OSCILLATORY SINGULAR INTEGRALS
ON HARDY SPACES ASSOCIATED WITH HERZ SPACES

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Abstract. In this paper, it is proved that the oscillatory singular integral operators of nonconvolution type are bounded from Hardy spaces associated with Herz spaces to Herz spaces.

1. Introduction

Let $T$ be an oscillatory singular integral operator defined by

$$Tf(x) = \text{p.v.} \int_{\mathbb{R}^n} e^{iP(x,y)} K(x-y) f(y) \, dy,$$

where $P(x,y)$ is a real-valued polynomial on $\mathbb{R}^n \times \mathbb{R}^n$ and $K$ is a Calderón-Zygmund kernel.

It is proved by D. H. Phong and E. M. Stein in [6] that $T$ is a bounded operator from $H^1_E$ to $L^1$ provided $P(x,y)$ is a real bilinear form, where $H^1_E$ is certain variant of the $H^1$ space. Later, this result is extended into the case of general $P(x,y)$ by Y. B. Pan in [5]. For general $P(x,y)$, it is still an interesting problem whether $T$ is a bounded operator from $H^1$ to $L^1$. Recently, some new Hardy spaces $HK_p$ associated with Herz spaces $K_p$ are introduced by the authors in [4] and [8]. The space $HK_p$ is defined by

$$HK_p = \{ f : \text{Gf} \in K_p \},$$

where $\text{Gf}$ is the Grand maximal function of $f$. An interesting fact shown in [8] is that $HK_p$ is the localization of $H^1$ at the origin. It is easy to see that the relation between $HK_p$ and $K_p$ is similar to one between $H^1$ and $L^1$.

In this paper, we shall prove that $T$ defined by (1.1) is a bounded operator from $HK_p$ to $K_p$. A counterexample shows that there exists an operator $T$ defined by (1.1), such that $T$ is not a bounded operator from $HK_p$ to itself. To formulate our result, let us first introduce some definitions.
Definition 1.1 (see [3]). Let $1 < p < \infty$ and $1/p + 1/p' = 1$. The Herz space $K_p(\mathbb{R}^n)$ consists of those functions $f \in L^p_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$ for which

$$
\|f\|_{K_p} := \sum_{k \in \mathbb{Z}} 2^{kn/p'} \|f \chi_k\|_p < \infty,
$$

where $\chi_k = \chi_{C_k}$, $C_k = Q_k \setminus Q_{k-1}$, and $Q_k = \{x : |x| < 2^k\}$.

Definition 1.2 (see [4]). Let $1 < p < \infty$. A function $a(x)$ on $\mathbb{R}^n$ is said to be a central $(1, p)$-atom if

1. $\text{Supp } a \subset Q$, where $Q$ is a ball centered at the origin;
2. $\|a\|_p \leq |Q|^{1/p-1}$;
3. $\int a(x) \, dx = 0$.

Now, we can state our result as follows.

Theorem. Let $1 < p < \infty$, $P(x, y)$ be a real-valued polynomial on $\mathbb{R}^n \times \mathbb{R}^n$, $\nabla_y P(0, y) = 0$, and $T$ be defined as in (1.1). Then $T$ maps $HK_p(\mathbb{R}^n)$ into $K_p(\mathbb{R}^n)$ and

$$
\|Tf\|_{K_p} \leq C\|f\|_{HK_p},
$$

where $C$ depends only on the total degree of $P(x, y)$ but not on the coefficients of $P(x, y)$.

2. Proof of the Theorem

To prove the Theorem, we need two lemmas.

Lemma 2.1. Let $f \in L^1(\mathbb{R}^n)$ and $1 < p < \infty$. Then $f \in HK_p(\mathbb{R}^n)$ if and only if $f$ can be represented as

$$
f(x) = \sum_i \lambda_i a_i(x),
$$

where each $a_i$ is a central $(1, p)$-atom and $\sum_i |\lambda_i| < \infty$. Moreover,

$$
\|f\|_{HK_p} := \|Gf\|_{K_p} \sim \inf \left\{ \sum_i |\lambda_i| \right\},
$$

where the infimum is taken over all decompositions of $f$ as above.


The following lemma belongs to Y. B. Pan [5].

Lemma 2.2. Let $\varphi \in C_0^\infty(\mathbb{R}^n)$ satisfy

$$
\varphi(x) = \begin{cases} 
1 & \text{for } |x| \leq 1, \\
0 & \text{for } |x| \geq 2,
\end{cases}
$$

and let $\psi \in C_0^\infty(\mathbb{R}^n)$ satisfy

$$
\psi(x) = \begin{cases} 
1 & \text{for } 1 \leq |x| \leq 2, \\
0 & \text{for } |x| \leq \frac{1}{4} \text{ or } |x| \geq 4.
\end{cases}
$$

Define $T_k$ by

$$
T_k f(x) = \psi(x/2^k) \int_{\mathbb{R}^n} e^{iP(x, y)} \varphi(y) f(y) \, dy.
$$
If \( P(x, y) \) satisfies
\[
P(x, y) = \sum_{|\alpha| \geq 1, |\beta| = l} a_{\alpha \beta} x^\alpha y^\beta + Q(x, y),
\]
where \( Q(x, y) \) is a polynomial with degree in \( y \) less than or equal to \( l - 1 \), then for each \( N > 0 \) (large enough) we have
\[
\| T_k \|_{L^2 \to L^2} \leq C 2^{nk} |a_{\alpha_0 \beta_0}|^{-1/2Nl} 2^{-k|\alpha_0|/2Nl},
\]
where \( |a_{\alpha_0 \beta_0}| = \max_{|\alpha| \geq 1, |\beta| = l} |a_{\alpha \beta}|. \)

**Proposition 2.1.** Let \( \delta > 0 \). Then we have
\[
\| f(\delta \cdot) \|_{K^p} \sim \delta^{-n} \| f \|_{K^p}.
\]

**Proof.** For any \( \delta > 0 \), there exists a \( k_0 \in \mathbb{Z} \) such that \( 2^{k_0} < \delta \leq 2^{k_0+1} \). By Definition 1.1,
\[
\| f(\delta \cdot) \|_{K^p} = \sum_{k \in \mathbb{Z}} 2^{kn/p'} \left( \int_{C_k} |f(\delta x)|^p \, dx \right)^{1/p}
\leq \sum_{k \in \mathbb{Z}} 2^{kn/p'} \delta^{-n/p} \left( \int_{2^{k_0} \delta^{n/p'} < |y| \leq 2^{k_0+1} \delta^{n/p'}} |f(y)|^p \, dy \right)^{1/p}
\leq \delta^{-n/p} 2^{-k_0 n/p'} \sum_{k \in \mathbb{Z}} 2^{(k+k_0)n/p'} \left( \int_{C_{k+k_0}} |f(y)|^p \, dy \right)^{1/p}
+ \delta^{-n/p} 2^{-(k_0+1)n/p'} \sum_{k \in \mathbb{Z}} 2^{(k+k_0+1)n/p'} \left( \int_{C_{k+k_0+1}} |f(y)|^p \, dy \right)^{1/p}
\leq C \delta^{-n} \| f \|_{K^p}.
\]

On the other hand,
\[
\| f \|_{K^p} = \| f(\delta^{-1} \delta \cdot) \|_{K^p} \leq C \delta^n \| f(\delta \cdot) \|_{K^p}.
\]

This finishes the proof of Proposition 2.1.

By Lemma 2.1, it is easy to see that the proof of the Theorem is reduced to the following proposition.

**Proposition 2.2.** Let \( 1 < p < \infty \), \( P(x, y) \) be a real-valued polynomial, \( \nabla_y P(0, y) = 0 \), and \( T \) be defined as in \( (1, 1) \). Then for any central \( (1, p) \)-atom \( a \),
\[
\| Ta \|_{K^p} \leq C,
\]
where \( C \) is independent of \( a \) and the coefficients of \( P(x, y) \).

**Proof.** Let \( \text{Supp} a \subset Q \) and \( Q \) be a ball centered at the origin with radius \( \delta \). If we write \( b(x) = \delta^n a(\delta x) \), then \( b(x) \) is a central \( (1, p) \)-atom supporting on unit ball \( B(0, 1) \). We also have
\[
Ta(\delta x) = \delta^{-n} T_1 b(x)
:= \delta^{-n} \text{p.v.} \int_{\mathbb{R}^n} e^{iP(\delta x, \delta y)} K_1(x - y) b(y) \, dy,
\]
where $K_1(x) = \delta^n K(\delta x)$. By Proposition 2.1, we obtain

$$\|T a\|_{K_p} \sim \|T_1 b\|_{K_p}.$$ 

Let $P_1(x, y) = P(\delta x, \delta y)$. Note that $\nabla_y P_1(0, y) = 0$ and $K_1$ is also a Calderón-Zygmund kernel. We may assume $T_1 = T$. Thus, it suffices to show

$$\|(T_b)\|_{K_p} \leq C,$$ 

where $C$ is independent of $b$ and the coefficients of $P(x, y)$ and $b$ is a central $(1, p)$-atom supporting on unit ball $B(0, 1)$.

We now turn to prove (2.1) by using induction on the degree of $y$, $l$, in $P(x, y)$. If $l = 0$, then

$$|T_b(x)| = \left| \text{p.v.} \int K(x, y)b(y)dy \right|.$$ 

Thus,

$$\|(T_b)\|_{K_p} = \sum_{k \in \mathbb{Z}} 2^{kn/p'} \|(T_b)\chi_k\|_p$$

$$= \sum_{k \leq 0} \cdots + \sum_{k > 0} \cdots := S_1 + S_2.$$ 

By $L^p$-boundedness of Calderón-Zygmund operators,

$$S_1 \leq C \sum_{k \leq 0} 2^{kn/p'} \|b\|_p = C \sum_{k \leq 0} 2^{kn/p'} = C.$$ 

From the condition of $K(x, y),

$$|K(x, y) - K(x, 0)| \leq C|y|/|x - y|^{n+1}, \quad \text{if } |y| < |x - y|/2,$$

it follows that

$$T_b(x) = \int b(y)[K(x, y) - K(x, 0)]dy$$

and

$$S_2 = \sum_{k > 0} 2^{kn/p'} \|(T_b)\chi_k\|_p$$

$$\leq C \sum_{k > 0} 2^{kn/p'} \left[ \int_{C_k} \left( \int_{B(0, 1)} \frac{|b(y)||y|}{|x - y|^{n+1}} dy \right)^{1/p} dx \right]$$

$$\leq C \sum_{k > 0} 2^{kn/p'} \left( \int_{C_k} \frac{dx}{|x|^{(n+1)p}} \right)^{1/p} \|b\|_p$$

$$\leq C \sum_{k > 0} 2^{kn/p'} 2^{-k(n+1)p-n}/p = C \sum_{k > 0} 2^{-k} = C.$$ 

Therefore, (2.1) holds for $l = 0$. Let us now consider the case of $l > 0$. We assume that (2.1) holds for $l - 1$ by induction. Since $\nabla_y P(0, y) = 0$, we can write

$$P(x, y) = \sum_{|\alpha| \geq 1, |\beta| = l} a_{\alpha\beta} x^\alpha y^\beta + Q(x, y),$$

where $Q(x, y)$ is a polynomial with degree in $y$ less than or equal to $l - 1$ and $\nabla_y Q(0, y) = 0$. Denote

$$|a_{\alpha\beta}| = \max_{|\alpha| \geq 1, |\beta| = l} |a_{\alpha\beta}|.$$
and

\begin{equation}
(2.2) \quad r = \max \{3, |a_{\alpha_0\beta_0}|^{-1/|\alpha_0|}\}.
\end{equation}

Since $r \geq 3$, we may assume $2^{j_0} < r \leq 2^{j_0+1}$ for some $j_0 \in \mathbb{N}$. We now write

$$
\|Tb\|_{K_p} = \sum_{j \leq 0} 2^{jn/p'} \|(Tb)\chi_j\|_p + \sum_{j = 1}^{j_0} 2^{jn/p'} \|(Tb)\chi_j\|_p
+ \sum_{j \geq j_0+1} 2^{jn/p'} \|(Tb)\chi_j\|_p
:= I_1 + I_2 + I_3.
$$

By $L^p$-boundedness of oscillatory singular integral operators (see [7]), we have

$$
I_1 \leq C \sum_{j \leq 0} 2^{jn/p'} \|b\|_p = C \sum_{j \leq 0} 2^{jn/p'} = C.
$$

To estimate $I_2$, we may assume $j_0 \geq 2$. In this case, $r = |a_{\alpha_0\beta_0}|^{-1/|\alpha_0|}$. By induction assumption,

$$
I_2 = \sum_{j = 1}^{j_0} 2^{jn/p'} \|(Tb)\chi_j\|_p
\leq \sum_{j = 1}^{j_0} 2^{jn/p'} \left\{ \int_{C_j} \left( \int_{\mathbb{R}^n} \left| e^{i\theta(x,y)} - e^{i\phi(x,y)} \right| K(x-y)b(y) \, dy \right)^p \, dx \right\}^{1/p}
+ \sum_{j = 1}^{j_0} 2^{jn/p'} \left\{ \int_{C_j} \left( \int_{\mathbb{R}^n} e^{i\phi(x,y)} K(x-y)b(y) \, dy \right)^p \, dx \right\}^{1/p}
\leq C \sum_{j = 1}^{j_0} 2^{jn/p'} \left\{ \int_{C_j} \left[ \int_{|y| \leq 1} \exp \left( i \sum_{|\alpha| \geq 1, |\beta| = l} a_{\alpha\beta} x^\alpha y^\beta \right) \right. \right.
- \left. \left. 1 \right| \frac{|b(y)|}{|x|} \, dy \right\}^p \, dx \right\}^{1/p} + C
\leq C \sum_{|\alpha| \geq 1, |\beta| = l} |a_{\alpha\beta}| \sum_{j = 1}^{j_0} 2^{jn/p'} \left( \int_{C_j} |x|^{(|\alpha| - n)p} \, dx \right)^{1/p} + C
\leq C \sum_{|\alpha| \geq 1, |\beta| = l} |a_{\alpha\beta}| \sum_{j = 1}^{j_0} 2^{j|\alpha|} + C
\leq C \sum_{|\alpha| \geq 1, |\beta| = l} |a_{\alpha\beta}| r^{|\alpha|} + C
\leq C |a_{\alpha_0\beta_0}| r^{|\alpha_0|} + C
\leq C_1 |a_{\alpha_0\beta_0}| r^{|\alpha_0|} + C = C_1 + C.
$$

It remains to estimate $I_3$. Let $\varphi$ and $\psi$ be the functions as in Lemma 2.2.
Then
\[ I_3 = \sum_{j \geq j_0 + 1} 2^{jn/p} \| (Tb) \chi_j \|_p \]
\[ \leq \sum_{j \geq j_0 + 1} 2^{jn/p} \left\{ \int_{C_j} \left( \int_{\mathbb{R}^n} |K(x - y) - K(x)| |b(y)| dy \right)^p dx \right\}^{1/p} \]
\[ + \sum_{j \geq j_0 + 1} 2^{jn/p} \left\{ \int_{C_j} \frac{1}{|x|^np} \left( \int_{\mathbb{R}^n} e^{iP(x, y)} b(y) dy \right)^p dx \right\}^{1/p} \]
\[ \leq \sum_{j \geq j_0 + 1} 2^{jn/p} \left( \int_{C_j} \frac{dx}{|x|^{(\alpha+1)p}} \right)^{1/p} + \sum_{j \geq j_0 + 1} 2^{-jn/p} \| T_j b \|_p \]
\[ \leq C + \sum_{j \geq j_0 + 1} 2^{-jn/p} \| T_j b \|_p. \]

By Lemma 2.2, we have
\[ \| T_j b \|_2 \leq C 2^{jn/2} |a_\alpha b_0|^{-1/2} 2^{-j|\alpha_0|/2N'} \| b \|_2. \]

It is easy to check from the definition of \( T_j \) that
\[ \| T_j b \|_\infty \leq C \| b \|_\infty \]
and
\[ \| T_j b \|_1 \leq C 2^{jn} \| b \|_1. \]

By the interpolation theorem, we obtain
\[ \| T_j b \|_p \leq \begin{cases} 
C 2^{jn/p} |a_\alpha b_0|^{-1/N'p} 2^{-j|\alpha_0|/N'p} \| b \|_p & \text{for } 1 < p < 2, \\
C 2^{jn/p} |a_\alpha b_0|^{-1/N'p} 2^{-j|\alpha_0|/N'p} \| b \|_p & \text{for } 2 < p < \infty.
\end{cases} \]

It follows from the above and (2.2) that if \( 1 < p < 2 \), then
\[ I_3 \leq C + C |a_\alpha b_0|^{-1/N'p} \sum_{j \geq j_0 + 1} 2^{-j|\alpha_0|/N'p} \]
\[ \leq C + C |a_\alpha b_0|^{1/|\alpha_0|} \|
\]
and if \( 2 < p < \infty \), then
\[ I_3 \leq C + C |a_\alpha b_0|^{-1/Np} \sum_{j \geq j_0 + 1} 2^{-j|\alpha_0|/Np} \]
\[ \leq C + C |a_\alpha b_0|^{1/|\alpha_0|} \]

This completes the proof of (2.1) and therefore the proof of Proposition 2.2.

Remark 2.1. Recently, Hardy spaces \( H \mathcal{A}^p(\mathbb{R}^n) \) related to the Beurling algebras \( \mathcal{A}^p(\mathbb{R}^n) \) have been introduced by Y. Z. Chen and K. S. Lau in [1] and independently by J. Garcia-Cuerva in [2]. It has been proved by the authors in [4] that
\[ HK_p \cap L^p = H \mathcal{A}^p \]
and
\[ \| f \|_{H \mathcal{A}^p} \sim \| f \|_{HK_p} + \| f \|_p. \]
On the other hand, it is easy to show that
\[(2.4) \quad \|f\|_{A^p} \sim \|f\|_{K_p} + \|f\|_p.\]
Thus, from (2.3), (2.4), and the Theorem, it is easy to see that under the conditions of Theorem, $T$ defined by (1.1) is a bounded operator from $HA^p(\mathbb{R}^n)$ to $A^p(\mathbb{R}^n)$ and
\[\|Tf\|_{A^p} \leq C\|f\|_{HA^p}.\]

Remark 2.2. A counterexample shows that there exists an operator $T$ defined by (1.1) such that $T$ is not a bounded operator from $HK_p$ to itself. Let us consider $n = 1$. Take a $g \in HK_p(\mathbb{R})$ such that $Hg(x) \neq 0$ a.e., where $Hg$ is the Hilbert transform of $g$. Let $P(x, y) = tx$, $t \in \mathbb{R}$. Suppose $T$ is a bounded operator from $HK_p$ into itself. Then $Tg \in HK_p(\mathbb{R})$. Thus, by Lemma 2.1, we have
\[\int Tg(x) \, dx = 0.\]
This is
\[\int e^{itx}Hg(x) \, dx = 0, \quad t \in \mathbb{R}.\]
Hence, $(Hg)^\vee(t) = 0$, $t \in \mathbb{R}$. It has been proved for the case of $l = 0$ in the proof of Theorem that $H$ maps $HK_p$ into $K_p$. Thus, $Hg \in K_p \subset L^1$. Combining it with $(Hg)^\vee(t) = 0$, $t \in \mathbb{R}$, we get a contradiction,
\[Hg(x) = 0 \quad \text{a.e.}\]
This confirms the above assertion. However, for the oscillatory integral operator $T$ of convolution type with $P(x, y) = P(x - y)$, the second-named author has proved that $T$ maps $HK_p$ into itself provided $\nabla P(0) = 0$. We omit it here.

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