OSCILLATORY SINGULAR INTEGRALS ON HARDY SPACES ASSOCIATED WITH HERZ SPACES

SHANZHEN LU AND DACHUN YANG

(Communicated by J. Marshall Ash)

Abstract. In this paper, it is proved that the oscillatory singular integral operators of nonconvolution type are bounded from Hardy spaces associated with Herz spaces to Herz spaces.

1. Introduction

Let $T$ be an oscillatory singular integral operator defined by

$$ Tf(x) = \text{p.v.} \int_{\mathbb{R}^n} e^{iP(x,y)} K(x-y)f(y) \, dy, $$

where $P(x,y)$ is a real-valued polynomial on $\mathbb{R}^n \times \mathbb{R}^n$ and $K$ is a Calderón-Zygmund kernel.

It is proved by D. H. Phong and E. M. Stein in [6] that $T$ is a bounded operator from $H^1_E$ to $L^1$ provided $P(x,y)$ is a real bilinear form, where $H^1_E$ is certain variant of the $H^1$ space. Later, this result is extended into the case of general $P(x,y)$ by Y. B. Pan in [5]. For general $P(x,y)$, it is still an interesting problem whether $T$ is a bounded operator from $H^1$ to $L^1$. Recently, some new Hardy spaces $HK_p$ associated with Herz spaces $K_p$ are introduced by the authors in [4] and [8]. The space $HK_p$ is defined by

$$ HK_p = \{ f : Gf \in K_p \}, $$

where $Gf$ is the Grand maximal function of $f$. An interesting fact shown in [8] is that $HK_p$ is the localization of $H^1$ at the origin. It is easy to see that the relation between $HK_p$ and $K_p$ is similar to one between $H^1$ and $L^1$.

In this paper, we shall prove that $T$ defined by (1.1) is a bounded operator from $HK_p$ to $K_p$. A counterexample shows that there exists an operator $T$ defined by (1.1), such that $T$ is not a bounded operator from $HK_p$ to itself. To formulate our result, let us first introduce some definitions.
Definition 1.1 (see [3]). Let $1 < p < \infty$ and $1/p + 1/p' = 1$. The Herz space $K_p(\mathbb{R}^n)$ consists of those functions $f \in L^p_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$ for which
\[
\|f\|_{K_p} := \sum_{k \in \mathbb{Z}} 2^{kn/p'} \|f\chi_k\|_p < \infty,
\]
where $\chi_k = \chi_{C_k}$, $C_k = Q_k \setminus Q_{k-1}$, and $Q_k = \{x : |x| < 2^k\}$.

Definition 1.2 (see [4]). Let $1 < p < \infty$. A function $a(x)$ on $\mathbb{R}^n$ is said to be a central $(1, p)$-atom if
\begin{enumerate}
\item $\text{Supp } a \subset Q$, where $Q$ is a ball centered at the origin;
\item $\|a\|_p \leq |Q|^{1/p-1}$;
\item $\int a(x) \, dx = 0$.
\end{enumerate}

Now, we can state our result as follows.

Theorem. Let $1 < p < \infty$, $P(x, y)$ be a real-valued polynomial on $\mathbb{R}^n \times \mathbb{R}^n$, $\nabla_y P(0, y) = O$, and $T$ be defined as in (1.1). Then $T$ maps $HK_p(\mathbb{R}^n)$ into $K_p(\mathbb{R}^n)$ and
\[
\|Tf\|_{K_p} \leq C\|f\|_{HK_p},
\]
where $C$ depends only on the total degree of $P(x, y)$ but not on the coefficients of $P(x, y)$.

2. PROOF OF THE THEOREM

To prove the Theorem, we need two lemmas.

Lemma 2.1. Let $f \in L^1(\mathbb{R}^n)$ and $1 < p < \infty$. Then $f \in HK_p(\mathbb{R}^n)$ if and only if $f$ can be represented as
\[
f(x) = \sum_i \lambda_i a_i(x),
\]
where each $a_i$ is a central $(1, p)$-atom and $\sum_i |\lambda_i| < \infty$. Moreover,
\[
\|f\|_{HK_p} := \|Gf\|_{K_p} \sim \inf \left\{ \sum_i |\lambda_i| \right\},
\]
where the infimum is taken over all decompositions of $f$ as above.


The following lemma belongs to Y. B. Pan [5].

Lemma 2.2. Let $\varphi \in C_0^\infty(\mathbb{R}^n)$ satisfy
\[
\varphi(x) = \begin{cases} 
1 & \text{for } |x| \leq 1, \\
0 & \text{for } |x| \geq 2,
\end{cases}
\]
and let $\psi \in C_0^\infty(\mathbb{R}^n)$ satisfy
\[
\psi(x) = \begin{cases} 
1 & \text{for } 1 \leq |x| \leq 2, \\
0 & \text{for } |x| \leq \frac{1}{4} \text{ or } |x| \geq 4.
\end{cases}
\]

Define $T_k$ by
\[
T_k f(x) = \psi(x/2^k) \int_{\mathbb{R}^n} e^{iP(x, y)} \varphi(y) f(y) \, dy.
\]
If \( P(x, y) \) satisfies
\[
P(x, y) = \sum_{|\alpha| \geq 1, |\beta| = l} a_{\alpha\beta} x^\alpha y^\beta + Q(x, y),
\]
where \( Q(x, y) \) is a polynomial with degree in \( y \) less than or equal to \( l - 1 \), then for each \( N > 0 \) (large enough) we have
\[
\|T_k\|_{L^2 \to L^2} \leq C 2^{nk} |a_{\alpha_0\beta_0}|^{-1/2Nl} 2^{-k|\alpha_0|/2Nl},
\]
where \( |a_{\alpha_0\beta_0}| = \max_{|\alpha| \geq 1, |\beta| = l} |a_{\alpha\beta}|. \)

**Proposition 2.1.** Let \( \delta > 0 \). Then we have
\[
\|f(\delta \cdot)\|_{K_p} \sim \delta^{-n} \|f\|_{K_p}.
\]
**Proof.** For any \( \delta > 0 \), there exists a \( k_0 \in \mathbb{Z} \) such that \( 2^{k_0} < \delta \leq 2^{k_0+1} \). By Definition 1.1,
\[
\|f(\delta \cdot)\|_{K_p} = \sum_{k \in \mathbb{Z}} 2^{kn/p'} \left( \int_{C_k} |f(\delta x)|^p \, dx \right)^{1/p} \leq \sum_{k \in \mathbb{Z}} 2^{kn/p'} \delta^{-np/p} \left( \int_{2^{k+k_0} \leq |y| \leq 2^{k+k_0+2}} |f(y)|^p \, dy \right)^{1/p} \leq \delta^{-np/p} 2^{-k_0n/p'} \sum_{k \in \mathbb{Z}} 2^{(k+k_0)n/p'} \left( \int_{C_{k+k_0}} |f(y)|^p \, dy \right)^{1/p} + \delta^{-np/p} 2^{-(k_0+1)n/p'} \sum_{k \in \mathbb{Z}} 2^{(k+k_0+1)n/p'} \left( \int_{C_{k+k_0+1}} |f(y)|^p \, dy \right)^{1/p} \leq C \delta^{-n} \|f\|_{K_p}.
\]
On the other hand,
\[
\|f\|_{K_p} = \|f(\delta^{-1} \cdot)\|_{K_p} \leq C \delta^{-n} \|f(\delta \cdot)\|_{K_p}.
\]
This finishes the proof of Proposition 2.1.

By Lemma 2.1, it is easy to see that the proof of the Theorem is reduced to the following proposition.

**Proposition 2.2.** Let \( 1 < p < \infty \), \( P(x, y) \) be a real-valued polynomial, \( \nabla_y P(0, y) = 0 \), and \( T \) be defined as in (1, 1). Then for any central \((1, p)\)-atom \( a \),
\[
\|Ta\|_{K_p} \leq C,
\]
where \( C \) is independent of \( a \) and the coefficients of \( P(x, y) \).

**Proof.** Let \( \text{Supp} a \subset Q \) and \( Q \) be a ball centered at the origin with radius \( \delta \). If we write \( b(x) = \delta^n a(\delta x) \), then \( b(x) \) is a central \((1, p)\)-atom supporting on unit ball \( B(0, 1) \). We also have
\[
Ta(\delta x) = \delta^{-n} T_1 b(x) = \delta^{-n} \int_{\mathbb{R}^n} e^{iP(\delta x, \delta y)} K_1(x - y) b(y) \, dy,
\]
where \( K_1 \) is the kernel of the \((1, p)\)-atom.
where $K_1(x) = \delta^n K(\delta x)$. By Proposition 2.1, we obtain
\[
\|Ta\|_{K_p} \sim \|T_1 b\|_{K_p}.
\]
Let $P_1(x, y) = P(\delta x, \delta y)$. Note that $\nabla_y P_1(0, y) = 0$ and $K_1$ is also a Calderón-Zygmund kernel. We may assume $T_1 = T$. Thus, it suffices to show
\[
(2.1) \quad \|Tb\|_{K_p} \leq C,
\]
where $C$ is independent of $b$ and the coefficients of $P(x, y)$ and $b$ is a central $(1, p)$-atom supporting on unit ball $B(0, 1)$.

We now turn to prove (2.1) by using induction on the degree of $y$, $l$, in $P(x, y)$. If $l = 0$, then
\[
|Tb(x)| = \left| \text{p.v.} \int K(x, y)b(y)\,dy \right|.
\]
Thus,
\[
\|Tb\|_{K_p} = \sum_{k \in \mathbb{Z}} 2^{kn/p'} \|(Tb)\chi_k\|_p
\]
\[
= \sum_{k \leq 0} \cdots + \sum_{k > 0} \cdots := S_1 + S_2.
\]

By $L^p$-boundedness of Calderón-Zygmund operators,
\[
S_1 \leq C \sum_{k \leq 0} 2^{kn/p'} \|b\|_p = C \sum_{k \leq 0} 2^{kn/p'} = C.
\]

From the condition of $K(x, y)$,
\[
|K(x, y) - K(x, 0)| \leq C|y|/|x - y|^{n+1}, \quad \text{if } |y| < |x - y|/2,
\]
it follows that
\[
Tb(x) = \int b(y)[K(x, y) - K(x, 0)]\,dy
\]
and
\[
S_2 = \sum_{k > 0} 2^{kn/p'} \|(Tb)\chi_k\|_p
\]
\[
\leq C \sum_{k > 0} 2^{kn/p'} \left[ \int_{C_k} \left( \int_{B(0, 1)} \frac{|b(y)||y|}{|x - y|^{n+1}} \,dy \right)^p \,dx \right]^{1/p}
\]
\[
\leq C \sum_{k > 0} 2^{kn/p'} \left( \int_{C_k} \frac{dx}{|x|^{(n+1)p}} \right)^{1/p} \|b\|_p
\]
\[
\leq C \sum_{k > 0} 2^{kn/p'} 2^{-k[(n+1)p-n]/p} = C \sum_{k > 0} 2^{-k} = C.
\]

Therefore, (2.1) holds for $l = 0$. Let us now consider the case of $l > 0$. We assume that (2.1) holds for $l - 1$ by induction. Since $\nabla_y P(0, y) = 0$, we can write
\[
P(x, y) = \sum_{|\alpha| \geq 1, \ |\beta| = l} a_{\alpha \beta} x^\alpha y^\beta + Q(x, y),
\]
where $Q(x, y)$ is a polynomial with degree in $y$ less than or equal to $l - 1$ and $\nabla_y Q(0, y) = 0$. Denote
\[
|a_{\alpha \beta} | = \max_{|\alpha| \geq 1, \ |\beta| = l} |a_{\alpha \beta}|
\]
and

\[ r = \max\{3, |a_{n_0}|^{-1/|\alpha_0|}\}. \]

Since \( r \geq 3 \), we may assume \( 2^{j_0} < r \leq 2^{j_0+1} \) for some \( j_0 \in \mathbb{N} \). We now write

\[
\|TB\|_{\mathcal{K}_p} = \sum_{j \leq 0} 2^{jn/p'} \|(TB)X_j\|_p + \sum_{j = 1}^{j_0} 2^{jn/p'} \|(TB)X_j\|_p \\
+ \sum_{j \geq j_0+1} 2^{jn/p'} \|(TB)X_j\|_p
: = I_1 + I_2 + I_3.
\]

By \( L^p \)-boundedness of oscillatory singular integral operators (see [7]), we have

\[
I_1 \leq C \sum_{j \leq 0} 2^{jn/p'} \|b\|_p = C \sum_{j \leq 0} 2^{jn/p'} = C.
\]

To estimate \( I_2 \), we may assume \( j_0 \geq 2 \). In this case, \( r = |a_{n_0}|^{-1/|\alpha_0|} \). By induction assumption,

\[
I_2 = \sum_{j = 1}^{j_0} 2^{jn/p'} \|(TB)X_j\|_p \\
\leq \sum_{j = 1}^{j_0} 2^{jn/p'} \left\{ \int_{C_j} \int_{\mathbb{R}^n} \left| e^{iP(x,y)} - e^{iQ(x,y)} \right| K(x - y)b(y) \, dy \, dx \right\}^{1/p} \\
+ \sum_{j = 1}^{j_0} 2^{jn/p'} \left\{ \int_{C_j} \int_{\mathbb{R}^n} e^{iQ(x,y)} K(x - y)b(y) \, dy \, dx \right\}^{1/p} \\
\leq C \sum_{j = 1}^{j_0} 2^{jn/p'} \left\{ \int_{C_j} \left[ \int_{|y| \leq 1} \exp \left( i \sum_{|\alpha| \geq 1, |\beta| = l} a_{\alpha\beta} x^\alpha y^\beta \right) \right. \\
\left. - 1 \left| \frac{b(y)}{|x|^n} \right| \right] \, dy \right\}^{1/p} + C
\]

\[
\leq C \sum_{|\alpha| \geq 1, |\beta| = l} |a_{\alpha\beta}| \sum_{j = 1}^{j_0} 2^{jn/p'} \left( \int_{C_j} |x|^{(|\alpha| - n)p} \, dx \right)^{1/p} + C
\]

\[
\leq C \sum_{|\alpha| \geq 1, |\beta| = l} |a_{\alpha\beta}| \sum_{j = 1}^{j_0} 2^{j|\alpha|} + C
\]

\[
\leq C \sum_{|\alpha| \geq 1, |\beta| = l} |a_{\alpha\beta}| r^{|\alpha|} + C
\]

\[
\leq C_1 |a_{n_0}| r^{|\alpha_0|} + C = C_1 + C.
\]

It remains to estimate \( I_3 \). Let \( \varphi \) and \( \psi \) be the functions as in Lemma 2.2.
Then
\[ I_3 = \sum_{j \geq j_0 + 1} 2^{jn/p'} \|(Tb) \chi_j\|_p \]
\[ \leq \sum_{j \geq j_0 + 1} 2^{jn/p'} \left\{ \int_{C_j} \left( \int_{\mathbb{R}^n} |K(x - y) - K(x)||b(y)| \, dy \right)^p \, dx \right\}^{1/p} \]
\[ + \sum_{j \geq j_0 + 1} 2^{jn/p'} \left\{ \int_{C_j} \frac{1}{|x|^n} \left( \int_{\mathbb{R}^n} e^{iP(x, y)} b(y) \, dy \right)^p \, dx \right\}^{1/p} \]
\[ \leq \sum_{j \geq j_0 + 1} 2^{jn/p'} \left( \int_{C_j} \frac{dx}{|x|^{(n+1)p'}} \right)^{1/p} + \sum_{j \geq j_0 + 1} 2^{-jn/p} \|T_j b\|_p \]
\[ \leq C + \sum_{j \geq j_0 + 1} 2^{-jn/p} \|T_j b\|_p. \]

By Lemma 2.2, we have
\[ \|T_j b\|_2 \leq C 2^{jn/2} |a_{\alpha_0 \beta_0}|^{-1/2} 2^{-j\alpha_0/2Nl} \|b\|_2. \]
It is easy to check from the definition of \( T_j \) that
\[ \|T_j b\|_\infty \leq C \|b\|_\infty \]
and
\[ \|T_j b\|_1 \leq C 2^j \|b\|_1. \]

By the interpolation theorem, we obtain
\[ \|T_j b\|_p \leq \begin{cases} 
C 2^{jn/p} |a_{\alpha_0 \beta_0}|^{-1/Nl} 2^{-j\alpha_0/Nl} \|b\|_p & \text{for } 1 < p \leq 2, \\
C 2^{jn/p} |a_{\alpha_0 \beta_0}|^{-1/Nl} 2^{-j\alpha_0/Nl} \|b\|_p & \text{for } 2 < p < \infty.
\end{cases} \]
It follows from the above and (2.2) that if \( 1 < p \leq 2 \), then
\[ I_3 \leq C + C |a_{\alpha_0 \beta_0}|^{-1/Nl} \sum_{j \geq j_0 + 1} 2^{-j\alpha_0/Nl} \]
\[ \leq C + C |a_{\alpha_0 \beta_0}|^{-1/Nl} \]
and if \( 2 < p < \infty \), then
\[ I_3 \leq C + C |a_{\alpha_0 \beta_0}|^{-1/Nl} \sum_{j \geq j_0 + 1} 2^{-j\alpha_0/Nl} \]
\[ \leq C + C |a_{\alpha_0 \beta_0}|^{-1/Nl} \leq C. \]

This completes the proof of (2.1) and therefore the proof of Proposition 2.2.

Remark 2.1. Recently, Hardy spaces \( HAP(\mathbb{R}^n) \) related to the Beurling algebras \( A_p(\mathbb{R}^n) \) have been introduced by Y. Z. Chen and K. S. Lau in [1] and independently by J. Garcia-Cuerva in [2]. It has been proved by the authors in [4] that \( HK_p \cap L^p = HAP \)
and
\[ (2.3) \quad \|f\|_{HAP} \sim \|f\|_{HK_p} + \|f\|_p. \]
On the other hand, it is easy to show that

\[(2.4) \quad \|f\|_{A^p} \sim \|f\|_{K^p} + \|f\|_{p}.
\]

Thus, from (2.3), (2.4), and the Theorem, it is easy to see that under the conditions of Theorem, $T$ defined by (1.1) is a bounded operator from $HAP(\mathbb{R}^n)$ to $A^p(\mathbb{R}^n)$ and

\[\|Tf\|_{A^p} \leq C\|f\|_{HAP}.
\]

**Remark 2.2.** A counterexample shows that there exists an operator $T$ defined by (1.1) such that $T$ is not a bounded operator from $HK^p$ to itself. Let us consider $n = 1$. Take a $g \in HK^p(\mathbb{R})$ such that $Hg(x) \neq 0$ a.e., where $Hg$ is the Hilbert transform of $g$. Let $P(x, y) = tx$, $t \in \mathbb{R}$. Suppose $T$ is a bounded operator from $HK^p$ into itself. Then $Tg \in HK^p(\mathbb{R})$. Thus, by Lemma 2.1, we have

\[\int Tg(x) \, dx = 0.
\]

This is

\[\int e^{itx}Hg(x) \, dx = 0, \quad t \in \mathbb{R}.
\]

Hence, $(Hg)^*(t) = 0$, $t \in \mathbb{R}$. It has been proved for the case of $l = 0$ in the proof of Theorem that $H$ maps $HK^p$ into $K^p$. Thus, $Hg \in K^p \subset L^1$. Combining it with $(Hg)^*(t) = 0$, $t \in \mathbb{R}$, we get a contradiction,

\[Hg(x) = 0 \quad \text{a.e.}
\]

This confirms the above assertion. However, for the oscillatory integral operator $T$ of convolution type with $P(x, y) = P(x - y)$, the second-named author has proved that $T$ maps $HK^p$ into itself provided $\nabla P(0) = 0$. We omit it here.

**ACKNOWLEDGMENT**

We thank Professor Yibiao Pan for providing us his paper [5].

**REFERENCES**