OSCI LLATION AND NONOSCILLATION CRITERIA
FOR DELAY DIFFERENTIAL EQUATIONS

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Abstract. Oscillation and nonoscillation criteria for the first-order delay differential equation
\[ x'(t) + p(t)x(\tau(t)) = 0, \quad t \geq t_0, \quad \tau(t) < t, \]
are established in the case where
\[ \int_{\tau(t)}^{t} p(s) \, ds \geq \frac{1}{e} \quad \text{and} \quad \lim_{t \to \infty} \int_{\tau(t)}^{t} p(s) \, ds = \frac{1}{e}. \]

1. Introduction

The qualitative properties of the solutions of the delay differential equation
\[ x'(t) + p(t)x(\tau(t)) = 0, \quad t \geq t_0, \]
where \( \tau(t) < t \), have been the subject of many investigations. The first systematic study was made by Myshkis [6]. Among others he has shown [5] that all solutions of (1) oscillate if
\[ p(t) > 0, \quad \lim_{t \to \infty} \sup \{t - \tau(t)\} < \infty, \quad \lim_{t \to \infty} \inf \{t - \tau(t)\} \cdot \lim_{t \to \infty} p(t) > \frac{1}{e}. \]
Later these conditions were improved, by Ladas [4] and Koplatadze and Chanturija [3], to
\[ \lim_{t \to \infty} \inf \int_{\tau(t)}^{t} p(s) \, ds > \frac{1}{e}. \]
Concerning the constant \( \frac{1}{e} \) in (2) we mention that if the inequality
\[ \int_{\tau(t)}^{t} p(s) \, ds \leq \frac{1}{e} \]
holds, then, according to a result in [3], (1) has a nonoscillatory solution. To the best of our knowledge there is no result in the case when we have
\[ \int_{\tau(t)}^{t} p(s) \, ds \geq \frac{1}{e} \quad \text{and} \quad \lim_{t \to \infty} \int_{\tau(t)}^{t} p(s) \, ds = \frac{1}{e}. \]

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In connection with the delay function $\tau(t)$ in (1) we suppose that $\tau(t)$ is strictly increasing on $[t_0, \infty)$, $\lim_{t \to \infty} \tau(t) = \infty$, and its inverse is $\tau_1(t)$ ($\tau_1(t) > t$). Let $\tau_{-k}(t)$ be defined on $[t_0, \infty)$ by
$$
\tau_{-k-1}(t) = \tau_1(\tau_{-k}(t)) \quad \text{for} \quad k = 1, 2, \ldots,
$$
and let
$$
k_k = \tau_{-k}(t_0), \quad k = 1, 2, \ldots.
$$
Clearly $t_k \to \infty$ as $k \to \infty$.

The coefficient $p(t)$ is assumed to be a piecewise continuous function and satisfies the relation
$$
\int_{\tau(t)}^t p(s) \, ds \geq \frac{1}{e}. \tag{4}
$$
Let $\phi(t)$ be a continuous function on $[\tau(t_0), t_0]$. A function $x(t)$ is a solution of (1), associated with the initial function $\phi(t)$, if $x(t) = \phi(t)$ on $[\tau(t_0), t_0]$, $x(t)$ is continuous on $[\tau(t_0), \infty)$, is differentiable almost everywhere on $[t_0, \infty)$, and satisfies (1).

As is customary, a solution is called oscillatory if it has arbitrarily large zeros. Otherwise it is called nonoscillatory.

Among the functions $p(t)$ we define a set $\mathcal{A}$ for $0 < \lambda \leq 1$ as follows.

**Definition.** The piecewise continuous function $p(t): [t_0, \infty) \to [0, \infty)$ belongs to $\mathcal{A}$ if
$$
\int_{\tau(t)}^t p(s) \, ds \geq \frac{1}{e}, \quad t \geq t_1, \tag{5}
$$
$$
\int_{\tau(t)}^t p(s) \, ds \geq \frac{1}{e} + \lambda_k \left( \int_{t_k}^{t_{k+1}} p(s) \, ds - \frac{1}{e} \right) \quad \text{for} \quad t_k < t \leq t_{k+1}, \quad k = 1, 2, \ldots,
$$
for some $\lambda_k \geq 0$, and
$$
\liminf_{k \to \infty} \lambda_k = \lambda > 0.
$$

We remark that if $\int_{\tau(t)}^t p(s) \, ds$ is a nonincreasing function and $\int_{\tau(t)}^t p(s) \, ds \geq \frac{1}{e}$, then $p(t) \in \mathcal{A}$, because we may have $\lambda_k = 1$ in (5). However, the monotonicity is not a necessary condition; e.g., in the case $\tau(t) = t - 1$ the function
$$
p(t) = \frac{1}{e} + (K \sin^2 \pi t/t^\alpha), \quad K > 0 \text{ and } 0 \leq \alpha \leq 2, \tag{6}
$$
belongs to $\mathcal{A}$ because $\int_{t-1}^{t} (\sin^2 \pi s/s^\alpha) \, ds$ is a nonincreasing function.

Our main results are

**Theorem 1.** Assume that the function $p(t)$ in (1) belongs to $\mathcal{A}$ for some $\lambda \in (0, 1]$ and
$$
\sum_{i=1}^{\infty} \left( \int_{t_{i-1}}^{t_i} p(s) \, ds - \frac{1}{e} \right) = +\infty. \tag{7}
$$

Then every solution of (1) oscillates.
In the next theorem we consider the case where the sum in (7) is convergent.

**Theorem 2.** Assume that \( p(t) \in \mathcal{A}_\lambda \), for some \( 0 < \lambda \leq 1 \) and either

\[
\lambda \limsup_{k \to \infty} k \sum_{i=k}^{\infty} \left( \int_{t_{i-1}}^{t_i} p(s) \, ds - \frac{1}{e} \right) > \frac{2}{e}
\]

or

\[
\lambda \liminf_{k \to \infty} k \sum_{i=k}^{\infty} \left( \int_{t_{i-1}}^{t_i} p(s) \, ds - \frac{1}{e} \right) > \frac{1}{2e}.
\]

Then every solution of (1) oscillates.

**Note.** If the function \( \int_{t(t)}^{t} p(s) \, ds \) is monotone, then the value of \( \lambda \) in conditions (8) and (9) of Theorem 2 is equal to one.

In the following theorem we give a criterion for nonoscillation.

**Theorem 3.** Let \( \tau(t) = t - 1 \), \( p(t) = \frac{1}{e} + a(t) \), and \( t_0 = 1 \) in (1); i.e., it has the form

\[
(1)' \quad x'(t) + \left[ \frac{1}{e} + a(t) \right] x(t-1) = 0, \quad t \geq 1.
\]

Assume that

\[
a(t) \leq \frac{1}{8e} t^2.
\]

Then (1)' has a solution \( x(t) \geq \sqrt{t} e^{-t} \).

The proofs of the above theorems and also some lemmas which will be used in these proofs will be given in the next section.

### 2. Lemmas and Proofs

The first two lemmas have origin in [3] (see also [2]).

**Lemma 1.** Assume that \( x(t) \) is a positive solution of (1) on \([t_{k-2}, t_k+1]\) for some \( k \geq 2 \). Let \( N \) be defined by

\[
N = \min_{t_k \leq t \leq t_k+1} \frac{x(\tau(t))}{x(t)}.
\]

Then \( N < (2e)^2 \).

**Proof.** Let \( L \) be the integral

\[
L = \int_{t_k}^{t_{k+1}} p(s) \, ds \geq \frac{1}{e}.
\]

By Lemma 3 in [2], we obtain \( N < \frac{((1 + \sqrt{1 - L})/L)^2}{L} \). Since the right-hand side is a decreasing function of \( L \), we get

\[
N < \frac{((1 + \sqrt{1 - (1/e)})/L)^2}{L} < (2e)^2.
\]

**Lemma 2.** Assume that \( x(t) \) is a positive solution of (1) on \([t_{k-3}, t_{k+1}]\) for some \( k \geq 3 \) and \( p(t) \in \mathcal{A}_\lambda \). Let \( M, N \) be defined by

\[
M = \min_{t_{k-1} \leq t \leq t_k} \frac{x(\tau(t))}{x(t)}, \quad N = \min_{t_k \leq t \leq t_{k-1}} \frac{x(\tau(t))}{x(t)}.
\]
Then

\[ M > 1 \quad \text{and} \quad N \geq \exp \left( M \left[ \frac{1}{e} + \lambda_k \left( \int_{t_k}^{t_{k+1}} p(s) \, ds - \frac{1}{e} \right) \right] \right) \geq M. \]

**Proof.** Following the lines of the proof of Lemma 1 in [2], we have \( \min\{ M, N \} = M \), and by (5) for \( t_k \leq t \leq t_{k+1} \)

\[ \frac{x(\tau(t))}{x(t)} \geq \exp \left( M \int_{\tau(t)}^{t} p(s) \, ds \right) \geq \exp \left( M \left[ \frac{1}{e} + \lambda_k \left( \int_{t_k}^{t_{k+1}} p(s) \, ds - \frac{1}{e} \right) \right] \right), \]

which implies the inequality concerning \( N \). On the other hand the solution \( x(t) \) is a strictly decreasing function on \( [t_{k-2}, t_{k+1}] \). Hence \( x(\tau(t))/x(t) > 1 \) on \( [t_{k-1}, t_k] \), and therefore \( M > 1 \). The proof of the lemma is complete.

The next lemma deals with some properties of the following sequence.

Let the sequence \( \{ n_i \} \) be defined by the recurrence relation

\[ r_0 = 1, \quad r_{i+1} = e^{r_i}/e \quad \text{for} \quad i = 0, 1, 2, \ldots. \]

**Lemma 3.** For the sequence \( \{ r_i \} \) in (10) the following relations hold:

(a) \( r_i < r_{i+1} \);
(b) \( r_i < e \);
(c) \( \lim_{i \to \infty} r_i = e \);
(d) \( r_i > e - \frac{2e}{i + 2} \).

**Proof.** The first two relations can be proved by induction. As a consequence of (a) and (b) the \( \lim_{i \to \infty} r_i = r \) exists and it is finite. Then by (10) we have

\[ r = e^{r}/e. \]

It is easy to check that

\[ e^{x}/e > x \quad \text{for} \quad x \neq e. \]

This inequality implies that the limit \( r \) equals \( e \).

Now we give the proof of (d). For \( i = 0 \) and \( i = 1 \) it is immediate. For \( i \geq 1 \) the proof goes by induction, so we have

\[ r_{i+1} = e^{r_i}/e > e^{1-2/(i+2)}, \]

and it is sufficient to show

\[ e^{1-2/(i+2)} > e - \frac{2e}{i + 3}, \]

or

\[ f(x) = e^{-2/x} + \frac{2}{x + 1} > 1 \quad \text{for} \quad x = i + 2. \]

Since

\[ f'(x) = \frac{2}{x^2} \left( e^{-1/x} + \frac{x}{x + 1} \right) \left( e^{-1/x} - \frac{x}{x + 1} \right) \]

and

\[ e^{1/x} > 1 + \frac{1}{x} = \frac{x + 1}{x}, \]

we have \( f'(x) < 0 \) and \( f(x) > \lim_{x \to \infty} f(x) = 1 \), which was to be shown.

The proof of the lemma is complete.
Proof of Theorem 1. Suppose the contrary. Then we may assume, without loss of
generality, that there exists a solution \( x(t) \) such that \( x(t) > 0 \) for \( t \geq t_{k-3} \)
for some \( k \geq 3 \). Let the sequence \( \{N_i\}_{i=0}^{\infty} \) be defined by

\[
N_i = \min_{t_{k+i-1} \leq t \leq t_{k+i}} \frac{x(t)}{\max_{t \leq t_{k+i}} x(t)}.
\]

By Lemma 2 we have \( N_0 > 1 \) and

\[
N_{i+1} \geq \exp \left( \frac{N_i}{e} \right) \exp \left( N_i \lambda_{k+i} \left( \int_{t_{k+i}}^{t_{k+i+1}} p(s) \, ds - \frac{1}{e} \right) \right) \geq N_i;
\]

therefore the sequence \( \{N_i\}_{i=0}^{\infty} \) is nondecreasing. On the other hand, by Lemma 1, it is bounded. Consequently the sequence converges. Let

\[
\lim_{i \to \infty} N_i = N.
\]

Then (13) implies

\[
N \geq \exp(N/e).
\]

Hence by (11) we have \( N = e \) and

\[
1 < N_0 < N_1 < \cdots < e.
\]

From (13), in view of (11), we obtain

\[
N_{i+1} \geq N_i \left( 1 + N_i \lambda_{k+i} \left( \int_{t_{k+i}}^{t_{k+i+1}} p(s) \, ds - \frac{1}{e} \right) \right).
\]

Thus

\[
N_{i+1} - N_i \geq N_i^2 \lambda_{k+i} \left( \int_{t_{k+i}}^{t_{k+i+1}} p(s) \, ds - \frac{1}{e} \right).
\]

From the definition of \( \lambda_k \) we know that \( \lambda = \liminf_{k \to \infty} \lambda_k > 0 \), so for any sufficiently small \( \varepsilon > 0 \) there exists a value \( \kappa_\varepsilon \) such that \( \lambda_{k+i} > \lambda - \varepsilon \)
for \( k + i > \kappa_\varepsilon \). Thus, for such \( i \)'s from (15) and (14), we have

\[
N_{i+1} - N_i \geq N_i^2 \lambda_{k+i} \left( \int_{t_{k+i}}^{t_{k+i+1}} p(s) \, ds - \frac{1}{e} \right).
\]

Summing up the inequalities above, we obtain

\[
e - N_i > N_i^2 (\lambda - \varepsilon) \sum_{j=i}^{\infty} \left( \int_{t_{k+j}}^{t_{k+j+1}} p(s) \, ds - \frac{1}{e} \right) \quad \text{for } k + i \geq \kappa_\varepsilon.
\]

The last inequality contradicts assumption (7). The proof is complete.
Proof of Theorem 2. Suppose the contrary. Then, as in the proof of Theorem 1, we have the sequence \( \{N_i\}_{i=0}^{\infty} \) such that inequalities (13)–(16) hold. In particular, from (13) we have

\[ N_{i+1} \geq \exp(N_i/e). \]

Comparing the last inequality with (10), we obtain by induction

\[ N_0 > r_0 = 1, \quad N_i > r_i \quad \text{for } i = 1, 2, \ldots. \]

Then by Lemma 3(d) we have

\[ (17) \quad e - N_i < e - r_i < 2e/(i + 2). \]

Multiplying (16) by \( k + i \) we obtain

\[
(k + i) \frac{2e}{i + 2} > N_i^2(\lambda - e)(k + i) \sum_{j=k+i}^{\infty} \left( \int_{t_j}^{t_{j+1}} p(s) \, ds - \frac{1}{e} \right) \quad \text{for } k + i \geq \kappa_e.
\]

Taking the limit as \( i \to \infty \) we get

\[
2e \geq e^2 \lambda \limsup_{k \to \infty} k \sum_{j=k}^{\infty} \left( \int_{t_j}^{t_{j+1}} p(s) \, ds - \frac{1}{e} \right),
\]

which contradicts (8).

Now let \( A \) be defined by

\[
A = \liminf_{k \to \infty} k \sum_{j=k}^{\infty} \left( \int_{t_j}^{t_{j+1}} p(s) \, ds - \frac{1}{e} \right).
\]

If \( A = \infty \), then, by (8), every solution oscillates. Therefore we consider the case \( 0 < A < \infty \). So for any sufficiently small \( \varepsilon > 0 \) there exists a value \( \hat{\kappa}_e \) such that for \( \lambda = \lambda - \varepsilon > 0 \) and \( \hat{A} = A - \varepsilon > 0 \)

\[ (18) \quad \lambda_k > \hat{\lambda} \quad \text{and} \quad \sum_{j=k}^{\infty} \left( \int_{t_j}^{t_{j+1}} p(s) \, ds - \frac{1}{e} \right) > \frac{\hat{A} \lambda}{k} \quad \text{for } k \geq \hat{\kappa}_e.
\]

If we use the inequality

\[
\exp \frac{x}{e} > x + \frac{1}{2} \exp \left( \frac{\xi}{e} \right) \left( 1 - \frac{x}{e} \right)^2 \quad \text{for } \xi < x < e
\]

in (13) we obtain for \( N_i > \xi \) and \( k + i > \hat{\kappa}_e \)

\[
N_{i+1} \geq \exp \left( \frac{N_i}{e} \right) \exp \left( N_i \hat{\lambda} \left( \int_{t_{k+i}}^{t_{k+i+1}} p(s) \, ds - \frac{1}{e} \right) \right)
\geq \left[ N_i + \frac{1}{2} \exp \left( \frac{\xi}{e} \right) \left( 1 - \frac{N_i}{e} \right)^2 \right] \left( 1 + N_i \hat{\lambda} \left( \int_{t_{k+i}}^{t_{k+i+1}} p(s) \, ds - \frac{1}{e} \right) \right).
\]

Consequently

\[
N_{i+1} - N_i > \frac{1}{2} \exp \left( \frac{\xi}{e} \right) \left( 1 - \frac{N_i}{e} \right)^2 + \xi^2 \hat{\lambda} \left( \int_{t_{k+i}}^{t_{k+i+1}} p(s) \, ds - \frac{1}{e} \right)
\]

\[ : \]
and summing up,
\[
e - N_i > \frac{1}{2} \exp \left( \frac{\xi}{e} \right) \sum_{j=i}^{\infty} \left( 1 - \frac{N_i}{e} \right)^2 + \xi^2 \frac{\lambda A}{k + i}.
\]
or
\[(19)\quad e - N_i > \frac{1}{2} \exp \left( \frac{\xi}{e} \right) \sum_{j=i}^{\infty} \left( 1 - \frac{N_i}{e} \right)^2 + \frac{\xi^2 \lambda A}{k + i}.
\]
In particular the last inequality yields
\[
e - N_i > \frac{U_0}{k + i}, \quad U_0 = \xi \frac{\lambda A}{2}
\]
By iteration we can improve this inequality to
\[(20)\quad e - N_i > \frac{U_n}{k + i}, \quad n = 0, 1, 2, \ldots.
\]
Namely by (19) we have
\[
e - N_i > \frac{1}{2} \exp \left( \frac{\xi}{e} \right) \sum_{j=i}^{\infty} \left( \frac{U_n}{e(k + j)} \right)^2 + \frac{\xi^2 \lambda A}{k + i}
\]
\[
> \frac{U_n^2}{2e^2} \exp \left( \frac{\xi}{e} \right) \frac{1}{k + i} + \frac{\xi^2 \lambda A}{k + i} = \frac{U_{n+1}}{k + i},
\]
where
\[(21)\quad U_{n+1} = \frac{U_n^2}{2e^2} \exp \left( \frac{\xi}{e} \right) + \xi^2 \frac{\lambda A}{k + i}, \quad n = 0, 1, 2, \ldots.
\]
From this it is clear that the sequence \( \{U_n\}_{n=0}^{\infty} \) is increasing. Moreover, comparing inequalities (17) and (20) we see that \( U_n \leq 2e \). Therefore the sequence has a limit, say \( U \), which satisfies the equation
\[
U = \frac{U^2}{2e^2} \exp \left( \frac{\xi}{e} \right) + \xi^2 \frac{\lambda A}{k + i}.
\]
This is a quadratic equation with real roots and therefore the discriminant is not negative; i.e.,
\[
1 - 2e^{\xi/e} - 2e^{-2} \xi^2 \frac{\lambda A}{k + i} \geq 0.
\]
Let \( \xi \to 0 \) and \( \xi \to e \). Then the last inequality becomes
\[
1 - 2e\lambda A \geq 0,
\]
which contradicts (9).

The proof of the theorem is complete.

**Proof of Theorem 3.** The proof is based on known comparison theorems (see Myshkis [6] or Elbert [1]). Let the functions \( A(t) \), \( B(t) \), \( C(t) \) be defined as
\[
A(t) = \frac{1}{e} + a(t),
\]
\[
B(t) = \frac{1}{e} + \frac{1}{8et^2},
\]
\[
C(t) = \frac{1}{e} \sqrt{1 - \frac{1}{t}}, \quad t > 1.
\]

By the assumption we have $A(t) \leq B(t)$. We are going to show that the inequality $B(t) < C(t)$ also holds. Namely, for $\theta = \frac{1}{2t} \in (0, \frac{1}{2})$, we have

$$C(t) - B(t) = \frac{\theta^3(\frac{1}{2}\theta^2 - \frac{1}{4}\theta + 2)}{e\sqrt{1 - 2\theta(1 - \theta + (1 + \frac{1}{2}\theta^2)\sqrt{1 - 2\theta}}} > 0.$$ 

Now we will compare the differential equations

$$x'(t) + A(t)x(t-1) = 0,$$
$$z'(t) + B(t)z(t-1) = 0,$$
$$u'(t) + C(t)u(t-1) = 0.$$ 

Let us observe that the function $u(t) = \sqrt{te^{-t}}$ is a solution of the last differential equation. Let the initial function $\phi(t)$ be the function $\sqrt{te^{-t}}$ on $[0, 1]$, and let $x(t)$ and $z(t)$ be the solutions of the first and the second differential equations respectively, associated with this initial function $\phi(t)$. Then by the comparison theorems mentioned above we have

$$x(t) > z(t) > u(t) = \sqrt{te^{-t}} \quad \text{for } t > 1,$$

which was to be shown.

**Remark 1.** For (1)' we have $t_k = k + 1$ and

$$\limsup_{k \to \infty} k \sum_{i=k}^{\infty} \left( \int_{t_{i-1}}^{t_i} p(s) \, ds - \frac{1}{e} \right) = \limsup_{k \to \infty} \int_{k}^{\infty} a(t) \, dt \leq \frac{1}{8e}.$$ 

Now the question arises naturally whether or not the bounds in conditions (8) and (9) of Theorem 2 can be replaced by smaller ones.

**Remark 2.** It is to be emphasized that in Theorem 3 we require neither

$$p(t) \geq 0 \quad \text{nor} \quad \int_{t(t)}^{t} p(s) \, ds \geq \frac{1}{e}.$$ 

**Remark 3.** Applying Theorems 1, 2 we see that, under (6), (1) oscillates for any $K > 0$ if $0 \leq \alpha < 2$ and for $K > \frac{1}{\alpha}$ if $\alpha = 2$. On the other hand it has a nonoscillatory solution for $K < \frac{1}{8e}$ if $\alpha = 2$.

**REFERENCES**


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