NORMAL DERIVATIONS IN NORM IDEALS

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Abstract. We establish the orthogonality of the range and the kernel of a normal derivation with respect to the unitarily invariant norms associated with norm ideals of operators. Related orthogonality results for certain nonnormal derivations are also given.

1. Introduction

Let $B(H)$ denote the algebra of all bounded linear operators on an infinite-dimensional complex separable Hilbert space $H$. For operators $A, B$ in $B(H)$, the generalized derivation $\delta_{A,B}$ as an operator on $B(H)$ is defined by

$$\delta_{A,B}(X) = AX - XB \quad \text{for all } X \in B(H).$$

When $A = B$, we simply write $\delta_A$ for $\delta_{A,A}$. If $N$ is a normal operator in $B(H)$, then $\delta_N$ is said to be a normal derivation.

In his investigation of normal derivations, Anderson [1, Theorem 1.7] proved that if $N$ and $S$ are operators in $B(H)$ such that $N$ is normal and $NS = SN$, then for all $X \in B(H)$

$$\|\delta_N(X) + S\| \geq \|S\|,$$

where $\| \cdot \|$ is the usual operator norm. Thus in the sense of [1, Definition 1.2], inequality (2) says that the range of $\delta_N$ is orthogonal to the kernel of $\delta_N$, which is just the commutant $\{N\}'$ of $N$.

It has been shown in [11, Theorem 1] that if $N$ and $S$ are operators in $B(H)$ such that $N$ is normal, $S$ is a Hilbert-Schmidt operator, and $S \in \{N\}'$, then for all $X \in B(H)$

$$\|\delta_N(X) + S\|_2^2 = \|\delta_N(X)\|_2^2 + \|S\|_2^2,$$

where $\| \cdot \|_2$ is the Hilbert-Schmidt norm. Thus in the usual Hilbert space sense, the Hilbert-Schmidt operators in the range of $\delta_N$ are orthogonal to those in the kernel of $\delta_N$.
It has also been shown recently in [12, Theorem 3.2] that if $N$ and $S$ are operators in $B(H)$ such that $N$ is normal and $S$ belongs to some Schatten $p$-class $C_p$ with $1 \leq p \leq \infty$ and $S \in \{N\}'$, then for all $X \in B(H)$

$$\|\delta_N(X) + S\|_p \geq \|S\|_p.$$  

The usual operator norm, the Hilbert-Schmidt norm, and the Schatten $p$-norms are only examples of a large family of unitarily invariant (or symmetric) norms on $B(H)$.

The purpose of this paper is to investigate the orthogonality of the range and the kernel of a normal derivation with respect to the wider class of unitarily invariant norms on $B(H)$. Derivations induced by certain nonnormal operators will also be discussed.

In §2 we will use a completely different analysis to extend (4) to all unitarily invariant norms defined on norm ideals of compact operators in $B(H)$. Extensions of this result to certain nonnormal operators will be the main theme of §3, in which we will treat derivations of the form $\delta_A B$, where $A$ is a dominant operator and $B^*$ is $M$-hyponormal. Moreover we will discuss the validity of (2) for various classes of derivations at the expense of requiring that $S$ is normal. A relevant example will also be presented.

Recall that each unitarily invariant norm $\|\| \cdot \|\|$ is defined on a natural subclass $J_{\|\| \cdot \|\|}$ of $B(H)$ called the norm ideal associated with the norm $\|\| \cdot \|\|$ and satisfies the invariance property $\|\|UV\|\| = \|\|A\|\|$ for all $A \in J_{\|\| \cdot \|\|}$ and for all unitary operators $U$, $V \in B(H)$. While the usual operator norm $\| \cdot \|$ is defined on all of $B(H)$, the other unitarily invariant norms are defined on norm ideals contained in the ideal of compact operators in $B(H)$. Given any compact operator $A \in B(H)$, denote by $s_1(A) \geq s_2(A) \geq \cdots$ the singular values of $A$, i.e., the eigenvalues of $|A| = (A^*A)^{1/2}$. There is a one-to-one correspondence between symmetric gauge functions defined on sequences of real numbers and unitarily invariant norms defined on norm ideals of operators. More precisely, if $\|\| \cdot \|\|$ is a unitarily invariant norm, then there is a unique symmetric gauge function $\Phi$ such that

$$\|\|A\|\| = \Phi(s_j(A))$$

for all $A \in J_{\|\| \cdot \|\|}$.

For $1 \leq p \leq \infty$, define

$$\|A\|_p = \left( \sum_j s_j^p(A) \right)^{1/p},$$

where, by convention, $\|A\|_\infty = s_1(A)$ is the usual operator norm of the compact operator $A$. These unitarily invariant norms are the well-known Schatten $p$-norms associated with the Schatten $p$-classes $C_p$, $1 \leq p \leq \infty$. Hence $C_1$, $C_2$, and $C_\infty$ are the trace class, the Hilbert-Schmidt class, and the class of compact operators, respectively. For good accounts on the theory of norm ideals and their associated unitarily invariant norms, the reader is referred to [9], [13], or [14] (see also [2] and references therein).

2. Normal derivations

In this section we present our main result of this paper. This result asserts that if $N$ is a normal operator in $B(H)$, then with respect to any unitarily
invariant norm $\|\cdot\|$, $\operatorname{ran} \delta_N \cap J_{\|\cdot\|}$ is orthogonal to $\ker \delta_N \cap J_{\|\cdot\|}$, where $\operatorname{ran} \delta_N$ and $\ker \delta_N$ are the range and the kernel of $\delta_N$, respectively.

To accomplish our goal we need two lemmas.

**Lemma 1.** Let $N \in B(H)$ be diagonal (normal with pure point spectrum), $S \in \{N\}'$, and $X \in B(H)$. If $\delta_N(X) + S \in J_{\|\cdot\|}$, then $S \in J_{\|\cdot\|}$ and

\[
\|\|\delta_N(X) + S\|\| \geq \|\|S\|\|. 
\]

**Proof.** Let $N$ have the distinct eigenvalues $\lambda_1, \lambda_2, \ldots$. Then, with respect to the decomposition $H = \bigoplus_{j=1}^{\infty} \ker(N - \lambda_j)$, $N$ has the operator matrix representation

\[
N = \begin{bmatrix}
\lambda_1 & 0 \\
\vdots & \ddots \\
0 & \lambda_d
\end{bmatrix}.
\]

Let $[S_{ij}]$ and $[X_{ij}]$ be the matrix representations of $S$ and $X$ with respect to the above decomposition of $H$. Then $NX - XN = [(\lambda_i - \lambda_j)X_{ij}]$, and in view of the assumption $S \in \{N\}'$ we have $S_{ij} = 0$ for $i \neq j$. Therefore,

\[
NX - XN + S = \begin{bmatrix}
S_{11} & * \\
S_{22} & * \\
* & \ddots
\end{bmatrix}.
\]

Since $\delta_N(X) + S \in J_{\|\cdot\|}$ and since the norm of an operator matrix always dominates the norm of its diagonal part (see [9, p. 82]), it follows that

\[
S \in J_{\|\cdot\|} \quad \text{and} \quad \|\|\delta_N(X) + S\|\| \geq \|\|S\|\|.
\]

**Lemma 2.** Let $N \in B(H)$ be normal, and set $H_1 = \bigvee_{\lambda \in \sigma} \ker(N - \lambda)$. If $S \in \{N\}'$ and there is an $X \in B(H)$ such that $\delta_N(X) + S \in C_\infty$, then $H_1$ reduces $S$ and $S|H_1^{\perp} = 0$.

**Proof.** Since $N$ is normal, $H_1$ reduces $N$ and $N|H_1$ is a diagonal operator. By Fuglede's theorem (see [10, p. 104]) $S^* \in \{N\}'$, so $H_1$ also reduces $S$. Let

\[
N = \begin{bmatrix}
N_1 & 0 \\
0 & N_2
\end{bmatrix}, \quad S = \begin{bmatrix}
S_1 & 0 \\
0 & S_2
\end{bmatrix}, \quad X = \begin{bmatrix}
X_{11} & X_{12} \\
X_{21} & X_{22}
\end{bmatrix}
\]

on $H = H_1 \oplus H_2$, where $H_2 = H_1^\perp$. The assumption $\delta_N(X) + S \in C_\infty$ implies $\delta_{N_1}(X_{22}) + S_2 \in C_\infty$. Anderson's result (2) (applied to the Calkin algebra $B(H_2)/C_\infty$) insures that $S_2 \in C_\infty$. Since the normal operator $N_2$ has no eigenvalues and since the compact selfadjoint operator $S_2^2$ belongs to $\{N_2\}'$, it follows that $S_2^2 = 0$. Hence $S_2 = 0$, as desired.

Now we are in a position to prove the main result of this paper.

**Theorem 1.** Let $N \in B(H)$ be normal, $S \in \{N\}'$, and $X \in B(H)$. If $\delta_N(X) + S \in J_{\|\cdot\|}$, then $S \in J_{\|\cdot\|}$ and

\[
\|\|\delta_N(X) + S\|\| \geq \|\|S\|\|.
\]

**Proof.** Since $\delta_N(X) + S \in J_{\|\cdot\|} \subseteq C_\infty$, it follows by Lemma 2 that on $H = H_1 \oplus H_1^\perp$,

\[
N = \begin{bmatrix}
N_1 & 0 \\
0 & N_2
\end{bmatrix}, \quad S = \begin{bmatrix}
S_1 & 0 \\
0 & 0
\end{bmatrix},
\]

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where \( H_1 = \bigvee_{\lambda \in \mathbb{C}} \ker(N - \lambda) \). If

\[
X = \begin{bmatrix}
X_{11} & X_{12} \\
X_{21} & X_{22}
\end{bmatrix}
\]
on \( H = H_1 \oplus H_1^\perp \), then

\[
\delta_N(X) + S = \begin{bmatrix}
\delta_{N_1}(X_{11}) + S_1 & * \\
* & * 
\end{bmatrix}.
\]

Since \( \delta_N(X) + S \in J_{|| \cdot ||} \), it follows that \( \delta_{N_1}(X_{11}) + S_1 \in J_{|| \cdot ||} \). But \( N_1 \) is diagonal and \( S_1 \in \{ N_1 \}' \). Thus, by Lemma 1, \( S_1 \in J_{|| \cdot ||} \) and \( ||\delta_{N_1}(X_{11}) + S_1|| \geq ||S_1|| \). Consequently, \( S \in J_{|| \cdot ||} \) and \( ||\delta_N(X) + S|| \geq ||\delta_{N_1}(X_{11}) + S_1|| \geq ||S_1|| = ||S|| \).

At the end of this section we use a familiar device of considering \( 2 \times 2 \) operator matrices to extend Theorem 1 to generalized normal derivations.

**Corollary 1.** Let \( N, M, S \in B(H) \) such that \( N \) and \( M \) are normal and \( NS = SM \). If \( X \in B(H) \) such that \( \delta_{N,M}(X) + S \in J_{|| \cdot ||} \), then \( S \in J_{|| \cdot ||} \) and

\[
||\delta_{N,M}(X) + S|| \geq ||S||.
\]

**Proof.** On \( H \oplus H \) consider the operators \( L, T, \) and \( Y \) defined as

\[
L = \begin{bmatrix}
N & 0 \\
0 & M
\end{bmatrix}, \quad T = \begin{bmatrix}
0 & S \\
0 & 0
\end{bmatrix}, \quad Y = \begin{bmatrix}
0 & X \\
0 & 0
\end{bmatrix}.
\]

Then \( L \) is normal, \( T \in \{ L \}' \), and

\[
\delta_L(Y) + T = \begin{bmatrix}
0 & \delta_{A,B}(X) + S \\
0 & 0
\end{bmatrix}.
\]

Thus by Theorem 1 applied to the operators \( L, T, \) and \( Y \) we have \( T \in J_{|| \cdot ||} \) and \( |||\delta_L(Y) + T||| \geq |||T||| \). Therefore \( S \in J_{|| \cdot ||} \) and \( ||\delta_{A,B}(X) + S|| \geq ||S|| \), as desired.

### 3. Nonnormal derivations

Extensions of (3) to certain subnormal operators have been given in [11, Theorems 2 and 3]. In the same vein we devote this section to the extension of the results in §2 to classes of operators larger than that of normal operators.

Recall that an operator \( A \in B(H) \) is called dominant (see [15]) if

\[
\text{ran}(A - \lambda) \subseteq \text{ran}(A - \lambda)^* \quad \text{for all } \lambda \in \mathbb{C}.
\]

In view of [6], \( A \) is dominant if and only if for each \( \lambda \in \mathbb{C} \) there exists a constant \( M_\lambda \) such that

\[
||(A - \lambda)^*f|| \leq M_\lambda ||(A - \lambda)f|| \quad \text{for all } f \in H.
\]

If there is a constant \( M \) such that \( M_\lambda \leq M \) for all \( \lambda \in \mathbb{C} \), then \( A \) is called \( M \)-hyponormal. If \( M = 1 \), then \( A \) is hyponormal.

Our promised generalization of (9) can be stated as follows.
Theorem 2. Let $A, B, S \in B(H)$ such that $A$ is dominant, $B^*$ is $M$-hyponormal, and $AS = SB$. If $X \in B(H)$ such that $\delta_{A, B}(X) + S \in J_{||| \cdot |||}$, then $S \in J_{||| \cdot |||}$ and

$$
||\delta_{A, B}(X) + S|| \geq ||S||. 
$$

Proof. Since the pair $(A, B)$ satisfies the Fuglede-Putnam property, it follows (see [16] or [19]) that $\overline{\text{ran} S}$ (the closure of $\text{ran} S$) reduces $A$, $\ker^\perp S$ (the orthogonal complement of $\ker S$) reduces $B$, and $A|\overline{\text{ran} S}$ and $B|\ker^\perp S$ are unitarily equivalent normal operators. Then, with respect to the orthogonal decompositions $H = \overline{\text{ran} S} \oplus (\overline{\text{ran} S})^\perp$ and $H = \ker^\perp S \oplus \ker S$, $A$ and $B$ can be respectively represented as

$$
A = \begin{bmatrix}
a_1 & 0 \\
0 & a_2
\end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix}
b_1 & 0 \\
0 & b_2
\end{bmatrix}.
$$

Now assume that the operators $S, X : \ker^\perp S \oplus \ker S \rightarrow \overline{\text{ran} S} \oplus (\overline{\text{ran} S})^\perp$ have the matrix representations

$$
S = \begin{bmatrix}
s_1 & 0 \\
0 & 0
\end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix}
x_1 & x_2 \\
x_3 & x_4
\end{bmatrix}.
$$

Then $A_1$ and $B_1$ are normal, and $A_1 S_1 = S_1 B_1$.

Applying Corollary 1 to the operators $A_1, B_1, S_1$, and $X_1$ we see that $S_1 \in J_{||| \cdot |||}$. Hence $S \in J_{||| \cdot |||}$ and

$$
||\delta_{A, B}(X) + S|| = \left||\begin{bmatrix}
\delta_{A_1, B_1}(X_1) + S_1 \\
* & *
\end{bmatrix}\right|| 
\geq ||\delta_{A_1, B_1}(X_1) + S_1|| \geq ||S_1|| = ||S||,
$$

which completes the proof of the theorem.

The usual operator norm version of (12) has been obtained by Elalami [7, Theorem 4.1] using a different method.

We would like to point out here that in view of [16] Theorem 2 is still valid for any pair of operators $(A, B)$ which satisfies the Fuglede-Putnam property, that is, $A^* S = SB^*$ whenever $AS = SB$, where $S \in B(H)$. For several such pairs, the reader is referred to [4] and references therein.

A closer look at the proof of Theorem 2 (see also [5, Theorem 1]) leads us to show that if $(A, B)$ satisfies the Fuglede-Putnam property and if $S \in C_2$ such that $AS = SB$, then for all $X \in B(H)$ we have

$$
||\delta_{A, B}(X) + S||^2 = ||\delta_{A, B}(X)||^2 + ||S||^2.
$$

This Hilbert space orthogonality result strengthens (3) and [11, Theorem 3].

It has been shown in [11, Theorem 4] that if $A \in B(H)$ is a cyclic subnormal operator and if $S \in C_2 \cap \{A\}'$, then for all $X \in B(H)$ we have

$$
||\delta_A(X) + S||^2 = ||\delta_A(X)||^2 + ||S||^2.
$$

In the same direction, it should be noted that the proof of Theorem 2 can be modified to insure that if $A \in B(H)$ is a cyclic subnormal operator and $S \in J_{||| \cdot |||} \cap \{A\}'$, then for all $X \in B(H)$ we have

$$
||\delta_A(X) + S|| \geq ||S||.
$$

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To verify (15) we need only show that \( \overline{\text{ran} S} \) reduces \( A \) and \( A \mid \overline{\text{ran} S} \) is normal, for then we can follow the arguments in the proof of Theorem 2. Since \( S \in \{ A \}' \) and \( A \) is a cyclic subnormal operator, it follows by Yoshino’s result [18] that \( S \) is also subnormal. This, together with the assumption \( S \in J_{\|\cdot\|} \subseteq C_\infty \), implies that \( S \) is in fact normal. Consequently \( S \in \{ A, A^* \}' \), and so \( \overline{\text{ran} S} \) reduces \( A \). If \( T = AS^* \), then \( T \in \{ A \}' \) and \( T^* T - TT^* = S(A^*A - AA^*)S^* \geq 0 \) (because \( A^*A - AA^* \geq 0 \)). Thus \( T \) is a compact hyponormal operator, and hence \( T \) is normal. Now we have \( AA^*S = ASA^* = AT^* = T^*A = SA^*A = A^*AS \), and so \( A \mid \overline{\text{ran} S} \) is normal.

In [8] an example is given to show that the cyclicity assumption on \( A \) is necessary for (14) to hold. This gives an affirmative answer to a question raised in [11]. The following example, which will also be used later in the paper, is simpler and shorter than the one given in [8].

**Example.** Let \( U \) be the unilateral shift operator of multiplicity one. On \( H \oplus H \), let \( A = \begin{bmatrix} U & 0 \\ 0 & 0 \end{bmatrix} \), \( S = \begin{bmatrix} 0 & 0 \\ P & 0 \end{bmatrix} \), and \( X = \begin{bmatrix} 0 & 0 \\ Q & 0 \end{bmatrix} \), where \( P = 1 - UU^* \) and \( Q = PU^* \). Then \( A \) is a noncyclic subnormal operator, \( S \in \{ A \}' \), and \( \delta_A(X) + S = 0 \), yet \( \|\delta_A(X)\| = \|S\| = 1 \) for every unitarily invariant norm \( \|\cdot\| \).

This example also indicates that Anderson’s result (2) cannot be extended to derivations induced by subnormal operators. However, if we require \( S \) to be normal, then in this case (2) works for several classes of operators. The list includes compact operators, dominant operators, quasinilpotent operators with positive real parts, and operators \( A \) for which \( p(A) = 0 \) for some quadratic polynomial \( p \) (see [7, 17]).

Another interesting class of operators for which (2) is true when \( S \) is normal is the class of operators \( A \) such that \( A^*A \) and \( A + A^* \) commute. It is well known that this class enjoys the property that \( \|A\| = r(A) \) (the spectral radius of \( A \)) (see [3]). Hence it is elementary to verify that (see [10, p. 130]) for all \( X \in B(H) \) and all \( \lambda \in \mathbb{C} \) we have

\[
\|\delta_A(X) + \lambda I\| \geq |\lambda|.
\]

Based on (16) and the spectral theorem for normal operators it can be shown that if \( A, S \in B(H) \) such that \( A^*A \) commutes with \( A + A^* \), \( S \) is normal, and \( S \in \{ A \}' \), then for all \( X \in B(H) \) we have

\[
\|\delta_A(X) + S\| \geq \|S\|.
\]

To prove (17) we first assume that \( S \) is a normal operator with finite spectrum. Then we use a continuity argument to establish the general case.

Finally, we remark that the example presented above shows that (2) fails to hold for an arbitrary (not necessarily normal) operator \( S \) in the commutant of \( A \).

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**References**


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