ANTIPODAL COINCIDENCE
FOR MAPS OF SPHERES INTO COMPLEXES

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ABSTRACT. This paper gives a partial answer to the question of whether there exists a Borsuk-Ulam type theorem for maps of \( S^n \) into lower-dimensional spaces, which are not necessarily manifolds. It is shown that for each \( k \) and \( n < 2k - 1 \), there exists a map \( f \) of \( S^n \) into a contractible \( k \)-dimensional complex \( Y \) such that \( fx \neq f(-x) \), for all \( x \in S^n \). In particular, there exists a map of \( S^3 \) into a 2-dimensional complex \( Y \) without an antipodal coincidence. This answers a question raised by Conner and Floyd in 1964. The complex \( Y \) provides also an example of a contractible \( k \)-dimensional complex whose deleted product has the Yang-index equal to \( 2k - 1 \).

1. Introduction and notation

We will say that a map \( f \) from \( S^n \) to a space \( Y \) has an antipodal coincidence if there exists a point \( x \in S^n \) such that \( fx = f(-x) \). The classical Borsuk-Ulam theorem says that every map \( f : S^n \rightarrow \mathbb{R}^k \) has an antipodal coincidence if \( k \leq n \). Conner and Floyd ([1], p. 85) proved that if \( k < n \), then every map \( f : S^n \rightarrow M^k \) of \( S^n \) into a \( k \)-dimensional differentiable manifold \( M^k \) has an antipodal coincidence. They also asked a specific question ([1], 89) of whether there is a map of \( S^3 \) into a 2-dimensional complex without an antipodal coincidence. We will construct such a map of \( S^3 \) into a 2-dimensional contractible complex \( Y \). The construction is extremely simple and geometric: in our example, \( Y \) is a subcomplex of the barycentric subdivision of a standard simplex. We will also show that 3 is the lowest dimension of the sphere for which such an example can be constructed.

If \( A \) and \( B \) are spaces, we will denote by \( A \ast B \) the join of \( A \) and \( B \). It consists of segments joining the points of \( A \) with the points of \( B \), the segments being mutually disjoint except at the endpoints. The join of \( A \) with the empty set is \( A \) itself. There is a standard (deformation) retraction of \((A \ast B) - A \) to \( B \) along the segments of the join.

If \( s \) is a simplex and \( t \) is a face of \( s \), we will denote by \( c(t) \) the face of \( s \) which is complementary to \( t \); i.e., \( s \) is the join of \( t \) and \( c(t) \); and

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\[ \dim c(t) = \dim s - \dim t - 1. \]
The barycenter of a simplex \( s \) will be denoted by \( b(s) \). By the carrier of a point \( x \) in a complex \( K \) we mean, as usual, the simplex of \( K \) containing \( x \) in the interior.

Let \( \Delta \) be a standard \((n+1)\)-dimensional simplex and let \( \Delta' \) denote its barycentric subdivision. We can think of \( \Delta \) as a subset of \( \mathbb{R}^{n+2} \) embedded in the standard way, so that the coordinates in \( \mathbb{R}^{n+2} \) are the barycentric coordinates of points of \( \Delta \). We will denote by \( \alpha: \mathbb{R}^{n+2} \to \mathbb{R}^{n+2} \) the symmetry of \( \mathbb{R}^{n+2} \) about the diagonal of \( \mathbb{R}^{n+2} \), i.e., the line where all the coordinates are equal to each other. Then \( \alpha \) corresponds to the antipodal map of the \((n+1)\)-plane \( P \) in which the simplex \( \Delta \) lies. Let \( o \) be the barycenter of \( \Delta \). Two points of \( \mathbb{R}^{n+2} \) are said to be antipodal (to each other) if they lie on opposite sides of the center \( o \). The simplex \( \Delta \) is not invariant under the central symmetry \( \alpha \), but the symmetry induces a simplicial map \( \beta: \Delta' \to \Delta' \) of the barycentric subdivision of \( \Delta \). In fact, if \( b = b(s) \) is a vertex of \( \Delta' \) and \( b \) is the barycenter of a face \( s \) of \( \Delta \), then \( \beta(b) \) is the barycenter of the face complementary to \( b \): \( \beta(b) = b(c(s)) \).

The vertices of the barycentric subdivision \( \Delta' \) are partially ordered, as usual, by their "ranks": the rank of a vertex \( b \) of \( \Delta' \) is the dimension of the face of \( \Delta \) of which \( b \) is the barycenter. Note the following facts:

**Lemma 1.** \( \beta: \Delta' \to \Delta' \) reverses the ranks of the vertices: if \( b \) is a vertex of \( \Delta' \) of rank \( q \), then the rank of \( \beta(b) \) is \( n - q \) (in the case \( q = n+1 \) it is understood that the center \( o \) is the barycenter of both \( \Delta \) and the empty simplex).

(2) \( \beta: \Delta' \to \Delta' \) is simplicially free outside the center \( o \) in the sense that for each simplex \( s \) of \( \Delta' \), the intersection of \( s \) with \( \beta(s) \) consists at most of the center \( o \).

(3) If \( x \) and \( y \) is an antipodal pair in \( \Delta \) and if \( u \) and \( v \) are the carriers of \( x \) and \( y \) in \( \Delta' \), respectively, then \( v = \beta(u) \).

The construction of our example rests on the fact that a simplex is so much "antisymmetric" with respect to the antipodal map.

**Lemma 2.** If \( s \) is a \( q \)-simplex of \( \Delta' \) which is a part of the \( q \)-skeleton \( \Delta^q \) of \( \Delta \), then the intersection \( (\beta(s)) \cap \Delta^q \) is a face of \( \beta(s) \) of dimension \( 2q - n \).

**Proof.** Since \( s \) is a simplex of \( \Delta' \) in \( \Delta^q \), the ranks of its vertices form a sequence \( (0, \ldots, q) \). Thus by Lemma 1, the ranks of \( \beta(s) \) form a sequence \( (n - q, \ldots, n) \). The face of \( \beta(s) \) which is in \( \Delta^q \) is made up of the vertices whose ranks are not more than \( q \). In the sequence \( (n - q, \ldots, n) \) there are exactly \( 2q - n + 1 \) integers not greater than \( q \).

**Corollary.** If \( 2q + 1 \leq n \), then \( \beta(\Delta^q) \cap \Delta^q = \emptyset \).

### 2. Construction of the complex \( Y \)

Throughout Sections 2 and 3, let \( q \) be an integer and \( k = n - q \).

**Definition.** Every simplex \( s \) of the barycentric subdivision \( \Delta' \) is the join of a unique pair \( u, v \) of simplices of \( \Delta' \), where \( u \) lies in the \( q \)-skeleton \( \Delta^q \) of \( \Delta \) and \( v \) is disjoint from the \( q \)-skeleton. The simplices \( v \) so obtained form a \( k \)-dimensional subcomplex \( Y^k \) of \( \Delta' \). We will call it the \( k \)-dimensional spine, denoted \( Y^k \), of \( \Delta \) (briefly, the \( k \)-spine) "dual" to the \( q \)-skeleton \( \Delta^q \). We can
also define the spine inductively by saying that the \( k \)-spine of a simplex is the union of the cones over the \((k - 1)\)-spines of the boundary simplices.

Let \( v_0, \ldots, v_{n+1} \) be the vertices of \( \Delta \). Given a set of \( q+1 \) indices \( i_0, \ldots, i_q \) out of \( (0, \ldots, n + 1) \), we will denote by \( b(i_0, \ldots, i_q) \) the center of the face spanned by \( v_{i_0}, \ldots, v_{i_q} \). The \( k \)-spine \( Y^k \) is disjoint with the \( q \)-skeleton of \( \Delta \) and intersects each \((q + 1)\)-face of \( \Delta \) spanned by \( v_{i_0}, \ldots, v_{i_{q+1}} \) in its center \( b(i_0, \ldots, i_{q+1}) \). These centers are "the extremities" of the spine \( Y^k \). There are \((n+2)\) of them.

Let \( C_k \) be the convex hull of the \( k \)-spine \( Y^k \). Then \( C_k \) is also the convex hull of the set of its extremities \( \{b(i_0, \ldots, i_{q+1})\} \). It is a convex \((n + 1)\)-dimensional polyhedron with vertices \( \{b(i_0, \ldots, i_{q+1})\} \) contained in \( \Delta - \Delta^q \). The boundary of \( C_k \), which we denote by \( \Sigma_k \), is a simplicial \( n \)-sphere.

Note that \( C_k \) is the intersection of the \((n+2)\) half-spaces given by \((q+2)x_i \leq 1, i = 0, \ldots, n + 1\), with the simplex \( \Delta \). The part of \( C_k \) lying on an \( m \)-face of \( C_k \) is just \( C_k + m - n - 1 \).

The special case of \( n = 3, k = 2, q = 1 \). In this case (which is actually the first interesting one), we can see that \( Y^2 \) is a 2-dimensional complex whose intersection with each 3-dimensional simplex \( s \) on the boundary of \( \Delta \) consists of four intervals going from the center of \( s \) to its 2-dimensional faces. The complex \( Y^2 \) itself is the cone over the graph made up by these four-legged objects. The convex hull \( C_2 \) of \( Y^2 \) is a 4-dimensional polyhedron with 10 vertices. It cannot be a regular polyhedron since there are no regular 4-dimensional polyhedra with 10 vertices. In fact, the boundary \( \Sigma_2 \) of \( C_2 \) has five tetrahedra (one in every 3-simplex of \( \Sigma_2 \)), five octahedra (one "opposite" each vertex of \( \Delta \)), thirty triangles, thirty edges and ten vertices (the Euler number being zero, as it should be).

3. The map of \( S^n \) into \( Y^k \)

Recall that every simplex \( s \) of \( \Delta' \) is the join of a unique simplex \( u \) of the barycentric subdivision of the \( q \)-skeleton of \( \Delta^q \) of \( \Delta \) with a unique simplex \( v \) of \( Y^k \). Thus there exists a standard deformation retraction of \( \Delta - \Delta^q \) to \( Y^k \) along the lines of the join in each simplex of \( \Delta' \) (with its face in \( \Delta^q \) deleted). Let \( f: \Sigma_k \cong S^n \rightarrow Y^k \) be the restriction of this map to \( \Sigma_k \). The antipodal symmetry \( \alpha \) induces a free involution on \( \Sigma_k \cong S^n \) which will also be called antipodal.

**Lemma 3.** If \( x \) and \( y \) is a pair of antipodal points of \( \Sigma_k \) such that \( fx = fy \), then \( fx = fy \) must be the center \( o \).

**Proof.** Let \( z = fx = fy \). Let \( t \) be the carrier of \( z \) in \( Y^k \) and let \( u \) and \( v \) be the carriers of \( x \) and \( y \), respectively, in \( \Delta' \). By the construction of \( f \), the simplex \( t \) is a face of both \( u \) and \( v \). Since \( x \) and \( y \) is an antipodal pair, Lemma 1 implies that the intersection \( u \cap v \) is just the center \( o \). Thus \( z = o \). \( \square \)

**Proposition.** If \( n \leq 2k - 1 \), then the map \( f: S^n \rightarrow Y^k \) has no antipodal coincidence: if \( x \) and \( y \) is an antipodal pair on \( S^n \), then \( fx \neq fy \).

**Proof.** By Lemma 3, \( f \) projects the points \( x \) and \( y \) to the center \( o \) from an antipodal pair of points lying in the \((n - k)\)-skeleton \( \Delta^{n-k} \) of \( \Delta \). However, if
\[ n \leq 2k - 1, \text{ then } 2(n - k) - 1 \leq n \text{ and by the corollary, } \Delta^{n-k} \text{ contains no antipodal pair. } \Box \]

4. CONCLUDING REMARKS

It is not hard to see that if \( n > 2k \), then every map \( f: S^n \to Y \) of a finite \( k \)-dimensional complex \( Y \) into \( S^n \) has an antipodal coincidence. If, in addition, \( H_k(Y;\mathbb{Z}_2) = 0 \), then the inequality \( n > 2k - 1 \) is already sufficient. For, if \( f: S^n \to Y \) is a map without an antipodal coincidence, then \( f \) defines a \( \mathbb{Z}_2 \)-equivariant map from \( S^n \) to the deleted product \( Y^* \) of \( Y \). By an argument similar to that used in [2] one can show that \( H^{2k}(Y^*;\mathbb{Z}_2) = 0 \). Thus the Yang \( \mathbb{Z}_2 \)-index, \( \text{Ind}^{\mathbb{Z}_2} Y^* \), of \( Y^* \) (see [1] and [3]) is less than \( 2k \leq n \) and hence an equivariant map from \( S^n \) to \( Y^* \) cannot exist.

Thus we have the following result.

Theorem. For each \( k \) and \( n \leq 2k - 1 \), there exists a map \( f \) of \( S^n \) into a contractible \( k \)-dimensional complex \( Y \) without an antipodal coincidence. In particular, there exists such a map of \( S^3 \) into a contractible \( 2 \)-dimensional complex. Moreover, \( n = 3 \) is the lowest integer for which there is a map of \( S^n \) into a complex of dimension less than \( n \) without an antipodal coincidence. \( \Box \)

In particular, the deleted product \( Y^* \) of the contractible complex \( Y \) constructed in Section 2 has the Yang \( \mathbb{Z}_2 \)-index equal to \( 2k - 1 \). We do not know whether there exist a \( k \)-dimensional complex whose deleted product has the Yang \( \mathbb{Z}_2 \)-index equal to \( 2k \).

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