NUMERICAL RADIUS PERSERVING OPERATORS ON $B(H)$

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(Communicated by Palle E. T. Jorgensen)

Abstract. Let $H$ be a Hilbert space over $\mathbb{C}$ and let $B(H)$ denote the vector space of all bounded linear operators on $H$. We prove that a linear isomorphism $T : B(H) \to B(H)$ is numerical radius-preserving if and only if it is a multiply of a $C^*$-isomorphism by a scalar of modulus one.

1. Introduction

Let $H$ be a Hilbert space over $\mathbb{C}$ and let $B(H)$ denote the vector space of all bounded linear operators on $H$. For every $A$ in $B(H)$, the numerical range and the numerical radius of $T$ are defined respectively by

$$W(A) = \{(Ax, x) : x \in H, \|x\| = 1\},$$
$$w(A) = \sup \{\|\lambda\| : \lambda \in W(A)\}.$$  

It is well known that $w(\cdot)$ is a norm on $B(H)$ and that this norm is equivalent to the usual operator norm. (See [4, p. 117].) A classical theorem of Kadison [4, Theorem 7] asserts that every linear isomorphism on $B(H)$ which is isometric with respect to the operator norm is a $C^*$-isomorphism followed by left multiplication by a fixed unitary operator. A $C^*$-isomorphism is a linear isomorphism of $B(H)$ such that $T(A^*) = T(A)^*$ for all $A$ in $B(H)$ and $T(A^n) = T(A)^n$ for all self-adjoint $A$ in $B(H)$ and all natural number $n$. A description of $C^*$-isomorphisms on $B(H)$ can be obtained. First of all we have from [6, Corollary 11] that a $C^*$-isomorphism on $B(H)$ is either a *-isomorphism or a *-anti-isomorphism. Suppose that $T$ is an algebra isomorphism on $B(H)$. Then by [3, Theorem 2], there is an invertible operator $V$ on $H$ such that $T(A) = VAV^{-1}$ for all $A$ in $B(H)$. If we also assume that $T(A^*) = T(A)^*$ for all $A$ in $B(H)$, then $VA^*V^{-1} = (V^{-1})^*A^*V^*$ and hence $(V^*V)A^* = A^*(V^*V)$ for all $A$ in $B(H)$. It follows that $V^*V$ is a scalar multiple of the identity operator $I$. Say $V^*V = kI$. As $V^*V$ is always a positive operator and $k$ cannot be zero, $k > 0$. Let $U = \sqrt[k]{k}V$. Then $U$ is unitary and $T(A) = UA^*U^*$ for all $A$ in $B(H)$. For a *-anti-isomorphism $T$, it can be shown (e.g., see [5, Remark 2]) that there is a unitary operator $U$ in $B(H)$ such that $T(A) = UA^*U^*$ for all $A$ in $B(H)$, where $A^t$ denotes the transpose.

Received by the editors June 15, 1993 and, in revised form, August 2, 1993.
1991 Mathematics Subject Classification. Primary 47B49, 47A12.

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of \( A \) relative to a fixed orthonormal basis of \( H \). Clearly operators of these two types are \( C^* \)-isomorphisms.

Let us turn to numerical range and numerical radius. Pellegrini \[9, \text{Theorem 3.1}\] proved that an operator \( T \) on \( B(H) \) is a \( C^* \)-isomorphism exactly when \( T \) preserves the "numerical range" of each element in \( B(H) \). It should be noted that Pellegrini obtained his result in a general Banach algebra, and his definition of numerical range is different from ours. In fact, for each \( A \) in \( B(H) \), the "numerical range" of \( A \) defined by Pellegrini reduces to the closure of \( W(A) \). When the underlying space \( H \) is finite-dimensional, \( W(A) \) is compact and hence the two sets are identical. Despite the discrepancy we still have that \( T \) is a \( C^* \)-isomorphism if and only if \( W(T(A)) = W(A) \) for every \( A \) in \( B(H) \).

For simplicity we shall call an operator \( T \) with the latter property numerical range-preserving. Likewise we say that \( T \) is numerical radius-preserving if \( w(T(A)) = w(A) \) for every \( A \) in \( B(H) \).

In the finite-dimensional situation, the above result was extended by Li. In \[1, \text{Theorem 1}\] he proved that \( T \) is numerical radius-preserving if and only if \( T \) is a scalar multiple of a \( C^* \)-isomorphism by a complex number of modulus one. It is immediate that if \( T \) is numerical range-preserving, then \( T \) is numerical radius-preserving and hence the scalar in question is one. In this note we prove that the conclusion of Li remains valid without the dimension constraint.

### 2. Results

In what follows \( T \) denotes a linear isomorphism on \( B(H) \) which is numerical radius-preserving on \( B(H) \). We shall prove that \( T \) maps the identity mapping \( I \) to a scalar multiple of \( I \). The scalar is necessarily of modulus one. Multiplying by the complex conjugate of the scalar, we get a numerical radius-preserving operator \( T_1 \) with an additional property that \( T_1(I) = I \). The result is concluded by showing that \( T_1 \) is a \( C^* \)-isomorphism.

We begin with a lemma which describes scalar multiples of \( I \) in terms of numerical radius. Let \( \Lambda = \{ \lambda \in \mathbb{C} : |\lambda| = 1 \} \).

**Lemma 1.** An operator \( A \in B(H) \) is a scalar multiple of \( I \) if and only if for every \( B \in B(H) \), there is a \( \lambda \in \Lambda \) such that \( w(A + \lambda B) = w(A) + w(B) \).

**Proof.** It is clear that if \( A \) is a scalar multiple of \( I \), then \( A \) satisfies the condition. For the converse we borrow the idea from Li and Tsing \[2, \text{p. 40}\]. We first show that elements in \( W(A) \) are of constant modulus; it follows then from the convexity of \( W(A) \) ([4, p. 113]) that the set is a singleton. Hence \( A \) is a scalar multiple of the identity \( I \). Now assume that there is an \( x \) in \( H \), \( ||x|| = 1 \), and \( \langle Ax, x \rangle < w(A) \). Let \( B \) be the orthogonal projection onto the linear span of \( x \). Then \( w(B) = 1 \). Fix any \( r \) such that \( \langle Ax, x \rangle < r < w(A) \). We can find an \( \varepsilon > 0 \) such that \( \langle Ay, y \rangle < r \) whenever \( ||y - x|| < \varepsilon \). In fact \( \langle Ay, y \rangle < r \) if there is a \( \lambda \in \Lambda \) such that \( ||y - \lambda x|| < \varepsilon \). Suppose that \( y \in H \), \( ||y|| = 1 \), and \( ||y - \lambda x|| \geq \varepsilon \) for every \( \lambda \in \Lambda \). Then

\[
\varepsilon^2 \leq \langle y - \lambda x, y - \lambda x \rangle = 2 - 2\text{Re}(y, \lambda x) \quad \text{for every} \quad \lambda \in \Lambda .
\]

It follows that \( \langle y, x \rangle \leq 1 - \frac{1}{2}\varepsilon^2 \). Let \( k = \min\{r + 1, w(A) + 1 - \frac{1}{2}\varepsilon^2\} \). Then for every \( \lambda \in \Lambda \) and \( y \in H \) with \( ||y|| = 1 \), we have

\[
||\langle (A + \lambda B)y, y \rangle|| \leq ||\langle Ay, y \rangle|| + ||\langle y, x \rangle|| \leq k .
\]

Hence \( w(A + \lambda B) < w(A) + w(B) \). \( \square \)
By the above lemma $T(I) = \lambda I$. Clearly we have $\lambda \in \Lambda$. Let $T_1 = \bar{\lambda}T$. Then $T_1(I) = I$. We need the following definitions. By a state on $B(H)$ we mean as usual a bounded linear functional $\rho$ on $B(H)$ such that $\rho(I) = \|\rho\| = 1$. The set $S$ of all states is called the state space of $B(H)$. A bounded linear operator $T : B(H) \to B(H)$ is said to be state-preserving if its adjoint $T'$ satisfies $T'(S) \subseteq S$. By [9, Theorem 2.3 and Theorem 3.1], $T$ is a $C^*$-isomorphism if and only if it is state-preserving. Let $x$ be a unit vector in $H$. The linear functional $\rho_x$ given by

$$\rho_x(A) = \langle A_x, x \rangle \quad \text{for every} \quad A \in B(H)$$

is a state of $B(H)$. States of this form are called vector states.

Lemma 2. The operator $T_1$ is state-preserving.

Proof. Let $w'$ denote the norm in $B(H)'$ dual to the numerical radius. Then $w'(\rho) \geq \|\rho\|$ for every $\rho$ in $B(H)'$. As $T_1$ is numerical radius-preserving, $w'(T_1(\rho)) = w'(\rho)$ for every $\rho$ in $B(H)'$. If $\rho_x$ is a vector state, then $w'(\rho_x) = 1$ and hence $\|T_1'(\rho_x)\| \leq w'(T_1(\rho_x)) = 1$. But $T_1'(\rho_x)(I) = \rho_x(T_1(I)) = \rho_x(I) = 1$. It follows that $T_1'(\rho_x)$ is a state of $B(H)$. By [4, Corollary 4.3.10] the state space is the closed convex hull of the vector states in the weak*-topology. This together with the fact that $T_1'$ is continuous in the weak*-topology entail that $T_1$ is state-preserving. □

By Lemma 1 and Lemma 2, we have proved

Theorem. A linear isomorphism $T$ on $B(H)$ is numerical radius-preserving if and only if $T$ is a multiple of a $C^*$-isomorphism by a scalar of modulus one.

In [1] Li also studied a numerical radius-preserving real-linear operator on the selfadjoint elements in $B(H)$. He proved ([1, Theorem 2]) that such an operator is the restriction of a $C^*$-isomorphism on $B(H)$ multiplied by $\pm 1$. Let us remark that as the numerical radius and the operator norm coincide on selfadjoint operators, this result can alternatively be deduced from [7, Theorem 2].

REFERENCES

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