ON A SEQUENCE TRANSFORMATION
WITH INTEGRAL COEFFICIENTS FOR EULER'S CONSTANT

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Abstract. Let \( \gamma \) denote Euler's constant, and let

\[
S_n = \left(1 + \frac{1}{2} + \cdots + \frac{1}{n-1}\right) - \log n \quad (n \geq 2).
\]

We prove by Ser's formula for the remainder \( \gamma - S_n \) that for all integers \( n \geq 1 \)
and \( t \geq 2 \) there are integers \( \mu_{n,0}, \mu_{n,1}, \ldots, \mu_{n,n} \) such that

\[
\mu_{n,0}S_t + \mu_{n,1}S_t + \cdots + \mu_{n,n}S_{t+n} = \gamma + O_t\left((n(n+1)(n+2)\cdots(n+t))^{-1}\right),
\]

where the constant in \( O_t \) depends only on \( t \).

The coefficients \( \mu_{n,k} \) are explicitly given and are bounded by \( 2^{3n+t-1} \).

By \( \gamma \) we denote Euler's constant; it is well known that the sequence \( (s_n)_{n \geq 0} \)
defined by

\[
s_n = \left(1 + \frac{1}{2} + \cdots + \frac{1}{n-1}\right) - \log n \quad (n \geq 2)
\]
tends to \( \gamma \), where

\[
s_n = \gamma + O(n^{-1}) \quad (n \geq 2).
\]

J. Ser [6] has proved that the remainder of \( \gamma - S_n \) \( (n \geq 2) \) can be expressed as
an infinite sum with rational terms: Let

\[
t_{m+2} = -\frac{1}{(m+1)!} \int_0^1 (0-x)(1-x)\cdots(m-x)dx \quad (m \geq 0).
\]

Then

\[
\gamma = \frac{1}{n} \sum_{m=0}^{\infty} \frac{t_{m+2}}{(m+n)} + \left(1 + \frac{1}{2} + \cdots + \frac{1}{n-1}\right) - \log n \quad (n \geq 2).
\]

(See also [3, pp. 14–15].)

But, of course, \( \gamma - S_n \) can be written in a lot of different ways. For example,
we get by Euler's summation formula for any positive integers \( n \geq 2 \) and \( k \):

\[
\gamma = S_n + \frac{1}{2n} + \sum_{j=1}^{k} \frac{B_{2j}}{2j \cdot n^{2j}} + R(n, k),
\]

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where $B_m$ are the Bernoulli numbers and
\[
|R(n, k)| \leq \frac{4}{n} \sqrt{\frac{k}{\pi n}} \left( \frac{k}{\pi n} \right)^{2k}
\]
(see [4]).

A historical remark. The representation of $\gamma$ by the right-hand side of (2) was the main tool in P. Appell's attempt to prove the irrationality of $\gamma$ in 1926 [1]. Appell himself quickly discovered his error and within a week he published a retraction. An outline of this incorrect proof is sketched in [2]. In what follows we apply a linear sequence transformation to the class of those sequences, where the error term can be expressed by a sum like (2). First we introduce some notation:
\[
(\alpha)_m = \alpha(\alpha + 1)(\alpha + 2) \cdots (\alpha + m - 1), \quad (\alpha)_0 = 1 \quad (\alpha \in \mathbb{R}, \ m \in \mathbb{Z}_{>0});
\]
\[
\mu_{n, k}(\tau) = (-1)^{n+k} \binom{\tau + k}{n} \binom{n}{k} \quad (n \in \mathbb{Z}_{\geq 0}, \ 0 \leq k \leq n),
\]
where $\tau \in \mathbb{Z}_{>0}$ is fixed. Note that $\mu_{n, k} \in \mathbb{Z}$ ($n \in \mathbb{Z}_{\geq 0}, \ 0 \leq k \leq n$).

**Theorem 1.** Let $(v_n)_{n \geq 0}$ be a sequence of real numbers such that
\[
\lim_{n \to \infty} v_n = s,
\]
(3)
where $(c_m)_{m \geq 1}$ denotes a sequence of real numbers satisfying
\[
0 \leq c_m \leq C \cdot (m + \sigma)! \quad (m \geq \max\{1; \ -\sigma\})
\]
for some constant $C > 0$ and some
\[
\sigma \in \mathbb{Z} \quad \text{with} \quad \sigma < \tau - 2.
\]
Then we have for
\[
(6)
\]
\[
|e_n| \leq C \cdot \left( \frac{(n + \sigma + 1)! \cdot (\tau - \sigma - 3)!}{(n + \tau - \sigma - 2)!} \right) \quad (n \geq \max\{0; \ -\sigma + 1\}).
\]

The linear sequence transformation given in (6) belongs to a certain class of so-called nonregular methods; a general theory of such transformations can be found in [7] (see Chapter 2.3.5).

**Theorem 2.** For $n \geq 1$ and $\tau \geq 2$ we have
\[
\left| \sum_{k=0}^{n} \mu_{n, k} s_{k+\tau} - \gamma \right| \leq \frac{(\tau - 1)!}{2n(n+1)(n+2) \cdots (n+\tau)}.
\]

From this theorem we get a very good approximation to $\gamma$ in terms of $s_n$, $s_{n+1}$, $s_{n+2}$ by choosing $\tau = n \geq 2$:
\[
\left| \sum_{k=0}^{n} \mu_{n, k} s_{n+k} - \gamma \right| \leq \frac{1}{2n^2(\frac{2n}{n})} \leq n^{-3/2} \cdot 4^{-n}.
\]
There are linear sequence transformations for \((s_n)_{n\geq 0}\) with nonintegral coefficients, which converge more rapidly to \(\gamma\) than the transformation given in Theorem 2 (see [5]). But from an arithmetical point of view in number theory it is much more attractive to accelerate the convergence by transformations with integral coefficients.

**Proof of the theorems.** From \(\sum_{k=0}^{n} \mu_{n,k} = 1\) we have by (3) and (6) for every \(n \geq 0:\)

\[
(7) \quad e_n = -\sum_{k=0}^{n} \sum_{m=1}^{\infty} (-1)^{n+k} \frac{(k+\tau)_n}{k! \cdot (n-k)! \cdot (k+\tau)_m} c_m
\]

\[
(8) \quad = -\sum_{m=1}^{\infty} c_m \sum_{k=0}^{n} (-1)^{n+k} \frac{(k+n+\tau-1)!}{k! \cdot (n-k)! \cdot (k+m+\tau-1)!}.
\]

From \(c_m \geq 0\) in (4) we conclude that the infinite series \(\sum_{m=1}^{\infty} \frac{c_m}{(n+m)!} (n \geq 0)\) converges absolutely, and so we may interchange the sums in (7). We express the terms in (8) again by Pochhammer's symbol; this gives for \(n \geq 0:\)

\[
(9) \quad e_n = (-1)^{n+1} \cdot \frac{(n+\tau-1)!}{n!} \sum_{m=1}^{\infty} \frac{c_m}{(m+\tau-1)!} \sum_{k=0}^{n} \frac{(n+k) \cdot (-n)_k}{k! \cdot (m+k)_k}
\]

(since \((-n)_k = 0\) if \(k > n\)). Let \(a, b, c\) be real numbers, \(c \neq 0, -1, -2, \ldots;\)

\[
(10) \quad F(a, b; c; x) = \sum_{k=0}^{\infty} \frac{(a)_k \cdot (b)_k}{k! \cdot (c)_k} x^k.
\]

We only treat the case \(c-a-b > 0;\) for this it is well known that

\[
(11) \quad F(a, b; c; 1) = \begin{cases} \frac{\Gamma(c) \cdot \Gamma(c-a-b)}{\Gamma(c-a) \cdot \Gamma(c-b)} & \text{if } c-a, c-b \neq 0, -1, -2, \ldots, \\ 0 & \text{otherwise.} \end{cases}
\]

The sum on the right-hand side of (10) occurs in (9) with

\(a = n+\tau, \quad b = -n, \quad c = m+\tau.\)

From \(m \geq 1\) in (9) we have \(m+\tau > \tau,\) hence \(c > a + b.\) Note \(c-a \leq 0 \Leftrightarrow m \leq n.\) By (11) we now see that \(e_n\) equals

\[
(12) \quad e_n = (-1)^{n+1} \cdot \frac{(n+\tau-1)!}{n!} \sum_{m=n+1}^{\infty} \frac{c_m}{(m+\tau-1)!} \frac{(m+\tau-1)! \cdot (m-1)!}{(m+n+\tau-1)!}
\]

\[
= (-1)^{n+1} \cdot \frac{(n+\tau-1)!}{n!} \sum_{m=0}^{\infty} \frac{c_{m+n+1}}{m! \cdot (m+2n+\tau)!} \quad (n \geq 0).
\]

Now let

\[n_0 = \max\{0; -(\sigma + 1)\}\].
$n \geq n_0$ implies $m+n+1 \geq n+1 \geq \max\{1, -\sigma\}$. Thus for $n \geq n_0$ we estimate $e_n$ from (12) by (4),

$$\left|e_n\right| \leq C \cdot \frac{(n+\tau-1)! \cdot (n+\sigma+1)! \cdot (m+n)!}{(m+2n+\tau)! \cdot m! \cdot (m+n+\sigma+1)!} \quad (n \geq n_0).$$

We treat the infinite sum in (13) in the same way as we did with the inner sum in (8). For $n \geq n_0$ we get

$$\left|e_n\right| \leq C \cdot \frac{(n+\tau-1)! \cdot (n+\sigma+1)!}{(2n+\tau)!} \sum_{m=0}^{\infty} \frac{(n+\sigma+2)_m \cdot (n+1)_m}{m! \cdot (2n+\tau+1)_m}.$$

To apply (11) again we now define in (10):

$$a = n + \sigma + 2, \quad b = n + 1, \quad c = 2n + \tau + 1.$$

From (5) we have $2n + \tau + 1 > 2n + \sigma + 3$, hence $c > a + b$. That gives

$$\left|e_n\right| \leq C \cdot \frac{(n+\tau-1)! \cdot (n+\sigma+1)! \cdot \Gamma(2n+\tau+1) \cdot \Gamma(\tau-\sigma-2)}{\Gamma(n+\tau-\sigma-1) \cdot \Gamma(n+\tau)}$$

$$= C \cdot \frac{(n+\sigma+1)! \cdot (\tau-\sigma-3)!}{(n+\tau-\sigma-2)!} \quad (n \geq n_0).$$

This proves the theorem.

Theorem 2 follows immediately from Theorem 1 and (2): Put

$$c_m = -\frac{1}{m} \int_0^1 (0-x)(1-x) \cdot \cdots \cdot (m-1-x) \, dx \quad (m \geq 1).$$

Hence

$$t_{m+1} = \frac{1}{(m-1)!} \cdot c_m \quad (m \geq 1);$$

from the definition of $s_n$ and (2) we get

$$s_{n+\tau} = \frac{1}{n+\tau} \sum_{m=1}^{\infty} \frac{t_{m+1}}{(m+n+\tau-1)} = \frac{1}{n+\tau} \sum_{m=1}^{\infty} \frac{c_m \cdot (n+\tau-1)!}{(m+n+\tau-1)!}$$

$$= \frac{1}{n+\tau} \sum_{m=1}^{\infty} \frac{c_m}{(n+\tau)_m} \quad (n \geq 0).$$

This is (3), where $\tau \geq \mathbb{Z}_{\geq 2}$. We get an integer $\sigma$ from (15) by

$$0 \leq c_m \leq \frac{1}{m} \int_0^1 x \cdot (m-1)! \, dx = \frac{(m-1)!}{2m} \leq \frac{(m-2)!}{2} \quad (m \geq 2).$$

Hence we may choose $\sigma = -2, \quad C = \frac{1}{2}, \quad n_0 = 1$; and (5) holds.

Theorem 2 now follows from Theorem 1, where $v_n = s_{n+\tau}$.

At last note that

$$\mu_{n,k} = (-1)^{n+k} \binom{\tau+k}{n} \binom{n}{k} = (-1)^{n+k} \binom{n+k+\tau-1}{n} \binom{n}{k}$$

$$= (-1)^{n+k} \frac{n+k+\tau-1}{n-k, k, k+\tau-1} \quad (n \geq 1, \quad 0 \leq k \leq n).$$

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1Note that (2) holds for $s_{n+\tau}$ with $n \geq 0$ and $\tau \geq 2$. 

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and
\[ |\mu_{n,k}| \leq 2^{n+k+1} \cdot 2^n \leq 2^{3n+1} \cdot 1. \]

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**References**


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