AN ESTIMATION OF SINGULAR VALUES
OF CONVOLUTION OPERATORS

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Abstract. In this paper we determine the asymptotic order of singular values
of convolution operators \( \int_0^x k(x - y) \, dy \), where \( k(x) = x^{\alpha-1} L(1/x) \) (0 < \( \alpha < 1/2 \)) and \( L \) is a slowly varying function from some class.

1. Introduction

Let \( \mathscr{H} \) be a separable Hilbert space over \( \mathbb{C} \) and \( A \) be a compact operator. The singular values of \( A \) \( (s_n(A)) \) are the eigenvalues of the operator \( (A^*A)^{1/2} \) (or \( (AA^*)^{1/2} \)).

V. Faber and G. M. Wing \[3, 4\] have found an upper bound on the singular values of fractional integral operators and of some other similar operators.

In \[2\] an exact asymptotic of the singular values of the fractional integral operator \( J^\alpha = \frac{1}{\Gamma(\alpha)} \int_0^x k(x - y)^{\alpha-1} \, dy \) is found. In this paper we find the asymptotic order of the singular values of the operator \( \int_0^x k(x - y) \, dy \) acting on \( \mathscr{H} = L^2(0, 1) \) whose kernel has power singularity and singularity arising from a slowly varying function \( L \) in the point \( x = 0 \). In what follows for given sequences \( \{a_n\}, \{b_n\} \) \( (a_n > 0, b_n > 0) \) we write \( a_n \asymp b_n \) if there exist constants \( c_1, c_2 > 0 \) such that \( c_1 \leq a_n/b_n \leq c_2 \) for all \( n \in \mathbb{N} \). By \( \int_a^b m(x, y) \, dy \) we denote the integral operator on \( L^2(a, b) \) with the kernel \( m(x, y) \).

2. Main result

Let \( L \in C^1[1, \infty) \) be a nondecreasing function on \( [1, \infty) \), let

\[
\lim_{x \to +\infty} xL'(x)/L(x) = 0,
\]

and let \( x \mapsto xL'(x)/L(x) \) be a nonincreasing function for \( x \) large enough. Define the operator \( A: L^2(0, 1) \to L^2(0, 1) \) by

\[
Af(x) = \int_0^x k(x - y)f(y) \, dy
\]

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where
\[ k(x) = x^{a-1}L \left( \frac{1}{x} \right) \quad (\alpha > 0). \]

Theorem 1. If 0 < \( \alpha < 1/2 \), then \( s_n(A) \propto L(n)/n^\alpha \).

Proof. Case A: \( L \) is not a bounded function. Then \( \lim_{x \to +\infty} L(x) = +\infty \).

Observe that if we smoothly extend \( L \) from \([1, \infty)\) to \([0, \infty)\) the new operator \( A \) has the same singular values as the old one (because it acts on \( \mathcal{H} = L^2(0, 1) \)). Because of that we can assume \( L \in C^{1}[0, \infty) \) and that \( L \) is a linear function on \([0, 1]\). Without loss of generality we can assume that \( L > 0 \) on \([0, \infty)\) (because \( s_n(I^\alpha) \propto 1/n^\alpha \) [2]). Let \( a > 1 \) be a fixed number and let

\[ L_a(x) = \begin{cases} L(x), & x \geq a, \\ L'(a)x + L(a) - aL'(a), & 0 < x < a. \end{cases} \]

Let \( B \) and \( B_a \) be linear operators on \( L^2(0, 1) \) defined by

\[ Bf(x) = \int_0^1 |x-y|^{\alpha-1}L \left( \frac{1}{|x-y|} \right) f(y) \, dy, \]
\[ B_a f(x) = \int_0^1 |x-y|^{\alpha-1}L_a \left( \frac{1}{|x-y|} \right) f(y) \, dy. \]

Before the proof of Theorem 1 we give the following lemma.

Lemma 1. If 0 < \( \alpha < 1/2 \) and \( a \) is large enough, then

\[ \lim_{n \to \infty} \frac{s_n(B)}{s_n(B_a)} = 1. \]

Proof. Let \( P: L^2(0, 1) \to L^2(0, 1) \) be a linear operator defined by \( Pf(x) = \chi[0,1/a](x)f(x) \) and \( Q = I - P \). (Here \( \chi[a,b] \) is the characteristic function of \([a,b]\).) Then

\[ B_a = (P + Q)B(P + Q) = PB_aP + QB_aP + PB_aQ + QB_aQ \]

and

\[ B = (P + Q)B(P + Q) = PBP + QBP + PBJQ + QBQ. \]

Since \( L_a(1/x) = L(1/x) \) for \( 0 < x \leq 1/a \), we obtain \( PB_aP = PBP \) and

\[ B = B_a + Q(B - B_a)P + P(B - B_a)Q + Q(B - B_a)Q. \]

From the definition of \( L_a \) it follows that \( Q(B - B_a)P \) and \( Q(B - B_a)Q \) are Hilbert Schmidt operators and hence

\[ s_n(Q(B - B_a)P + P(B - B_a)Q + Q(B - B_a)Q) = \sigma(n^{-1/2}) = \sigma \left( \frac{L(n)}{n^\alpha} \right) \quad (0 < \alpha < \frac{1}{2}) \]

If we show that

\[ \lim_{n \to \infty} \frac{n^\alpha}{L(n)} s_n(B_a) = c_0 \neq 0, \]
then from (2), (3), (4), and the Ky Fan Theorem [5] follows (1). Now we prove (4) (with \( c_0 = \pi^{-\alpha}\Gamma(\alpha)\cos(\alpha\pi/2) \)) if \( 0 < \alpha < 1/2 \) and \( a \) is large enough.

Consider the operator \( B'_a : L^2(-1, 1) \rightarrow L^2(-1, 1) \) defined by

\[
B'_a f(x) = \int_{-1}^{1} k_a(|x-y|) f(y) \, dy
\]

where

\[
k_a(t) = t^{\alpha-1} L_a \left( \frac{1}{t} \right) \quad (t > 0).
\]

Let

\[
K_a(\xi) = \int \epsilon^{i\xi} k_a(|t|) \, dt
\]

and

\[
H_a(x, y) = \sum_{n=-\infty}^{\infty} (k_a(|x - y + 4n|) - k_a(|x + y + 4n + 2|)).
\]

By direct computation we conclude that

\[
\int_{-1}^{1} H_a(x, y) \varphi_n(y) \, dy = K_a \left( \frac{n\pi}{2} \right) \varphi_n(x)
\]

where

\[
\varphi_n(x) = \sin \frac{n\pi(1+x)}{2}, \quad n \in \mathbb{N}.
\]

(The system \( \{\varphi_n\}_{n=1}^{\infty} \) is an orthonormal basis of \( L^2(-1, 1) \).) We shall demonstrate that the following conditions are satisfied:

1°. \( K_a(\xi) \sim \text{const} \cdot L(\xi)/\xi^\alpha \) when \( \xi \rightarrow +\infty \).

2°. If \( a \) is large enough, the function \( K_a \) is decreasing if \( \xi \) is large enough.

3°. The operator \( D : L^2(0, 2) \rightarrow L^2(0, 2) \) defined by

\[
Df(x) = \int_{0}^{2} k_a(x+y) f(y) \, dy
\]

has the property \( s_n(D) = \sigma(L(n)/n^\alpha) \) \( (0 < \alpha < 1/2) \).

4°. The function \( \sum_{n \neq 0; n \neq -1} (k_a(|x - y + 4n|) - k_a(|x + y + 4n + 2|)) \) is bounded on \( [-1, 1]^2 \).

The property 4° is the consequence of the linearity of \( L_a \) on \( [0, a] \). Simple computation yields

\[
\int_{0}^{2} \int_{0}^{2} |k_a(x+y)|^2 \, dx \, dy < \infty \quad \text{(for every } \alpha > 0)\]

and hence

\[
s_n(D) = \sigma(n^{-1/2}) = \sigma \left( \frac{L(n)}{n^\alpha} \right) \quad \left( 0 < \alpha < \frac{1}{2} \right).
\]

By a substitution we obtain

\[
K_a(\xi) = 2\xi^{-\alpha} \int_{0}^{\infty} t^{-\alpha-1} \cos \frac{1}{t} \cdot L_a(\xi t) \, dt.
\]

We shall prove now

\[
Q(\xi) = \int_{0}^{\infty} x^{-\alpha-1} \cos \frac{1}{x} L_a(\xi x) \, dx \sim L(\xi) \int_{0}^{\infty} x^{-\alpha-1} \cos \frac{1}{x} \, dx \quad (\xi \rightarrow +\infty).
\]
Let
\[ Q_1(\xi) = \int_{1/\pi}^{+\infty} x^{-\alpha-1} \cos \frac{1}{x} L_\alpha(\xi x) \, dx \]
and
\[ Q_2(\xi) = \int_{0}^{1/\pi} x^{-\alpha-1} \cos \frac{1}{x} L_\alpha(\xi x) \, dx. \]

Then by Theorem 2.6, p. 63 in [6] we get
\[ Q_1(\xi) = L_\alpha(\xi) \left( \int_{1/\pi}^{\infty} x^{-\alpha-1} \cos \frac{1}{x} \, dx + \sigma(1) \right) \]
\[ = L(\xi) \left( \int_{1/\pi}^{\infty} x^{-\alpha-1} \cos \frac{1}{x} \, dx + \sigma(1) \right). \tag{7} \]

By partial integration, we get
\[ Q_2(\xi) = \int_{0}^{1/\pi} \sin \frac{1}{x} \cdot x^{-\alpha} L_\alpha(\xi x) \left[ 1 - \alpha \frac{x L'_\alpha(\xi x)}{L_\alpha(\xi x)} \right] \, dx, \]
i.e.,
\[ \frac{Q_2(\xi)}{L_\alpha(\xi)} = (1 - \alpha) \int_{0}^{1/\pi} x^{-\alpha} \sin \frac{1}{x} \frac{L_\alpha(\xi x)}{L_\alpha(\xi)} \, dx + \int_{0}^{1/\pi} x^{-\alpha} \sin \frac{1}{x} \frac{L_\alpha(\xi x)}{L_\alpha(\xi)} \frac{\xi x L'_\alpha(\xi x)}{L_\alpha(\xi x)} \, dx. \]
Since \( L_\alpha \) is a nondecreasing function and \( \lim_{t \to +\infty} t L'_\alpha(t)/L_\alpha(t) = 0 \), by the Lebesgue Dominated Convergence Theorem
\[ \frac{Q_2(\xi)}{L_\alpha(\xi)} \to (1 - \alpha) \int_{0}^{1/\pi} x^{-\alpha} \sin \frac{1}{x} \, dx = \int_{0}^{1/\pi} x^{-\alpha-1} \cos \frac{1}{x} \, dx. \]

Therefore
\[ Q_2(\xi) = L_\alpha(\xi) \left( \int_{0}^{1/\pi} x^{-\alpha-1} \cos \frac{1}{x} \, dx + \sigma(1) \right) \]
\[ = L(\xi) \left( \int_{0}^{1/\pi} x^{-\alpha-1} \cos \frac{1}{x} \, dx + \sigma(1) \right). \tag{8} \]

Since
\[ \int_{0}^{\infty} x^{-\alpha-1} \cos \frac{1}{x} \, dx = \Gamma(\alpha) \cos \frac{\alpha \pi}{2} \]
and \( Q = Q_1 + Q_1 \), from (7) and (8) we obtain (6) and
\[ K_\alpha(\xi) = 2\Gamma(\alpha) \cos \frac{\alpha \pi}{2} L(\xi) \frac{\xi}{\xi^\alpha} (1 + \sigma(1)), \quad \xi \to +\infty, \]
which proves 1°.
Now we prove property 2° of $K_a$. Since

$$K'_a(\xi) = -2\alpha\xi^{-\alpha-1}\int_0^\infty x^{-\alpha-1} \cos \frac{1}{x} L_a(\xi x) \, dx$$

$$+ 2\xi^{-\alpha}\int_0^{+\infty} x^{-\alpha} \cos \frac{1}{x} L'_a(\xi x) \, dx$$

$$= 2\frac{L_a(\xi)}{\xi^{\alpha+1}} \left(-2\alpha\Gamma(\alpha) \cos \frac{\alpha\pi}{2} + \sigma(1) + \int_0^\infty x^{-\alpha-1} \cos \frac{1}{x} L'_a(\xi x) \, dx \right),$$

it suffices to prove that

$$\left| \int_0^\infty x^{-\alpha-1} \cos \frac{1}{x} \frac{\xi L'_a(\xi x)}{L_a(\xi)} \, dx \right| < 2\alpha\Gamma(\alpha) \cos \frac{\alpha\pi}{2}$$

if $a$ and $\xi$ are large enough. Since

$$\int_1^\infty x^{-\alpha-1} \cos \frac{1}{x} \frac{\xi L'_a(\xi x)}{L_a(\xi)} \, dx \to 0 \quad (\xi \to +\infty)$$

and

$$\int_{a/\xi}^{a/\xi} x^{-\alpha-1} \cos \frac{1}{x} \frac{\xi L'_a(\xi x)}{L_a(\xi)} \, dx \to 0 \quad (\xi \to +\infty),$$

inequality (10) will follow from

$$\left| \int_0^1 x^{-\alpha-1} \cos \frac{1}{x} \frac{L_a(\xi x)}{L_a(\xi)} \cdot \frac{\xi L'_a(\xi x)}{L_a(\xi)} \, dx \right| \quad (a \text{ large enough and } \xi > a).$$

Applying twice the Bonnet Mean Value Theorem (the functions $x \mapsto L_a(x)$ and $x \mapsto x \cdot L'_a(x)/L_a(x)$ are nondecreasing and decreasing on $[0, \infty)$ and $[x_0, \infty)$ respectively) we get

$$\left| \int_0^1 x^{-\alpha-1} \cos \frac{1}{x} \frac{L_a(\xi x)}{L_a(\xi)} \cdot \frac{\xi L'_a(\xi x)}{L_a(\xi)} \, dx \right| \leq \frac{aL'(a)}{L(a)} \left| \int_{c_1}^{c_2} x^{-\alpha-1} \cos \frac{1}{x} \, dx \right|$$

where $a/\xi < c_1 < c_2 < 1$.

Since the integral $\int_0^\infty x^{-\alpha-1} \cos \frac{1}{x} \, dx$ is convergent and

$$\lim_{x \to +\infty} xL'(x) = 0,$$

(11) holds if $a$ is fixed and large enough and $\xi > a$ is large enough. From 1°, 2°, and (5) it follows that

$$s_n \left( \int_{-1}^1 H_a(x, y) \cdot dy \right) \sim 2\Gamma(\alpha) \cos \frac{\alpha\pi}{2} \frac{L(n)}{(n\pi/2)^\alpha}. $$

From 3° we get

$$s_n \left( \int_{-1}^1 k_a(|x + y + 2|) \cdot dy \right) = \sigma \left( \frac{L(n)}{n^2} \right),$$

$$s_n \left( \int_{-1}^1 k_a(|x + y - 2|) \cdot dy \right) = \sigma \left( \frac{L(n)}{n^2} \right).$$
The function

\[ R(x, y) = ka(|x - y - 4|) + \sum_{n \neq 0; n \neq -1} (ka(|x - y + 4n|) - ka(|x + y + 4n + 2|)) \]

is bounded on \([-1, 1]^2\) (a consequence of \(4^0\)), hence \(\int_{-1}^{1} R(x, y) \cdot dy\) is a Hilbert Schmidt operator and

\[
s_n \left( \int_{-1}^{1} R(x, y) \cdot dy \right) = \sigma(n^{-1/2}) = \sigma \left( \frac{L(n)}{n^\alpha} \right).
\]

From (12), (13), (14), and the Ky Fan Theorem it follows that

\[
s_n(B_a') \sim 2\Gamma(\alpha) \cos \frac{\alpha \pi}{2} \frac{L(n)}{(n\pi/2)^\alpha}.
\]

From (15) and from

\[
\int_R ka \left( \frac{|t|}{2} \right) e^{itx} dt = 2\Gamma(\alpha) \cos \frac{\alpha \pi}{2} \frac{L(2\xi)}{(2\xi)^\alpha}(1 + \sigma(1))
\]

by substitution in the eigenvalue relation \(B_a e_n = \lambda_n e_n\), we obtain

\[
s_n(B_a) \sim \Gamma(\alpha) \cos \frac{\alpha \pi}{2} \frac{L(n)}{(n\pi)^\alpha}.
\]

Lemma 1 is proved.

From now on suppose \(a\) is a fixed and large enough number such that (1) holds.

**Proof of Theorem 1 in Case A.** Since

\[ Af(x) = \int_0^x k(x - y) f(y) dy, \]

we have

\[(A + A^*) f = B f = \int_0^1 k(|x - y|) f(y) dy.\]

By Lemma 1 we get

\[
\lim_{n \to \infty} \frac{s_n(B)}{s_n(B_a)} = 1
\]

and so

\[ s_{2n}(B) \geq c_1 s_{2n}(B_a) \]

\((c_1'\) does not depend on \(n\)). The last inequality and (16) imply

\[ s_{2n}(B) \geq c_1 \frac{L(n)}{n^\alpha} \]

\((c_1'\) does not depend on \(n\)).

Since \(s_{2n}(B) \leq s_n(A) + s_n(A^*) = 2s_n(A)\), we obtain

\[
s_n(A) \geq \frac{c_1}{2} \frac{L(n)}{n^\alpha}.
\]

Now we prove the following inequality

\[
s_n(A) \leq \text{const} \frac{L(n)}{n^\alpha} \]

\((\text{const}\) does not depend on \(n\)).

Here we use the following lemma proved in [4].
Lemma 2. Let $K_n(x, y)$ be a sequence of functions integrable to $x$ and to $y$ individually, $0 \leq x, y \leq 1$. Let $K(x, y)$ be a similar function, and suppose that for almost all $y$

$$\int_0^1 |K(x, y) - K_n(x, y)| \, dx \leq \beta_n \quad (\beta_n \to 0)$$

and also that for almost all $x$

$$\int_0^1 |K(x, y) - K_n(x, y)| \, dy \leq \gamma_n \quad (\gamma_n \to 0).$$

Finally, suppose that for each $n$

$$\mathcal{H}_n = \int_0^1 K_n(x, y) \cdot dy$$

is a compact operator on $L^2(0, 1)$. Then $\mathcal{H} = \int_0^1 K(x, y) \cdot dy$ is also a compact operator on $L^2(0, 1)$ and

$$s_n(\mathcal{H}) \leq s_n(\mathcal{H}_n) + \sqrt{\beta_n \gamma_n}.$$

Now let us put

$$K_n(x, y) = \begin{cases} (x - y + \frac{1}{n})^{\alpha-1} L \left( \frac{1}{x-y+1/n} \right), & y < x, \\ 0, & y \geq x, \end{cases}$$

and

$$K(x, y) = \begin{cases} (x - y)^{\alpha-1} L \left( \frac{1}{x-y} \right), & y < x, \\ 0, & y \geq x. \end{cases}$$

The function $t \mapsto t^{\alpha-1} L(1/t)$ is decreasing (for $0 < \alpha < 1$) and hence

$$\int_0^1 |K(x - y) - K_n(x, y)| \, dy = \int_0^{1/n} t^{\alpha-1} L \left( \frac{1}{t} \right) \, dt - \int_x^{x+1/n} t^{\alpha-1} L \left( \frac{1}{t} \right) \, dt < \int_0^{1/n} t^{\alpha-1} L \left( \frac{1}{t} \right) \, dt.$$

Since

$$\int_0^{1/n} L^{\alpha-1} \left( \frac{1}{t} \right) \, dt = \int_n^{+\infty} t^{-\alpha-1} L(t) \, dt$$

and

$$\int_x^{+\infty} t^{-\alpha-1} L(t) \, dt \sim \frac{1}{\alpha} \frac{L(x)}{x^{\alpha}} \quad (x \to +\infty),$$

we get

$$\int_0^1 |K(x, y) - K_n(x, y)| \, dy \leq c_3 \frac{L(n)}{n^\alpha} \quad (c_3 \text{ does not depend on } n).$$

Similarly,

$$\int_0^1 |K_n(x, y) - K_n(x, y)| \, dx \leq c_4 \frac{L(n)}{n^\alpha} \quad (c_4 \text{ does not depend on } n).$$
From (19), (20), and Lemma 2 we obtain

\[ s_n(A) \leq \sqrt{c_3 c_4 \frac{L(n)}{n^\alpha}} + s_n(\mathcal{H}_n). \]  

Now, we can estimate the norm \( \| \mathcal{H}_n \|_2 \) (Hilbert Schmidt norm). We have

\[ \| \mathcal{H}_n \|_2^2 = \int_0^1 \int_0^1 |K_n(x, y)|^2 \, dx \, dy \]
\[ = \int_{1/n}^{1+1/n} y^{2\alpha-2} \left( L \left( \frac{1}{y} \right) \right)^2 \cdot \left( 1 - y + \frac{1}{n} \right) \, dy \]
\[ \leq \int_{1/n}^{1+1/n} y^{2\alpha-2} \left( L \left( \frac{1}{y} \right) \right)^2 \, dy. \]

From this inequality by simple computation we get

\[ \| \mathcal{H}_n \|_2^2 \leq c_5 n^{1-2\alpha} (L(n))^2 \] (\( c_5 \) does not depend on \( n \)).

Since \( n s_n^2 (\mathcal{H}_n) \leq \| \mathcal{H}_n \|_2^2 \), we obtain

\[ s_n(\mathcal{H}_n) \leq c_6 \frac{L(n)}{n^\alpha} \] (\( c_6 \) does not depend on \( n \)).

Now (18) follows from (21) and (22). The theorem is proved for the case when the function \( L \) is not bounded.

Case B: The function \( L \) is bounded. Since \( L \) is nondecreasing we have \( \lim_{x \to +\infty} L(x) = d < \infty \). By assumption of Theorem 1 we get \( d > 0 \).

Lemma 3. Suppose \( r \in C[0, 1] \), \( r(0) = 0 \), and \( G \) is a linear operator on \( L^2(0, 1) \) defined by

\[ Gf(x) = \int_0^x (x - y)^{\alpha-1} r(x - y) f(y) \, dy. \]

If \( 0 < \alpha < 1/2 \), then

\[ \lim_{n \to \infty} n^\alpha s_n(G) = 0. \] (23)

Proof of Lemma 3. Let us represent \( G \) as

\[ Gf(x) = \int_0^1 |x - y|^{\alpha-1} M(x, y) f(y) \, dy \]

where

\[ M(x, y) = \begin{cases} r(x - y), & 0 < y \leq x < 1, \\ 0, & 1 \geq y \geq x \geq 0. \end{cases} \]

Let \( \epsilon > 0 \). Then there exists \( \delta > 0 \) such that \( |M(x, y)| < \epsilon \) if \( |x - y| < \delta \).

Put

\[ \Omega_1 = [0, 1]^2 \setminus \{(x, y): |x - y| < \delta \}, \quad \Omega_2 = [0, 1]^2 \setminus \Omega_1. \]

Suppose \( G_1, G_2 \) are linear operators on \( L^2(0, 1) \) defined by

\[ G_if(x) = \int_0^1 |x - y|^{\alpha-1} \chi_{\Omega_i}(x, y) M(x, y) f(y) \, dy, \quad i = 1, 2 \]

(\( \chi_{\Omega_i} \) are characteristic functions of \( \Omega_i \), \( i = 1, 2 \)).
Then \( G = G_1 + G_2 \) and
\[
(24) \quad s_{2n}(G) \leq s_n(G_1) + s_n(G_2).
\]
By Lemma 1 from [1] we obtain
\[
s_n(G_1) \leq \text{const} \cdot \varepsilon \left[ \int_0^{1/n} t^{a-1} dt + n^{-1/2} \left( \int_{1/n}^\infty t^{2a-2} dt \right)^{1/2} \right],
\]
i.e. (since \( 0 < \alpha < 1/2 \)),
\[
(25) \quad s_n(G_1) \leq \text{const} \cdot \varepsilon \cdot \frac{1}{n^\alpha} \quad \text{(const does not depend on } n\text{)}.
\]
On the other hand, \( G_2 \) is a Hilbert Schmidt operator and hence
\[
s_n(G_2) \leq c_7(\delta) \cdot n^{-1/2}.
\]
From the previous inequality we get (for \( 0 < \alpha < 1/2 \))
\[
(26) \quad n^\alpha s_n(G_2) < \varepsilon
\]
if \( n \) is large enough.

From (24), (25), and (26) we obtain
\[
\lim_{n \to \infty} n^\alpha s_{2n}(G) = 0
\]
and
\[
\lim_{n \to \infty} n^\alpha s_n(G) = 0.
\]

**Proof of Theorem 1 in Case B.** Put \( r(x) = L(1/x) - d \). Applying Lemma 3 we get
\[
(27) \quad \lim_{n \to \infty} n^\alpha s_n \left( \int_0^x (x - y)^{a-1} \left( L \left( \frac{1}{x - y} \right) - d \right) \cdot dy \right) = 0.
\]
In [2] it is proved that
\[
s_n \left( \int_0^x (x - y)^{a-1} \cdot dy \right) \sim \Gamma(\alpha)(n\pi)^{-\alpha}.
\]
From (27), the previous asymptotic formula, and the Ky Fan Theorem we conclude
\[
s_n \left( \int_0^x (x - y)^{a-1} \left( L \left( \frac{1}{x - y} \right) \cdot dy \right) \sim d \cdot \Gamma(\alpha)(n\pi)^{-\alpha}.
\]

Theorem 1 is proved.

**Remark.** From the proof it is evident that if \( L \) is bounded, then it is enough to suppose that \( L \) is continuous and \( \lim_{x \to \infty} L(x) = d \neq 0 \).

**Theorem 2.** Suppose function \( L \) satisfies conditions from the beginning of this paper. Let \( r \in C^1[0, 1] \), \( r(0) = 0 \), \( k_1(x) = k(x)(1+r(x)) \) \( (k(x) = x^{a-1}L(1/x)) \), and let \( A_1: L^2(0, 1) \to L^2(0, 1) \) be a linear operator defined by
\[
A_1 f(x) = \int_0^x k_1(x - y) f(y) \, dy.
\]
If \( 0 < \alpha < 1/2 \), then \( s_n(A_1) \sim L(n)/n^\alpha \).
Lemma 4. Suppose $A$ and $B$ are composed operators on Hilbert space $\mathcal{H}$ such that $s_n(A) \asymp L(n)/n^\beta$ ($L$ is a slowly varying function, $\beta > 0$) and $\lim_{n \to \infty} \frac{n^\beta}{L(n)} s_n(B) = 0$. Then $s_n(A + B) \asymp L(n)/n^\beta$.

Proof of Lemma 4. From conditions $s_n(A) \asymp L(n)/n^\beta$ it follows that there exists constants $d_1 > 0$ and $d_2 > 0$ such that

$$d_2 \frac{L(n)}{n^\beta} \leq s_n(A) \leq d_1 \frac{L(n)}{n^\beta}. \quad (28)$$

For arbitrary $k \in \mathbb{N}$, $n = (k + 1)m + j$, $j = 0, 1, 2, \ldots, k$, by properties of singular values [5], we have

$$s_{(k+1)m+j}(A + B) \leq s_{km+j}(A) + s_{m+1}(B),$$

i.e.,

$$\frac{s_{(k+1)m+j}(A + B)}{s_{(k+1)m+j}(A)} \leq \left(1 + \frac{s_{m+1}(B)}{s_{km+j}(A)}\right) \cdot \frac{s_{km+j}(A)}{s_{(k+1)m+j}(A)}. \quad (29)$$

From (28) we get

$$\frac{s_{(k+1)m+j}(A + B)}{s_{(k+1)m+j}(A)} \leq \left(1 + \frac{s_{m+1}(B)}{s_{km+j}(A)}\right) \cdot \frac{d_1}{d_2} \left(\frac{(k + 1)m + j}{km + j}\right)^\beta \frac{L(km + j)}{L((k + 1)m + j)}. \quad (30)$$

Since $\frac{n^\beta}{L(n)} s_n(A) \to 0$ (or equivalently $s_n(B)/s_n(A) \to 0$) we obtain

$$\lim_{n \to \infty} \frac{s_n(A + B)}{s_n(A)} \leq \frac{d_1}{d_2} \left(\frac{k + 1}{k}\right)^\beta. \quad (31)$$

As $k$ is arbitrary, we get

$$\lim_{n \to \infty} \frac{s_n(A + B)}{s_n(A)} \leq \frac{d_1}{d_2}. \quad (32)$$

Similarly, we get

$$\lim_{n \to \infty} \frac{s_n(A + B)}{s_n(A)} \geq \frac{d_2}{d_1}. \quad (33)$$

Lemma 4 is proved.

Proof of Theorem 2. Since $r \in C^1[0, 1]$ and $r(0) = 0$, $\int_0^x k(x - y)r(x - y) \cdot dy$ is a Hilbert Schmidt operator and therefore

$$s_n \left(\int_0^x k(x - y)r(x - y) \cdot dy\right) = \sigma(n^{-1/2}) = \sigma \left(\frac{L(n)}{n^\alpha}\right) \quad (0 < \alpha < \frac{1}{2}). \quad (34)$$

From Theorem 1 we have

$$s_n \left(\int_0^x k(x - y) \cdot dy\right) \leq \frac{L(n)}{n^\alpha}. \quad (35)$$

The statement of Theorem 2 follows from (34), (35), and Lemma 4.

Example. Let $L(x) = (\ln x)^\beta$, $\beta \geq 0$, and let the function $r$ satisfy $r \in C^1[0, 1]$, $r(0) \neq 0$. We consider the operator $T: L^2(0, 1) \to L^2(0, 1)$ defined by

$$Tf(x) = \int_0^x (x - y)^{\alpha - 1}(-\ln(x - y))^\beta r(x - y)f(y)dy \quad (0 < \alpha < 1/2).$$

Then $s_n(T) \asymp (\ln n)^\beta/n^\alpha$. 

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