ON TREE IDEALS

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(Communicated by Andreas R. Blass)

Abstract. Let \( I^0 \) and \( m^0 \) be the ideals associated with Laver and Miller forcing, respectively. We show that \( \text{add}(I^0) < \text{cov}(I^0) \) and \( \text{add}(m^0) < \text{cov}(m^0) \) are consistent. We also show that both Laver and Miller forcing collapse the continuum to a cardinal \( \leq \aleph_1 \).

Introduction and notation

In this paper we investigate the ideals connected with the classical tree forcings introduced by Laver [La] and Miller [Mi]. Laver forcing \( L \) is the set of all trees \( p \) on \( \omega^\omega \) such that \( p \) has a stem and whenever \( s \in p \) extends \( \text{stem}(p) \) then \( \text{Succ}_p(s) := \{ n : s^\uparrow n \in p \} \) is infinite. Miller forcing \( M \) is the set of all trees \( p \) on \( \omega^\omega \) such that \( p \) has a stem and for every \( s \in p \) there is \( t \in p \) extending \( s \) such that \( \text{Succ}_p(t) \) is infinite. We denote the set of all these splitting nodes in \( p \) by \( \text{Split}(p) \). For any \( t \in \text{Split}(p) \), \( \text{Split}_p(t) \) is the set of all minimal (with respect to extension) members of \( \text{Split}(p) \) which properly extend \( t \). For both \( L \) and \( M \) the order is inclusion.

The Laver ideal \( I^0 \) is the set of all \( X \subseteq \omega^\omega \) with the property that for every \( p \in L \) there is \( q \in L \) extending \( p \) such that \( X \cap [q] = \emptyset \). Here \( [q] \) denotes the set of all branches of \( q \). The Miller ideal \( m^0 \) is defined analogously, using conditions in \( M \) instead of \( L \). By a fusion argument one easily shows that \( I^0 \) and \( m^0 \) are \( \sigma \)-ideals.

The additivity (\( \text{add} \)) of any ideal is defined as the minimal cardinality of a family of sets belonging to the ideal whose union does not. The covering number (\( \text{cov} \)) is defined as the least cardinality of a family of sets from the ideal whose union is the whole set on which the ideal is defined—in our case. Clearly \( \omega_1 \leq \text{add}(I^0) \leq \text{cov}(I^0) \leq \omega \) and \( \omega_1 \leq \text{add}(m^0) \leq \text{cov}(m^0) \leq \omega \) hold.
The main result in this paper says that there is a model of ZFC where \( \text{add}(l^0) < \text{cov}(l^0) \) and \( \text{add}(m^0) < \text{cov}(m^0) \) hold. The motivation was that by a result of Plewik [P1] it was known that the additivity and the covering number of the ideal connected with Mathias forcing are the same and they are equal to the cardinal invariant \( h \)—the least cardinality of a family of maximal antichains of \( \mathcal{P}(\omega)/\text{fin} \) without a common refinement. On the other hand, in [JuMiSh] it was shown that \( \text{add}(s^0) < \text{cov}(s^0) \) is consistent, where \( s^0 \) is Marczewski’s ideal—the ideal connected with Sacks forcing \( S \). Intuitively, \( L \) and \( M \) sit somewhere between Mathias forcing and \( S \). In [GoJoSp] it was shown that under Martin’s axiom \( \text{add}(l^0) = \text{add}(m^0) = \epsilon \), whereas this is false for \( s^0 \) (see [JuMiSh]).

The method of proof for \( \text{add}(s^0) < \text{cov}(s^0) \) in [JuMiSh] is the following: For a forcing \( P \) denote by \( \kappa(P) \) the least cardinal to which forcing with \( P \) collapses the continuum. In [JuMiSh] it is shown that \( \text{add}(s^0) \leq \kappa(S) \). In [BaLa] it was shown that in \( V^{S_{\omega_2}} \), \( \kappa(S) = \omega_1 \) holds, where \( S_{\omega_2} \) is the countable support iteration of length \( \omega_2 \) of \( S \). Hence \( V^{S_{\omega_2}} \models \text{add}(s^0) = \omega_1 \). On the other hand, a Löwenheim-Skolem argument shows that \( V^{S_{\omega_2}} \models \text{cov}(s^0) = \omega_2 \).

Our method of proof is similar. Denoting by \( P_{\omega_2} \) a countable support iteration of length \( \omega_2 \) of \( L \) and \( M \) (each occurring on a stationary set), in §2 we prove the following:

**Theorem.**

\[ V^{P_{\omega_2}} \models \omega_1 = \text{add}(l^0) = \text{add}(m^0) < \text{cov}(l^0) = \text{cov}(m^0) = \omega_2. \]

The crucial steps in the proof are to show that \( \kappa(L) \), \( \kappa(M) \) equal \( \omega_1 \) and \( \text{add}(l^0) \leq \kappa(L) \), \( \text{add}(m^0) \leq \kappa(M) \) hold.

We will use the standard terminology for set theory and forcing. By \( b \) we denote the least cardinality of a family of functions in \( ^{\omega_0}\omega \) which is unbounded with respect to eventual dominance and \( d \) will be the least cardinality of a dominating family in \( ^{\omega_1}\omega \). Moreover, \( p \) is the least cardinality of a filter base on \( (\omega)^\omega \) without any lower bound, and \( t \) is the least cardinality of a decreasing chain in \( (\omega)^\omega \) without any lower bound. It is easy to see that \( \omega_1 \leq p \leq t \leq b \leq d \leq c \).

### 1. Upper and lower bounds

**Theorem 1.1.**

1. \( t \leq \text{add}(l^0) \leq \text{cov}(l^0) \leq b. \)
2. \( p \leq \text{add}(m^0) \leq \text{cov}(m^0) \leq d. \)

**Proof of Theorem 1.1(1).** We have to prove the first and the third inequality. For the third inequality, let \( \{f_\alpha : \alpha < b\} \) be an unbounded family. Define

\[ X_\alpha := \{f \in ^{\omega_0}\omega : (\exists k) f(k) < f_\alpha(k)\}. \]

Clearly \( \bigcup\{X_\alpha : \alpha < b\} = ^{\omega_0}\omega \). We claim \( X_\alpha \in l^0 \). Let \( p \in L \). We define \( q \in L \) as follows: \( \text{stem}(q) := \text{stem}(p) \), and for any \( s \) extending \( \text{stem}(q) \) we have \( s \in q \) if and only if \( s \in p \) and \( (\forall k) \) if \( |\text{stem}(q)| \leq k < |s| \), then \( s(k) \geq f_\alpha(k) \). Then clearly \( q \in L \), \( q \) extends \( p \), and \( [q] \cap X_\alpha = \emptyset. \)

In order to prove the first inequality we use the following notation from [JuMiSh]: Let \( Q := \{\overline{A} = \langle A_s : s \in ^{<\omega}\omega \rangle : (\forall s) A_s \in [\omega]^\omega\} \). For \( \overline{A} \in Q \) we
define a sequence of Laver trees \( \langle p_s(\overline{A}) : s \in <\omega \omega \rangle \) as follows: \( p_s(\overline{A}) \) is the unique Laver tree such that \( \text{stem}(p_s(\overline{A})) = s \) and if \( t \in p_s(\overline{A}) \) extends \( s \), then \( \text{Succ}_{p_s(\overline{A})}(t) = A_t \).

For \( \overline{A}, \overline{B} \in Q \) we define:
\[
\overline{A} \subseteq \overline{B} \iff (\forall s) A_s \subseteq B_s,
\]
\[
\overline{A} \preceq^* \overline{B} \iff (\forall s) A_s \preceq^* B_s,
\]
\[
\overline{A} \preceq \overline{B} \iff (\forall s) A_s \preceq B_s \land (\forall \omega s) A_s \subseteq B_s.
\]

Here \( \preceq \) is a slight but important modification of \( \preceq^* \) from [JuMiSh].

**Fact 1.2.** \( (Q, \preceq^*) \) is t-closed.

**Proof of Fact 1.2.** Suppose \( \langle A_\alpha : \alpha < \gamma \rangle \), where \( \gamma < t \) is a decreasing sequence in \( (Q, \preceq^*) \). Let \( \overline{A}_\alpha := (A^s_\alpha : s \in <\omega \omega) \). Since \( \gamma < t \), there is \( \overline{B} = (B^s_\alpha : s \in <\omega \omega) \in Q \) such that \( (\forall \alpha < \gamma) \overline{B} \preceq^* \overline{A}_\alpha \). Define \( f_\alpha : <\omega \omega \rightarrow \omega \) such that \( (\forall s) B^s_\alpha \preceq f_\alpha(s) \). Now let \( B_s := B^s_\alpha \setminus f(s) \) and \( \overline{B} := (B_s : s \in <\omega \omega) \). It is easy to check that \( (\forall \alpha < \gamma) \overline{B} \preceq \overline{A}_\alpha \).

**Fact 1.3.** Suppose \( X \in l^0 \) and \( \overline{A} \in Q \). There exists \( \overline{B} \in Q \) such that \( \overline{B} \preceq \overline{A} \) and \( (\forall s < \omega \omega)[p_s(\overline{B})] \cap X = \emptyset \).

**Proof of Fact 1.3.** First note that if \( D := \{ p \in L : [p] \cap X = \emptyset \} \), then \( D \) is open dense and even 0-dense, i.e., for every \( p \in L \) there exists \( q \in D \) extending \( p \) such that \( \text{stem}(q) = \text{stem}(p) \). The proof of this is similar to Laver’s proof in [La] that the set of Laver trees deciding a sentence in the language of forcing with \( L \) is 0-dense: Suppose \( p \in L \) has no 0-extension whose branches are not in \( X \). Then inductively we can construct \( q \in L \) extending \( p \) such that every extension of \( q \) has a branch in \( X \), contradicting \( X \in l^0 \).

Using this it is straightforward to construct \( \overline{B} \) as desired.

**Fact 1.4.** Suppose \( X \subseteq <\omega \omega, \overline{A}, \overline{B} \in Q, \overline{B} \preceq^* \overline{A}, \) and \( (\forall s < \omega \omega)[p_s(\overline{A})] \cap X = \emptyset \).

Then \( (\forall s < \omega \omega)[p_s(\overline{B})] \cap X = \emptyset \).

**Proof of Fact 1.4.** Clearly, if \( F \subseteq p_s(\overline{B}) \) is finite, then
\[
[p_s(\overline{B})] = \bigcup \{ [p_t(\overline{B})] : t \in p_s(\overline{B}) \setminus F \}.
\]
But for almost all \( t \in p_s(\overline{B}) \), \( p_t(\overline{B}) \) extends \( p_t(\overline{A}) \). So clearly \( [p_s(\overline{B})] \subseteq [p_s(\overline{A})] \) and hence \( [p_s(\overline{B})] \cap X = \emptyset \).

**End of the proof of Theorem 1.1(1).** Suppose we are given \( \langle X_\alpha : \alpha < \gamma \rangle \) and \( q \in L \), where \( \gamma < t \) and \( (\forall \alpha) X_\alpha \in l^0 \). Choose \( \overline{A} \in Q \) such that \( p_{\text{stem}(q)}(\overline{A}) = q \), and let \( \overline{B}_0 \) be the \( \overline{B} \) given by Fact 1.3 for \( \overline{A} \) and \( X_0 \). If \( \langle \overline{A}_\alpha : \alpha < \beta \rangle \) is constructed for \( \beta \leq \gamma \) and \( \beta \) is a successor, then choose \( \overline{B}_\beta \) as given by Fact 1.3 for \( \overline{A} = \overline{B}_{\beta - 1} \) and \( X = X_\beta \). If \( \beta \) is a limit, then by Fact 1.2 choose first \( \overline{A} \) such that \( (\forall \alpha < \beta) \overline{A} \preceq^* \overline{B}_\alpha \) and then find \( \overline{B}_\beta \subseteq \overline{A} \) as given by Fact 1.3 for \( \overline{A} \) and \( X = X_\beta \). Finally, if we have constructed \( \overline{B}_\beta = (B^s_\beta : s \in <\omega \omega) \), define \( \overline{B} := (B_s : s \in <\omega \omega) \) by \( B_s := B^s_\beta \cap \text{Succ}_q(s) \) if \( s \in q \) extends \( \text{stem}(q) \), and \( B_s := B^s_\beta \) otherwise. It is easy to check that \( \overline{B} \in Q, p_{\text{stem}(q)}(\overline{B}) \) extends \( q \) and \( (\forall \alpha < \gamma) [p_{\text{stem}(q)}(\overline{B})] \cap X_\alpha = \emptyset \).
Proof of Theorem 1.1(2). The proof is similar to (1). For the third inequality, let \( \langle f_\alpha : \alpha < \delta \rangle \) be a dominating family. Define

\[ X_\alpha := \{ f \in \omega^\omega : (\forall k)(f(k) < f_\alpha(k)) \}. \]

Then \( \bigcup\{X_\alpha : \alpha < \delta\} = \omega^\omega \) and in an analogous way as in (1) it can be seen that \( X_\alpha \in \mathcal{m}^0 \).

In order to prove the first inequality we need the following concept from [GoJoSp]. Let \( R \) be the set of all \( \bar{P} = (P_s : s \in \omega^\omega) \) where each \( P_s \subseteq \omega^\omega \) is infinite, \( t \in P_s \) implies \( s \subseteq t \), and if \( t, t' \in P_s \) are distinct, then \( t(|s|) \neq t'(|s|) \).

Given \( \bar{P} \in R \) we can define \( \langle p_s(\bar{P}) : s \in \omega^\omega \rangle \) as follows: \( p_s(\bar{P}) \) is the unique Miller tree with stem \( s \) such that if \( t \in \text{Split}(p_s(\bar{P})) \), then \( \text{Split}_{p_s}(t) = P_t \).

Define the following relations on \( R \):

\[ \bar{P} \leq \bar{Q} \Leftrightarrow (\forall s)p_s(\bar{P}) \leq p_s(\bar{Q}), \]
\[ \bar{P} \approx \bar{Q} \Leftrightarrow (\forall s)p_s(\bar{P}) =^* Q_s \land (\forall s)p_s(\bar{Q}) = Q_s, \]
\[ \bar{P} \leq^* \bar{Q} \Leftrightarrow (\exists \bar{P}')(\bar{P} \leq \bar{P}' \land \bar{P}' \leq \bar{Q}). \]

Fact 1.5 [GoJoSp, 4.14]. Assume \( MA_\kappa(\sigma\text{-centered}) \). If \( \langle P_\alpha : \alpha < \kappa \rangle \) is a \( \leq^* \)-decreasing sequence in \( R \), then there exists \( \bar{Q} \in R \) such that \( (\forall \alpha < \kappa)\bar{Q} \leq^* \bar{P}_\alpha \).

The following two facts have proofs similar to those of Facts 1.3 and 1.4.

Fact 1.6. Suppose \( X \in \mathcal{m}^0 \) and \( \bar{P} \in R \). There exists \( \bar{Q} \leq \bar{P} \) such that \( (\forall s)[p_s(\bar{Q})] \cap X = \emptyset \).

Fact 1.7. Suppose \( X \in \mathcal{m}^0, \bar{P}, \bar{Q} \in R, \bar{P} \leq^* \bar{Q}, \) and \( (\forall s)[p_s(\bar{Q})] \cap X = \emptyset \). Then \( (\forall s)[p_s(\bar{P})] \cap X = \emptyset \).

Now using, Facts 1.5, 1.6, 1.7 and the well-known result that for all \( \kappa < \lambda \) \( MA_\kappa(\sigma\text{-centered}) \) holds, a similar construction as in Theorem 1.1(1) shows that \( \kappa < \text{add}(\mathcal{m}^0) \).

2. ADD AND COV ARE DISTINCT

Definition 2.1. A set \( A \subseteq \omega^\omega \) is called strongly dominating if and only if

\[ (\forall f \in \omega^\omega)(\exists \eta \in A)(\forall k)f(k(\eta - 1)) < \eta(k). \]

Definition 2.2. For any set \( A \subseteq \omega^\omega \), we define the domination game \( D(A) \) as follows:

There are two players, GOOD and BAD. GOOD plays first. The game lasts \( \omega \) moves.

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<tr>
<th>GOOD</th>
<th>BAD</th>
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<tbody>
<tr>
<td>( s )</td>
<td>( n_0 )</td>
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<tr>
<td>( m_0 )</td>
<td>( n_1 )</td>
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<tr>
<td>( m_1 )</td>
<td>( \vdots )</td>
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The rules are: $s$ is a sequence in $<\omega_\omega$, and the $n_i$ and $m_i$ are natural numbers. (Whoever breaks these rules first, loses immediately.)

The GOOD player wins if and only if:

(a) For all $i$, $m_i > n_i$.
(b) The sequence $s^{-m_0}m_1^{-} \cdots$ is in $A$.

Lemma 2.3. Let $A \subseteq \omega_\omega$ be a Borel set. Then the following are equivalent:

1. There exists a Laver tree $p$ such that $[p] \subseteq A$.
2. $A$ is strongly dominating.
3. GOOD has a winning strategy in the game $D(A)$.

Remark. Strongly dominating is not the same as dominating. For example, the closed set

$$A := \{ n \in \omega_\omega : (\forall k)(n(2k) = n(2k + 1)) \}$$

is dominating but is not strongly dominating.

Proof of Lemma 2.3. We consider the following condition:

4. (For all $F : \omega_\omega \times \omega \to \omega$)(exists $\eta \in A$)(exists $k$)(exists $i$)(for all $k$)(for all $i$) $\eta(k) > F(\eta \upharpoonright k, i)$.

We will show (1) $\rightarrow$ (2) $\rightarrow$ (4) $\rightarrow$ (3) $\rightarrow$ (1).

(1) $\rightarrow$ (2) is clear.

(2) $\rightarrow$ (4): Given $F$, define $f$ by

$$f(m) := \max\{ F(s, i) : i < m, s \in m^{-m+1} \} + m;$$

$f$ is increasing. $f(m) \geq m$ for all $m$.

Find $\eta$ such that $(\forall k)(\eta(k) > f(\eta(k - 1)))$. Then $\eta$ is increasing. For almost all $k$ we have, letting $m := \eta(k - 1)$: $m \geq k - 1$, so $\eta \upharpoonright k \in m^{-m+1}$, so by the definition of $f$ we get $f(m) \geq F(\eta \upharpoonright k, i)$ for any $i \leq k$. So $\eta(k) > f(\eta(k - 1) \geq F(\eta \upharpoonright k, i)$.

(4) $\rightarrow$ (3): Assume that GOOD has no winning strategy. Then BAD has a winning strategy $\sigma$ (since the game $D(A)$ is Borel, hence determined).

We can find a function $F : \omega_\omega \times \omega \to \omega$ such that for all $s, m_0, \ldots, m_k$ we have

$$\sigma(s, m_0, \ldots, m_k) = F(s^{-m_0} \cdots m_k, |s|).$$

Find $\eta \in A$ as in (4). So there is $k_0$ such that $\forall k \geq k_0 \eta(k) \geq F(\eta \upharpoonright k, k_0)$. So in the play

<table>
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<td>$n_0 := \sigma(s) = F(\eta \upharpoonright k_0, k_0)$</td>
</tr>
<tr>
<td>$m_0 := \eta(k_0 + 1)$</td>
<td>$n_1 := \sigma(s, m_0) = F(\eta \upharpoonright (k_0 + 1), k_0)$</td>
</tr>
<tr>
<td>$m_1 := \eta(k_0 + 2)$</td>
<td>$\vdots$</td>
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player BAD followed the strategy $\sigma$, but player GOOD won, a contradiction.

(3) $\rightarrow$ (1): Let $B$ be the set of all sequences $s^{-m_0}m_1^{-} \cdots$ that can be played when GOOD follows a specific winning strategy. Clearly $B \subseteq A$, and for some Laver tree $p$, $B = [p]$.
Lemma 2.4 [Ke]. Let \( A \subseteq \omega_\omega \) be an analytic set. Then the following are equivalent:

1. There exists a Miller tree \( p \) such that \([p] \subseteq A\).
2. \( A \) is unbounded in \((\omega_\omega, \leq^*)\).

Lemma 2.5. (1) Suppose \( b = c \). For every dense open \( D \subseteq L \) there exists a maximal antichain \( A \subseteq D \) such that

\[
\forall q \in L([q] \subseteq \bigcup \{[p] : p \in A\} \Rightarrow \exists A' \in [A]^{<c} \forall p \in A \setminus A' \ p \perp q).
\]

(2) The same is true for \( M \).

Proof. Let \( L = \{q_\alpha : \alpha < c\} \). Inductively we will define a set \( S \subseteq c \) and sequences \( \langle x_\gamma : \gamma < c \rangle \) and \( \langle p_\gamma : \gamma \in S \rangle \). Finally we will let \( A = \{p_\gamma : \gamma \in S\} \).

Let \( 0 \in S \) and choose \( x_0 \in [q_0] \) arbitrarily.

It can easily be seen that every Laver tree contains \( c \) extensions such that every two of them do not contain a common branch. So clearly we may find \( p_0 \in D \) such that \( x_0 \notin [p_0] \).

Now suppose that \( \langle x_\gamma : \gamma < \alpha \rangle \) and \( \langle p_\gamma : \gamma \in S \cap \alpha \rangle \) have been constructed for \( \alpha < c \).

First choose \( x_\alpha \in [q_\alpha] \) arbitrarily, but such that, if \([q_\alpha] \notin \bigcup \{[p_\gamma] : \gamma < \alpha\}\), then \( x_\alpha \notin \bigcup \{[p_\gamma] : \gamma < \alpha\} \).

In order to decide whether \( \alpha \in S \) or not we distinguish the following two cases:

Case 1. \( q_\alpha \) is compatible with some \( p_\gamma, \gamma < \alpha \). In this case \( \alpha \notin S \).

Case 2. \( q_\alpha \) is incompatible with all \( p_\gamma, \gamma < \alpha \). Now we let \( \alpha \in S \), and we define \( p_\alpha \) as follows:

By Lemma 2.3 for each \( \gamma \in \alpha \) we may find \( f_\gamma : \omega \to \omega \) such that

\[
(\forall \eta \in [p_\gamma] \cap [q_\alpha])(\exists k)(\eta(k) \leq f_\gamma(\eta(k - 1))).
\]

By our assumption on \( b \) there exists a strictly increasing \( f \) which dominates all the \( f_\gamma \)'s. Now define \( p_\alpha' \in L \) as follows: \( stem(p_\alpha') = stem(q_\alpha) \), and for \( t \in p_\alpha' \), if \( t \supseteq stem(p_\alpha') \) and \( |t| = n \), we require

\[
Succ_{p_\alpha'}(t) = Succ_{q_\alpha}(t) \cap [f(t(n - 1)), \infty).
\]

Clearly \( p_\alpha' \in L \), \( p_\alpha' \subseteq q_\alpha \), and by (**) and our assumption on \( f \) we conclude \([p_\gamma] \cap [p_\alpha'] = \emptyset\) for every \( \gamma < \alpha \).

By the remark above that every Laver tree contains \( c \) extensions such that every two of them do not contain a common branch, we may find \( p_\alpha \in D \) such that \( p_\alpha \) extends \( p_\alpha' \) and \([p_\alpha] \) and \( \{x_\gamma : \gamma \leq \alpha\} \) are disjoint.

This finishes the construction. Now let \( A := \{p_\gamma : \gamma \in S\} \).

Since every \( q_\alpha \) is either compatible with some \( p_\gamma, \gamma < \alpha \) (Case 1) or contains the condition \( p_\alpha \) (Case 2), and for \( \alpha \neq \gamma \) with \( \alpha, \gamma \in S \) we have \([p_\alpha] \cap [p_\gamma] = \emptyset\), we conclude that \( A \) is a maximal antichain.

\( A \) also satisfies condition (\*) : Let \( q = q_\alpha \). By construction, if \([q_\alpha] \notin \bigcup \{[p_\gamma] : \gamma \in S \cap \alpha\} \), then \([q_\alpha] \notin \bigcup \{[p_\gamma] : \gamma \in S\} \).

The proof of (2) is analogous, but instead of Lemma 2.3 we use Lemma 2.4.

Lemma 2.6. Suppose \( b = c \). Then \( \text{add}(\lambda^0) \leq \kappa(\lambda) \) and \( \text{add}(\lambda^0) \leq \kappa(M) \).
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Proof. We may assume \( \kappa(\mathbb{L}) < \varepsilon \). Let \( \dot{\mathcal{F}} \) be a \( \mathbb{L} \)-name such that \( \Vdash_{\mathbb{L}} \text{"} \dot{\mathcal{F}} : \kappa(\mathbb{L}) \to \varepsilon \text{ is onto"} \). For \( \alpha < \kappa(\mathbb{L}) \) let

\[
D_\alpha := \{ p \in \mathbb{L} : (\exists \beta) p \Vdash_{\mathbb{L}} \dot{\mathcal{F}}(\alpha) = \beta \}.
\]

For \( p \in D_\alpha \) we write \( \beta_p = \beta_p(\alpha) \) for the unique \( \beta \) satisfying \( p \Vdash_{\mathbb{L}} \dot{\mathcal{F}}(\alpha) = \beta \).

Clearly \( D_\alpha \) is dense and open. So we may choose a maximal antichain \( A_\alpha \subseteq D_\alpha \) as in Lemma 2.5. Let

\[
X_\alpha := \omega \setminus \bigcup \{ \{ p \} : p \in A_\alpha \}.
\]

Then \( X_\alpha \in \mathcal{L}^0 \). We claim that \( X = \bigcup_{\alpha < \kappa(\mathbb{L})} X_\alpha \notin \mathcal{L}^0 \). Suppose on the contrary \( X \in \mathcal{L}^0 \). So we may find \( q \in \mathbb{L} \) such that \( [q] \cap X = \emptyset \) and hence \( [q] \subseteq \bigcup \{ \{ p \} : p \in A_\alpha \} \) for each \( \alpha \). By the choice of \( A_\alpha \) each of the sets

\[
B_\alpha := \{ \beta_p(\alpha) : p \in A_\alpha \text{, } p \text{ compatible with } q \}
\]

is bounded in \( \varepsilon \). Since \( \varepsilon \) is regular by our assumption \( b = \varepsilon \), we can find \( \nu < \varepsilon \) such that for all \( \alpha < \kappa(\mathbb{L}) \), \( B_\alpha \subseteq \nu \). So easily conclude that

\( q \Vdash_{\mathbb{L}} \text{"} \text{ran}(\dot{\mathcal{F}}) \subseteq \nu < \varepsilon \text{"} \).

This is a contradiction.

The proof for \( \mathbb{M} \) is similar.

Theorem 2.7. \( \kappa(\mathbb{L}) \leq \mathfrak{h} \) and \( \kappa(\mathbb{M}) \leq \mathfrak{h} \).

Proof. We prove it only for \( \mathbb{L} \). The proof for \( \mathbb{M} \) is very similar. We work in \( V \). Let \( (\mathcal{A}_\alpha : \alpha < \mathfrak{h}) \) be a family of maximal almost disjoint families such that:

1. if \( \alpha < \beta < \mathfrak{h} \), then \( \mathcal{A}_\beta \) refines \( \mathcal{A}_\alpha \);
2. there exists no maximal almost disjoint family refining all the \( \mathcal{A}_\alpha \);
3. \( \bigcup \{ \mathcal{A}_\alpha : \alpha < \mathfrak{h} \} \) is dense in \( ([\omega]^{\omega}, \subseteq^*) \).

That such a sequence exists was shown in \([\text{BaPeSi}]\).

Since \( \mathfrak{h} \) is regular, for every \( p \in \mathbb{L} \) there exists \( \alpha < \mathfrak{h} \) such that for each \( s \in \text{Split}(p) \) there is \( A \in \mathcal{A}_\alpha \) with \( A \subseteq^* \text{Succ}_p(s) \). Hence, writing \( \mathcal{L}_\alpha \) for the set of those \( p \in \mathbb{L} \) for which \( \alpha \) has the property just stated, we conclude \( \mathbb{L} = \bigcup \{ \mathcal{L}_\alpha : \alpha < \mathfrak{h} \} \).

For each \( A \in \mathcal{A}_\alpha \) choose \( \mathcal{B}_A = \{ B^A(p) : p \in \mathbb{L} \} \), a maximal almost disjoint family on \( A \).

Now we will define \( \mathbb{L}_\alpha := \{ q^\alpha(p) : p \in \mathcal{L}_\alpha \} \) such that \( q^\alpha(p) \) extends \( p \) for every \( p \in \mathcal{L}_\alpha \) and \( p_1 \neq p_2 \) implies \( q^\alpha(p_1) \perp q^\alpha(p_2) \). For \( p \in \mathcal{L}_\alpha \), \( q^\alpha(p) \) will be defined as follows:

For each \( s \in \text{Split}(p) \) let \( C^\alpha_s(p) := \text{Succ}_p(s) \cap B^A(p) \) where \( A \in \mathcal{A}_\alpha \) is such that \( A \subseteq^* \text{Succ}_p(s) \). So clearly \( C^\alpha_s(p) \) is infinite. Now \( q^\alpha(p) \) is the unique Laver tree \( \leq p \) satisfying \( \text{stem}(q^\alpha(p)) = \text{stem}(p) \) and for each \( s \in \text{Split}(q^\alpha(p)) \) we have \( \text{Succ}_{q^\alpha(p)}(s) = C^\alpha_s(p) \).

It is not difficult to see that \( \mathbb{L}_\alpha \) has the stated properties.

Now we are ready to define a \( \mathbb{L} \)-name \( \dot{\mathcal{F}} \) such that \( \Vdash_{\mathbb{L}} \text{"} \dot{\mathcal{F}} : \mathfrak{h} \to \varepsilon \text{ is onto"} \):

For each \( p \in \mathcal{L}_\alpha \), let \( \{ r^\xi(p) : \xi < \varepsilon \} \subseteq \mathbb{L} \) be a maximal antichain below \( q^\alpha(p) \), and define \( \dot{\mathcal{F}} \) in such a way that \( r^\xi(p) \Vdash_{\mathbb{L}} \text{"} \dot{\mathcal{F}}(\alpha) = \xi \text{"} \). As \( \bigcup \{ \mathcal{L}_\alpha : \alpha < \mathfrak{h} \} \) is dense in \( \mathbb{L} \), it is easy to check that \( \dot{\mathcal{F}} \) is as desired.
Theorem 2.8. Let $\omega_2 = S_M \cup S_L$, where the sets $S_M$ and $S_L$ are disjoint and stationary. Let $(P_\alpha, Q_\alpha : \alpha < \omega_2)$ be a countable support iteration of length $\omega_2$ such that for all $\alpha$ we have $\Vdash_{P_\alpha} Q_\alpha = M$ whenever $\alpha \in S_M$, and $\Vdash_{P_\alpha} Q_\alpha = L$ otherwise. Also suppose that $V$ satisfies CH. Then in $V_{P_\omega}$, $\mathfrak{p} = \omega_1$ holds.

Proof. Both $M$ and $L$ have the property $(*)_1$ of [JuSh]. (For $L$, this was proved in [JuSh] and for $M$ this was proved in [BaJuSh].) [JuSh] also showed that this property is preserved under countable support iterations, so also $P_{\omega_2}$ has this property. Hence, the reals of $V$ do not have measure zero in $V_{P_\omega}$, so from $\mathfrak{p} \leq \mathfrak{s} \leq \text{unif}(\mathcal{L})$ (where $\mathfrak{s}$ is the splitting number and $\text{unif}(\mathcal{L})$ is the smallest cardinality of a set of reals which is not null) we get the desired conclusion.

Theorem 2.9. Let $P_{\omega_2}$ be as in Theorem 2.8. Then

$$V_{P_{\omega_2}} \models \omega_1 = \text{add}(\mathcal{I}^0) = \text{add}(\mathcal{M}^0) < \text{cov}(\mathcal{I}^0) = \text{cov}(\mathcal{M}^0) = \omega_2.$$ 

Proof. Since $L$ adds a dominating real, we have $V_{P_{\omega_2}} \models b = c$; so by Lemma 2.6 and Theorems 2.7 and 2.8 it suffices to prove that the covering coefficients are $\omega_2$ in the respective models. The proof of this is similar to the proof of [JuMiSh, Theorem 1.2] that cov of the Marczewski ideal is $\omega_2$ in the iterated Sacks's forcing model.

We give the proof only for $\mathcal{I}^0$. Suppose $(\mathcal{X}_\alpha : \alpha < \omega_1) \in V_{P_{\omega_2}}$ is a sequence of $\mathcal{I}^0$-sets. In $V_{P_{\omega_2}}$ let $f_\alpha : L \to L$ be such that for every $p \in L$, $f_\alpha(p)$ extends $p$ and $[f_\alpha(p)] \cap \mathcal{X}_\alpha = \emptyset$. Since $P_{\omega_2}$ has the $\omega_2$-chain condition, by a L"owenheim-Skolem argument it is possible to find $\gamma < \omega_2$ such that

$$(f_\alpha \upharpoonright L_{V_\gamma} : \alpha < \omega_1) \in V_{P_\gamma},$$

where $V_\gamma := V_{P_\gamma}$. Moreover, it is possible to find such a $\gamma$ in $S_L$. We claim that the Laver real $x_\gamma$ (which is added by $Q_\gamma = L^{P_\omega}$) is not in $\bigcup_{\alpha < \omega_1} X_\alpha$, which will finish the proof. Otherwise, for some $p \in L_{P_{\omega_2}}$, where $L_{P_{\omega_2}} := L_{\omega_2}/G_\gamma$ and some $\alpha < \omega_1$ we would have $p \Vdash x_\gamma \in X_\alpha$. But letting $q := p(\gamma) \in L$ and letting $r(\gamma) := f_\alpha(q)$ and $r(\beta) := p(\beta)$ for $\beta > \gamma$ we see that $r \Vdash x_\gamma \notin X_\alpha$, a contradiction.

References


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