formed by removing from \( n \) any prime also dividing \( l \).
Lemma 1.5. Let \( n \neq 0, l \in \mathbb{N} \). Then \( \mathbb{Z}[l^{-1}]/n\mathbb{Z}[l^{-1}] \cong \mathbb{Z}/t\mathbb{Z} \) where \( t = n : l \).

Proof. Let \( H_k \) be the subgroup of \( Z_k \cong \mathbb{Z} \) generated by \( n \) \((k \in \mathbb{N})\). The map \( \varphi_k : Z_k \to Z_{k+1} \) given by multiplication by \( l \) defines a map \( \overline{\varphi}_k : Z_k/H_k \to Z_{k+1}/H_{k+1} \). We have \( \mathbb{Z}[l^{-1}] \cong \lim(Z_k, \varphi_k) \) and \( \mathbb{Z}[l^{-1}]/n\mathbb{Z}[l^{-1}] \cong \lim(Z_k/H_k, \overline{\varphi}_k) \). Each \( Z_k/H_k \) is isomorphic to the cyclic group \( \mathbb{Z}/n\mathbb{Z} \) and \( \overline{\varphi}_k \) is multiplication by \( l \). If \( b \) is a generator of \( \mathbb{Z}/n\mathbb{Z} \), then \( l^m b \) has order \( n(l^m, n)^{-1} \). For \( m \) large enough (\( m \) larger than the maximum multiplicity of any prime dividing \( n \)) \( l^m b \) has order \( t = n : l \). It follows that \( \mathbb{Z}[l^{-1}]/n\mathbb{Z}[l^{-1}] \cong \lim(\mathbb{Z}/t\mathbb{Z}, m_t) \cong \mathbb{Z}/t\mathbb{Z} \) (where \( m_t \) is multiplication by \( l \)). \( \square \)

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Let \( \wedge^{ev} \Lambda_g \) denote \( \bigoplus_{j=0}^{[d/2]} \wedge^{2j} \Lambda_g \) and \( \wedge^{odd} \Lambda_g \) denote \( \bigoplus_{j=0}^{[d/2]} \wedge^{2j+1} \Lambda_g \). The maps \( \wedge^{ev} M_a \) and \( \wedge^{odd} M_a \) are interpreted similarly. The Pimsner-Voiculescu six-term exact sequence for the \( K \)-groups of the crossed product \( C^* \)-algebra \( B_g = C^*(\Lambda_g) \times \mathbb{Z} \) is:

\[
\begin{array}{cccccc}
\wedge^{ev} \Lambda_g & \xrightarrow{1-\wedge^{ev} M_a} & \wedge^{ev} \Lambda_g & \xrightarrow{i_*} & K_0(B_g) \\
\downarrow \delta_1 & & \downarrow \delta_0 & & \\
K_1(B_g) & \xleftarrow{i_*} & \wedge^{odd} \Lambda_g & \xrightarrow{1-\wedge^{odd} M_a} & \wedge^{odd} \Lambda_g
\end{array}
\]

The cases \( \deg(g) = 1, 2, \) and \( 3 \) are dealt with separately.

The case \( \deg(g) = 1 \). The single root \( a \) of \( g(x) = a_0 + a_1 x \) is \(-a_0a_1^{-1} \) and the six-term exact sequence is:

\[
\begin{array}{cccccc}
\mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \xrightarrow{i_*} & K_0(B_g) \\
\delta_1 & & \downarrow \delta_0 & & \\
K_1(B_g) & \xleftarrow{i_*} & \Lambda_g & \xrightarrow{1-M_a} & \Lambda_g
\end{array}
\]

If \( a \neq 1 \), the map \( 1-M_a \) is injective (\( \Lambda_g \) is an integral domain) and \( i_* : \mathbb{Z} \to K_0(B_g) \) is an isomorphism. To compute \( K_1(B_g) \), first note that \( \ker \delta_1 = \text{im } i_\ast \cong \text{coker}(1-M_a) \). Also, \( 0 \to \ker \delta_1 \to K_1(B_g) \xrightarrow{\delta_1} \mathbb{Z} \to 0 \) splits since \( \mathbb{Z} \) is projective. Thus, \( K_1(B_g) \cong \mathbb{Z} \oplus \ker \delta_1 \cong \mathbb{Z} \oplus \mathbb{Z}/(a_0 + a_1)\mathbb{Z} \) by Proposition 1.4.

If \( a = 1 \), then \( \Lambda_g = \mathbb{Z}, B_g = C^*(\mathbb{T}^2), K_0(B_g) \cong \mathbb{Z} \oplus \mathbb{Z}, \) and \( K_1(B_g) \cong \mathbb{Z} \oplus \mathbb{Z} \).

Proposition 2.1. If \( g = a_0 + a_1 x \) is a degree 1, irreducible polynomial in \( \mathbb{Z}[x] \), then

\[
K_0(B_g) = \begin{cases} 
\mathbb{Z} \oplus \mathbb{Z} & \text{if } a_0 + a_1 = 0, \\
\mathbb{Z} & \text{otherwise}
\end{cases}
\]

and

\[
K_1(B_g) = \mathbb{Z} \oplus \mathbb{Z}/(a_0 + a_1)\mathbb{Z}.
\]

The case \( \deg(g) = 2 \). Let \( g(x) = \sum_{i=0}^{2} a_i x^i \in \mathbb{Z}[x] \) be irreducible with \( a \in \mathbb{C} \) a root and \( l = \{a_0, a_2\} \). The six-term exact sequence becomes:
Since \( a \neq 1 \), the map \( 1 - M_a \) is injective and \( \ker \delta_1 = 0 \). Thus, \( \delta_1 \) is surjective and \( K_0(B_g) \cong \mathbb{Z} \oplus \mathbb{Z}[l^{-1}]/(1 - aoa^{-1})\mathbb{Z}[l^{-1}] \). To compute \( K_1(B_g) \), note that 
\[
0 \to \ker \delta_1 \to K_1(B_g) \to \im \delta_1 \to 0
\]
is exact and multiplication by \( 1 - aoa^{-1} \) is either injective (if \( a_2 \neq ao \)) or zero (if \( a_2 = ao \)). In the first case, \( \im \delta_1 = \mathbb{Z} \); in the second case, \( \im \delta_1 = \mathbb{Z}/(\sum_{i=0}^2 a_i)\mathbb{Z} \). Also note that \( \ker \delta_1 = \im i_* \cong \mathbb{C}(1 - M_a) = \mathbb{Z}/(\sum_{i=0}^2 a_i)\mathbb{Z} \) by Proposition 1.4. Thus, if \( a_2 \neq ao \), then \( \im \delta_1 = \mathbb{Z} \) is projective and \( K_1(B_g) \cong \mathbb{Z} \oplus \mathbb{Z}/(\sum_{i=0}^2 a_i)\mathbb{Z} \).

If \( a_2 = ao \), then \( K_1(B_g) \) is an extension of \( \mathbb{Z} \oplus \mathbb{Z}[l^{-1}] \) by \( \mathbb{Z}/(\sum_{i=0}^2 a_i)\mathbb{Z} \). It is still possible to determine \( K_1(B_g) \) by computing the abelian group \( \text{Ext}(\mathbb{Z}[l^{-1}], \mathbb{Z}/m\mathbb{Z}) \) for \( m, l \) relatively prime in \( \mathbb{N} \).

**Proposition 2.2.** If \( m, l \in \mathbb{N} \) with \( (m, l) = 1 \), then \( \text{Ext}(\mathbb{Z}[l^{-1}], \mathbb{Z}/m\mathbb{Z}) = 0 \).

**Proof.** Let \( Z_k \cong \mathbb{Z} \) (with unit \( e_k \)) for \( k \in \mathbb{N}_0 \) and \( D \) the (free) submodule of the free \( \mathbb{Z} \)-module \( P = \bigoplus_{k \in \mathbb{N}_0} Z_k \) generated by the independent set \( \{h_i | h_i = e_i - le_{i+1}, i \in \mathbb{N}_0\} \). Since \( \mathbb{Z}[l^{-1}] \) is isomorphic to \( \text{lim}(Z_k, M_l) \) (where \( M_l \) denotes multiplication by \( l \)), we obtain the projective presentation \( 0 \to D \overset{p}{\to} \mathbb{Z}[l^{-1}] \to 0 \) of \( \mathbb{Z}[l^{-1}] \). Thus, \( \text{Ext}(\mathbb{Z}[l^{-1}], \mathbb{Z}/m\mathbb{Z}) = \text{coker} \mu^* \) where \( \mu^* : \text{Hom}_{\mathbb{Z}}(P, \mathbb{Z}/m\mathbb{Z}) \to \text{Hom}_{\mathbb{Z}}(D, \mathbb{Z}/m\mathbb{Z}) \). We show \( \mu^* \) is onto. Choose \( \varphi = (\varphi_k) \in \text{Hom}(P, \mathbb{Z}/m\mathbb{Z}) \cong \prod_{k \in \mathbb{N}_0} \text{Hom}(Z_k, \mathbb{Z}/m\mathbb{Z}) \cong \prod_{k \in \mathbb{N}_0} \mathbb{Z}/m\mathbb{Z} \) where \( \varphi_k \in \text{Hom}(Z_k, \mathbb{Z}/m\mathbb{Z}) \) may be identified with the element \( \varphi_k(e_k) \) of \( \mathbb{Z}/m\mathbb{Z} \). The image of \( \mu^* \) in \( \prod_{k \in \mathbb{N}_0} \text{Hom}(Z_k, \mathbb{Z}/m\mathbb{Z}) \) consists of \( \{\psi = (\psi_k)|\psi_k = \varphi_k - l\varphi_{k+1}, (\varphi_k) \in \prod_{k \in \mathbb{N}_0} \mathbb{Z}/m\mathbb{Z}\} \). Since \( l \) and \( m \) are relatively prime, multiplication by \( l \) maps \( \mathbb{Z}/m\mathbb{Z} \) onto itself and the equations \( \psi_k = \varphi_k - l\varphi_{k+1} \) can be solved for \( \varphi_k \in \mathbb{Z}/m\mathbb{Z} \) given \( \psi = (\psi_k) \in \text{Hom}(Z_k, \mathbb{Z}/m\mathbb{Z}) \). Thus, \( \mu^* \) is onto and the result follows. \( \square \)

**Theorem 2.3.** Let \( g = \sum_{i=0}^2 a_ix_i \in \mathbb{Z}[x] \) be a degree 2 irreducible polynomial and \( l = \{a_0, a_2\} \). If \( a_2 \neq a_0 \), then \( K_0(B_g) \cong \mathbb{Z} \oplus \mathbb{Z}/t\mathbb{Z} \) with \( t = (a_2 - a_0) : l \) and \( K_1(B_g) \cong \mathbb{Z} \oplus \mathbb{Z}/(\sum_{i=0}^2 a_i)\mathbb{Z} \). If \( a_2 = a_0 \), then \( K_0(B_g) \cong \mathbb{Z} \oplus \mathbb{Z}[l^{-1}] \) and \( K_1(B_g) \cong \mathbb{Z} \oplus \mathbb{Z}[l^{-1}] \oplus \mathbb{Z}/(\sum_{i=0}^2 a_i)\mathbb{Z} \).

**Proof.** Since \( a_2 \) and \( a_2^{-1} \) are in \( \mathbb{Z}[l^{-1}] \), the ideal generated by \( 1 - aoa^{-1} \) in \( \mathbb{Z}[l^{-1}] \) is the same as that generated by \( a_2 - a_0 \). Thus, \( \mathbb{Z}[l^{-1}]/(1 - aoa^{-1})\mathbb{Z}[l^{-1}] \cong \mathbb{Z}/t\mathbb{Z} \) with \( t = (a_2 - a_0) : l \) by Lemma 1.5, and the result for \( K_0 \) follows. It remains to compute \( K_1(B_g) \) when \( a_2 = a_0 \). In this case, \( l = ao \) and \( \{l, \sum_{i=0}^2 a_i\} = \{ao, 2ao + a_1\} = (ao, a_1) = (ao, a_1) = 1 \) since \( \text{cont}(g) = 1 \). Thus,

\[
\text{Ext} \left( \mathbb{Z} \oplus \mathbb{Z}[l^{-1}], \mathbb{Z}/ \left( \sum_{i=0}^2 a_i \right) \mathbb{Z} \right)
= \text{Ext} \left( \mathbb{Z}, \mathbb{Z}/ \left( \sum_{i=0}^2 a_i \right) \mathbb{Z} \right) \oplus \text{Ext} \left( \mathbb{Z}[l^{-1}], \mathbb{Z}/ \left( \sum_{i=0}^2 a_i \right) \mathbb{Z} \right) = 0
\]

by Proposition 2.2. The result for \( K_1 \) follows. \( \square \)
The case $\text{deg}(g) = 3$. Let $g = \sum_{i=0}^{3} a_i x^i \in \mathbb{Z}[x]$ be irreducible, $a \in \mathbb{C}$ a root, and $l = \{a_0, a_3\}$. The six-term exact sequence is:

$$
\begin{array}{c l}
\mathbb{Z} \oplus \wedge^2 \Lambda_g & \xrightarrow{0 \oplus (1 - \wedge^2 M_a)} \mathbb{Z} \oplus \wedge^2 \Lambda_g \\
\delta_1 & \xrightarrow{i_*} K_0(B_g) \\
K_1(B_g) & \xleftarrow{i_*} \Lambda_g \oplus \mathbb{Z}[l^{-1}] \xrightarrow{(1 - M_a) \oplus (1 + M_{a_0 a_3^{-1}})} \Lambda_g \oplus \mathbb{Z}[l^{-1}]
\end{array}
$$

Again $a \neq 1$, so $1 - M_a : \Lambda_g \to \Lambda_g$ is injective. Thus, $\text{im} \delta_0 = \text{ker} M_{(1 + a_0 a_3^{-1})}$ which is either $0$ (if $a_3 + a_0 \neq 0$) or $\mathbb{Z}[l^{-1}]$ (if $a_3 + a_0 = 0$). Since $\text{ker} \delta_0 = \text{im} i_* \simeq \mathbb{Z} \oplus \text{coker}(1 - \wedge^2 M_a)$, we have $K_0(B_g)$ is either $\mathbb{Z} \oplus \text{coker}(1 - \wedge^2 M_a)$ (if $a_3 + a_0 \neq 0$) or an extension of $\mathbb{Z}[l^{-1}]$ by $\mathbb{Z} \oplus \text{coker}(1 - \wedge^2 M_a)$ (if $a_3 + a_0 = 0$).

The group $K_1(B_g)$ is an extension of $\text{im} \delta_1 = \mathbb{Z} \oplus \text{ker}(1 - \wedge^2 M_a)$ by $\text{ker} \delta_1$ where $\text{ker} \delta_1 = \text{im} i_* \simeq \text{coker}(1 - M_a) \oplus \text{coker} M_{(1 + a_0 a_3^{-1})} \simeq \mathbb{Z}/(\sum_{i=0}^{3} a_i) \mathbb{Z} \oplus \text{coker} M_{(1 + a_0 a_3^{-1})}$. By Lemma 1.5, $\text{coker} M_{(1 + a_0 a_3^{-1})} \simeq \mathbb{Z}/t \mathbb{Z}$ with $t = (a_3 + a_0) : 1$ if $a_3 + a_0 \neq 0$. If $a_3 + a_0 = 0$, then $\text{coker} M_{(1 + a_0 a_3^{-1})} = \mathbb{Z}[l^{-1}]$.

Proposition 2.4. Let $g(x) = \sum_{i=0}^{3} a_i x^i \in \mathbb{Z}[x]$ be a degree 3 irreducible polynomial and $l = \{a_0, a_3\}$.

If $a_3 + a_0 \neq 0$, then $K_0(B_g) = \mathbb{Z} \oplus \wedge^2 \Lambda_g / (1 - \wedge^2 M_a) \wedge^2 \Lambda_g$ and $0 \to \mathbb{Z}/(\sum_{i=0}^{3} a_i) \mathbb{Z} \oplus \mathbb{Z}/l \mathbb{Z} \to K_1(B_g) \to 0$ where $t = (a_3 + a_0) : 1$.

If $a_3 + a_0 = 0$, then $0 \to \mathbb{Z} \oplus \wedge^2 \Lambda_g / (1 - \wedge^2 M_a) \wedge^2 \Lambda_g \to K_0(B_g) \to 0$ and $0 \to \mathbb{Z}/(\sum_{i=0}^{3} a_i) \mathbb{Z} \oplus \mathbb{Z}[l^{-1}] \to K_1(B_g) \to 0$.

It is straightforward to compute these groups if we impose the restriction that both $a_3$ and $a_0 \in \{1, -1\}$ (so $l = 1$). In this case, $\Lambda_g$ has a basis $e_i = a^i$ ($i = 0, 1, 2$) and is isomorphic to $\mathbb{Z}^3$. Identifying $\wedge^2 \mathbb{Z}^3$ with $\mathbb{Z}^3$ ($E_i = e_j \wedge e_k$ with $i, j, k \in \{0, 1, 2\}$ in cyclic order is a basis of $\wedge^2 \mathbb{Z}^3$), the map $\wedge^2 M_a$ has matrix form

$$
\begin{pmatrix}
a_1 a_3^{-1} & a_2 a_3^{-1} & 1 \\
-a_0 a_3^{-1} & 0 & 0 \\
0 & -a_0 a_3^{-1} & 0
\end{pmatrix}.
$$

The first, second, and third determinantal divisors of $1 - \wedge^2 M_a$, i.e., the invariants of the submodule $(1 - \wedge^2 M_a) \wedge^2 \Lambda_g$ in $\wedge^2 \Lambda_g$, are $1, 1,$ and $a_0 a_2 - a_1 a_3^{-1}$ respectively. Thus, $\text{coker}(1 - \wedge^2 M_a) = \mathbb{Z}/(a_2 - a_1) \mathbb{Z}$ if $a_0 + a_3 \neq 0$, and $\text{coker}(1 - \wedge^2 M_a) = \mathbb{Z}/(a_2 + a_1) \mathbb{Z}$ if $a_0 + a_3 = 0$. We also have, if $a_0 + a_3 \neq 0$, that

$$
\text{ker}(1 - \wedge^2 M_a) \simeq \begin{cases}
\mathbb{Z} & \text{if } a_1 = a_2, \\
0 & \text{otherwise}.
\end{cases}
$$

If $a_0 + a_3 = 0$, then

$$
\text{ker}(1 - \wedge^2 M_a) \simeq \begin{cases}
\mathbb{Z} & \text{if } a_1 + a_2 = 0, \\
0 & \text{otherwise}.
\end{cases}
$$

Proposition 2.5. Let $g = \sum_{i=0}^{3} a_i x^i$ be a degree 3 irreducible polynomial in $\mathbb{Z}[x]$ with $|a_0| = |a_3| = 1$. License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
If \( a_0 + a_3 \neq 0 \), then

\[
K_0(B_g) = \mathbb{Z} \oplus \mathbb{Z}/(a_2 - a_1)\mathbb{Z}
\]

and

\[
K_1(B_g) = \begin{cases} 
\mathbb{Z}^2 \oplus \mathbb{Z}/(\sum_{i=0}^{3} a_i)\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } a_1 = a_2, \\
\mathbb{Z} \oplus \mathbb{Z}/(\sum_{i=0}^{3} a_i)\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{otherwise}.
\end{cases}
\]

If \( a_0 + a_3 = 0 \), then

\[
K_0(B_g) = \mathbb{Z}^2 \oplus \mathbb{Z}/(a_1 + a_2)\mathbb{Z}
\]

and

\[
K_1(B_g) = \begin{cases} 
\mathbb{Z}^3 \oplus \mathbb{Z}/(\sum_{i=0}^{3} a_i)\mathbb{Z} & \text{if } a_1 + a_2 = 0, \\
\mathbb{Z}^2 \oplus \mathbb{Z}/(\sum_{i=0}^{3} a_i)\mathbb{Z} & \text{otherwise}.
\end{cases}
\]

Proposition 3.1. Let \( g \in \mathbb{Z}[x] \) be nonconstant and irreducible. If \( \tau \) is a tracial state on \( B_g \), then the range of \( \tau \) on \( K_0(B_g) \) is \( \mathbb{Z} \).

Proof. Identify \( \Lambda_g \) with \( \mathbb{Z}[a, a^{-1}] \) for \( a \in \mathbb{C} \setminus \{0\} \) a root of \( g \). If \( a = 1 \), then \( B_g = C(T^2) \) and \( \tau(K_0(B_g)) = \mathbb{Z} \), so assume \( a \neq 1 \). The map \( 1 - \alpha_* \) of \( K_1(C(\Lambda_g)) \) restricts to \( 1 - M_a \) on \( \Lambda_g = H^1(\Lambda_g, \mathbb{Z}) \) (by viewing an element of \( \Lambda_g \) as an element of \( \mathcal{A}_g \) one obtains a unitary in \( C(\Lambda_g) \)). Since \( a \neq 1 \), \( 1 - M_a \) is injective on \( \Lambda_g \) and thus \( \Delta^a_\sigma(\ker(1 - \alpha_*)) = 0 \) where \( \Delta^a_\sigma \) is the group homomorphism from \( \ker(1 - \alpha_*) \) to \( \mathbb{R}/\tau(K_0(C(\Lambda_g))) \) described in [6]. Thus \( \tau(K_0(B_g)) = \tau(K_0(C(\Lambda_g))) \) ([6]). Since \( \mathcal{A}_g \) is compact and connected \( (\text{cont}(g) = 1) \), \( \tau(K_0(C(\Lambda_g))) = \mathbb{Z} \).

We briefly consider how well the \( K \)-groups reflect the \(*\)- or anti-\(*\)-isomorphism classes of these algebras. Already, if \( \deg(g) = 1 \), there are many examples of non-\(*\)- or anti-\(*\)-isomorphic algebras with isomorphic \( K \)-groups (even if we view \( K_0(B_g) \) as an ordered group). By results in [1], it is enough to find \( g, h \in \mathbb{Z}[x] \) irreducible of degree one with \( |g(1)| = |h(1)| \neq 0 \) and \( g \neq \pm h \) and \( g \neq \pm h^0 \). There are also many examples if the degree of the polynomials are two. For example, it is enough to find \( g = \sum_{i=0}^{2} a_ix^i \) and \( h = \sum_{i=0}^{2} b_ix^i \) irreducible in \( \mathbb{Z}[x] \) with \( a_0 \neq a_2, b_0 \neq b_2, |\sum_{i=0}^{2} a_i| = |\sum_{i=0}^{2} b_i| \) and \( |a_2 - a_0| : \{a_0, a_2\} = |b_2 - b_0| : \{b_0, b_2\} \) but \( g \neq \pm h \) and \( g \neq \pm h^0 \). Choose \( a_0, a_1, a_2 \in \mathbb{Z} \) with \( a_0 + a_2 \neq 0, a_0 + a_1 + a_2 \neq 0 \), and both \( a_1^2 - 4a_0a_2 \) and \( (2(a_0 + a_2) + a_1)^2 - 4a_0a_2 \) not squares. Letting \( b_0 = a_0, b_2 = a_2, \) and \( b_1 = -2(a_0 + a_2) - a_1 \) yields one example.

References


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