A NOTE ON MORITA EQUIVALENCE
OF TWISTED C*-DYNAMICAL SYSTEMS

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Abstract. We present an elementary proof that every twisted C*-dynamical
system is Morita equivalent to an ordinary system. As a corollary we prove
the equivalence $C_0(G/H, A) \times_{\alpha, u} G \sim A \times_{\alpha, u} H$ for Busby-Smith twisted
dynamical systems, generalizing an important result of Green.

It is essentially the content of a recent theorem of Echterhoff [5, Theorem 1] that every twisted dynamical system is Morita equivalent to an ordinary system. By avoiding Green's imprimitivity theorem [6, Corollary 5] and appealing directly to the stabilization trick [8, Theorem 3.4] of Packer and Raeburn, we provide an elementary proof of this fact, at the same time generalizing it (in the separable case) to Busby-Smith twisted systems. Thus our main theorem provides a way of lifting much of the theory for ordinary and Green-twisted systems to the more general systems. As an example of its utility, we prove an analog of Green's important equivalence $C^*(G, C_0(G/H, A); \tau) \sim C^*(H, A, \tau)$ [6] for Busby-Smith twisted systems. This, in turn, will form the basis for a process of inducing representations, an imprimitivity theorem, and ultimately a version of the Mackey-Green machine for these twisted systems.

1. Preliminaries

Throughout this note $G$ will be a second-countable locally compact group; $A$ and $B$ will always be separable C*-algebras. The multiplier algebra of $A$ is denoted by $\mathcal{M}(A)$ and its unitary group by $\mathcal{U}(A)$. If $C^*$-algebras $A$ and $B$ are (strongly) Morita equivalent via an equivalence bimodule $X$ (see [10, 11]), we will write $A \sim X B$ or simply $A \sim B$.

A twisted action of a group $G$ on a C*-algebra $A$ is a pair $(\alpha, u)$ consisting of a strongly Borel map $\alpha: G \to \text{Aut}(A)$ and a strictly Borel map $u: G \times G \to \mathcal{U}(A)$ such that $\alpha_s \circ \alpha_t = \text{Ad} u(s, t) \circ \alpha_{st}$ and $\alpha_r(u(s, t))u(r, st) = u(r, s)u(rs, t)$ for all $r, s, t$ in $G$. We call the quadruple $(A, G, \alpha, u)$ a (Busby-Smith) twisted dynamical system. (See [2; 8, Definition 2.1].) If the cocycle $u$ is trivial (i.e., identically 1), then we say $(A, G, \alpha, u)$ is an ordinary
dynamical system and write \((A, G, \alpha, u)\) for short. Note that then \(\alpha\) is a Borel
homomorphism into the Polish group \(\text{Aut}(A)\), so is actually continuous [7, Proposition 5]. Thus the ordinary systems are the objects studied in [3] and
[4], among others. For every twisted dynamical system there is a (unique)
crossed product \(C^*\)-algebra \(A \times_{\alpha, u} G\) which is a universal object for covariant
representations of \((A, G, \alpha, u)\) in multiplier algebras [9].

Two twisted actions \((\beta, v)\) and \((\alpha, u)\) of \(G\) on \(A\) are exterior equivalent
if there is a strictly Borel map \(w : G \to \mathcal{U}(A)\) such that \(\alpha_s = \text{Ad} w_s \circ \beta_s\) and
\(u(s, t) = w_s \beta_s(w_t)v(s, t)w_t^*\) for any \(s, t\) in \(G\) [8, Definition 3.1].

More generally, suppose \((B, G, \beta, v)\) and \((A, G, \alpha, u)\) are twisted sys-
tems and \(X\) is a \(B - A\) equivalence bimodule. Let \(\text{Aut}(X)\) denote the set
of bicontinuous linear bijections \(\phi\) of \(X\) which satisfy the ternary homomor-
phism identity \(\phi(x \cdot (y, z)_A) = (\phi(x) \cdot (\phi(y), \phi(z)))_A\). (The analogous identity
using \(B\)-valued inner products is equivalent.) Then following [1, Definition
2.1], we will say \((B, G, \beta, v)\) and \((A, G, \alpha, u)\) are Morita equivalent if there
is a strongly Borel map \(\gamma : G \to \text{Aut}(X)\) such that for \(s, t\) in \(G\) and \(x, y\) in
\(X\):

\[
\begin{align*}
(1) \quad & \alpha_s((x, y)_A) = (\gamma_s(x), \gamma_s(y))_A. \\
(2) \quad & \beta_s(b(x, y)) = b(\gamma_s(x), (x, y)). \\
(3) \quad & \gamma_s \circ \gamma_t(x) = v(s, t) \cdot \gamma_{st}(x) \cdot u(s, t)^*.
\end{align*}
\]

We write \((B, G, \beta, v) \sim_{X, \gamma} (A, G, \alpha, u)\) and call \((X, \gamma)\) a system of imprim-
itivity implementing the equivalence.

2. The main theorem

Theorem 2.1. Let \((A, G, \alpha, u)\) be a twisted dynamical system, and let \(\mathcal{H}\) de-
note the compact operators on \(\mathcal{H} = L^2(G)\). Then there is an ordinary action \(\beta\)
of \(G\) on \(A \otimes \mathcal{H}\) and a map \(\delta : G \to \text{Aut}(A \otimes \mathcal{H})\) such that

\[(A \otimes \mathcal{H}, G, \beta) \sim_{A \otimes \mathcal{H}, \delta} (A, G, \alpha, u).\]

Proof. We appeal to the Packer-Raeburn stabilization trick [8, Theorem 3.4].
Thus we have a Borel map \(w : G \to \mathcal{U}(A \otimes \mathcal{H})\) which implements an exterior
equivalence between an ordinary action \((\beta, 1)\) of \(G\) on \(A \otimes \mathcal{H}\) and \((\alpha \otimes id_{\mathcal{H}}, u \otimes 1)\).

Let \(A \otimes \mathcal{H}\) have the canonical \(A \otimes \mathcal{H} - A\) equivalence bimodule structure; so
\(A \otimes \mathcal{H}\) is the completion of the algebraic tensor product \(A \otimes \mathcal{H}\) with respect to
the norm induced by the \(A\)-valued inner product \(\langle a \otimes \xi, b \otimes \eta \rangle_A = \langle \eta, \xi \rangle_{\mathcal{H}} a^* b\).
For \(s\) in \(G\), the rule \(a \otimes \xi \mapsto \alpha_s(a) \otimes \xi\) defines an automorphism of \(A \otimes \mathcal{H}\)
which satisfies condition (1) for this product, so is isometric with respect to
the induced norm, and thus extends to a map \(\alpha_s \otimes \text{id}_{\mathcal{H}}\) of \(A \otimes \mathcal{H}\) into itself.
Then for \(x\) in \(A \otimes \mathcal{H}\), the map \(s \mapsto \alpha_s \otimes \text{id}_{\mathcal{H}}(x)\) is Borel, using the fact that
\(s \mapsto \alpha_s(a)\) is Borel for \(a\) in \(A\), together with a routine density argument.

Now define \(\delta_s : A \otimes \mathcal{H} \to A \otimes \mathcal{H}\) by

\[
\delta_s(x) = w_s^* \cdot \alpha_s \otimes \text{id}_{\mathcal{H}}(x).
\]

Then straightforward calculations on elementary tensors in \(A \otimes \mathcal{H}\) verify that each \(\delta_s\)
satisfies the ternary homomorphism identity, and that the map \(s \mapsto \delta_s\)
satisfies conditions (1)–(3) above. For example, for any \(s, t\) in \(G\) and \(a \otimes \xi\)
in $A \otimes \mathcal{H}$ we have
\[
\delta_{s} \circ \delta_{t}(a \otimes \xi) = w_{s}^{*} \cdot \alpha_{s} \otimes \text{id}_{\mathcal{H}}(w_{t}^{*} \cdot \alpha_{t} \otimes \text{id}_{\mathcal{H}}(a \otimes \xi)) \\
= w_{s}^{*} \alpha_{s} \otimes \text{id}_{\mathcal{H}}(w_{t})^{*} \cdot \alpha_{s} \circ \alpha_{t}(a) \otimes \xi \\
= (w_{s}^{*} \alpha_{s} \otimes \text{id}_{\mathcal{H}}(w_{t})^{*} u(s, t) \otimes 1) \cdot \alpha_{st} \otimes \text{id}_{\mathcal{H}}(a \otimes \xi) \cdot u(s, t)^{*} \\
= w_{st}^{*} \cdot \alpha_{st} \otimes \text{id}_{\mathcal{H}}(a \otimes \xi) \cdot u(s, t)^{*} \\
= \delta_{st}(a \otimes \xi) \cdot u(s, t)^{*}.
\]
In particular, condition (1) (or (2)) implies each $\delta_{s}$ is isometric, and therefore bicontinuous since each $\delta_{s}$ is invertible. Thus each $\delta_{s}$ belongs to $\text{Aut}(A \otimes \mathcal{H})$.

It only remains to show that the map $s \mapsto \delta_{s}$ is strongly Borel. To see this, fix $x$ in $A \otimes \mathcal{H}$ and let $\{e_{i}\}$ be a countable approximate identity for $A \otimes \mathcal{H}$. Then each of the maps $s \mapsto w_{s}^{*}e_{i}$ and $s \mapsto \alpha_{s} \otimes \text{id}_{\mathcal{H}}(x)$ are Borel, so that $s \mapsto w_{s}^{*}e_{i} \cdot \alpha_{s} \otimes \text{id}_{\mathcal{H}}(x)$ is Borel for each $i$. Since $s \mapsto \delta_{s}(x)$ is the pointwise limit of these Borel maps, it too is Borel, and the theorem follows. □

We remark that the analogous theorem for separable Green-twisted systems can be derived from Theorem 2.1, proving in essence Echterhoff's [5, Theorem 1]. This just requires verifying that the process of converting Green-twisted systems into Busby-Smith systems described in [8, §5] preserves the respective notions of Morita equivalence. While not complicated, the proof is lengthy, so we will not include it here.

Now let $H$ be a closed subgroup of $G$, and denote an element $tH$ of the quotient space $G/H$ by $t$. We define the diagonal twisted action $(\tilde{\alpha}, \tilde{\psi})$ of $G$ on $C_{0}(G/H, A)$ as follows:

\[
\tilde{\alpha}_{s}(f)(i) = \alpha_{s}(f(s^{-1}i)) \quad \text{and} \quad [\tilde{\psi}(s, t)f](r) = u(s, t)f(r).
\]

Then we have the promised analog of Green's result [6, Corollary 5].

**Corollary 2.2.** Let $(A, G, \alpha, u)$ be a twisted dynamical system, and let $H$ and $(\tilde{\alpha}, \tilde{\psi})$ be as above. Then

\[
C_{0}(G/H, A) \times \tilde{\alpha}, \tilde{\psi} \rightarrow G \times \alpha, u H.
\]

**Proof.** Let $(A \otimes \mathcal{H}, G, \beta)$ be the ordinary system which by Theorem 2.1 is Morita equivalent to $(A, G, \alpha, u)$. Then it is straightforward to check that $C_{0}(G/H, A \otimes \mathcal{H})$ is a $C_{0}(G/H, A \otimes \mathcal{H}) - C_{0}(G/H, A)$ equivalence bimodule when equipped with pointwise actions and inner products. Moreover, calculations similar to those in the proof of Theorem 2.1 verify that the diagonal action $\tilde{\delta}$ of $G$ on $C_{0}(G/H, A \otimes \mathcal{H})$ defined by $\tilde{\delta}_{s}(x)(i) = \delta_{s}(x(s^{-1}i))$ yields

\[
(C_{0}(G/H, A \otimes \mathcal{H}), G, \tilde{\beta}) \sim C_{0}(G/H, A \otimes \mathcal{H}), \tilde{\delta} \circ (C_{0}(G/H, A), G, \tilde{\alpha}, \tilde{\psi}).
\]

Because Morita equivalent twisted systems have Morita equivalent crossed products [1, Theorem 2.3], we have $C_{0}(G/H, A) \times \tilde{\alpha}, \tilde{\psi} G \sim C_{0}(G/H, A \otimes \mathcal{H}) \times \beta G$.

Next, notice that restricting the twisted actions $(\alpha, u)$ and $(\beta, 1)$ to $H$ yields Morita equivalent systems $(A, H, \alpha, u)$ and $(A \otimes \mathcal{H}, H, \beta)$. Again using [1, Theorem 2.3], we have $A \otimes \mathcal{H} \times \beta H \sim A \times \alpha, u H$. But Green's [6, Corollary 5] applied to $(A \otimes \mathcal{H}, G, \beta)$ gives us $C_{0}(G/H, A \otimes \mathcal{H}) \times \beta G \sim A \otimes \mathcal{H} \times \beta H$; the corollary now follows by the transitivity of Morita equivalence. □

The development of a theory of induced representations for Busby-Smith twisted systems—in particular, for the Mackey-Green machine—will require
not only the abstract Morita equivalence of Corollary 2.2 but a concrete equivalence bimodule which implements it. To be sure, the use of transitivity in the above proof implicitly gives such a bimodule; namely, the balanced tensor product of the three bimodules involved. This three-fold tensor product bimodule turns out to be extremely unpleasant to work with. We would prefer a bimodule completion of $B_c(G, A)$, the bounded Borel functions with compact support, which would be analogous to Green's $C_c(G, A)$. Such a bimodule does exist; however, technical difficulties arise in proving this which are beyond the scope of this note. The general process of inducing covariant representations of Busby-Smith twisted dynamical systems, as well as the particular problem of providing a workable equivalence bimodule for Corollary 2.2 are addressed in work currently in preparation.

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REFERENCES


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