OMITTED VALUES IN DOMAINS OF NORMALITY

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Abstract. It is proved that if \( U \) and \( V \) are connected components of the Fatou set of an entire function \( f \) and if \( f(U) \subset V \), then \( V \setminus f(U) \) contains at most one point.

Let \( f \) be an entire (or rational) function. The Fatou set \( F \) of \( f \) is the subset of the plane (or sphere) where the iterates of \( f \) form a normal family. It is easy to see that if \( U \) is a connected component of \( F \), then \( f(U) \) is contained in some connected component \( V \) of \( F \). For rational functions, we have \( f(U) = V \) (see [2, §5.4]). For transcendental entire functions, it is possible that \( f(U) \neq V \). A simple example is given by \( f(z) = \lambda e^z \), \( 0 < \lambda < e^{-1} \), where \( F \) is connected and \( 0 \in F \setminus f(F) \).

Theorem. Let \( f \) be an entire transcendental function and let \( U \) and \( V \) be connected components of \( F \) satisfying \( f(U) \subset V \). Then \( V \setminus f(U) \) contains at most one point.

To prove the theorem, let \( f \), \( U \), and \( V \) be as required and suppose that \( V \setminus f(U) \) is not empty. It is easy to see that this implies that there exists a curve \( \gamma \) tending to \( \infty \) in \( U \) such that \( f(z) \) tends to a value in \( V \setminus f(U) \) as \( z \to \infty \) in \( \gamma \). In particular, \( f(z) \) is bounded on some curve tending to \( \infty \). A result of Baker [1, §3] now implies that \( U \) and \( V \) are simply-connected. Hence there exist conformal maps \( \varphi \) and \( \psi \) from the unit disk \( D \) onto \( U \) and \( V \). We define \( g = \psi^{-1} \circ f \circ \varphi \) so that \( g(D) \subset D \). Clearly, it suffices to prove that \( D \setminus g(D) \) contains at most one point.

By a result of Beurling (see [4, Theorems 11.5 and 11.9]) there exists a set \( A \subset [0, 2\pi) \) of capacity zero with the property that if \( \theta \notin A \), then there exists \( a_\theta \in \partial U \setminus \{ \infty \} \) such that \( \varphi(re^{i\theta}) \to a_\theta \) as \( r \to 1 \). It follows that \( f(\varphi(re^{i\theta})) \to f(a_\theta) \in \partial V \setminus \{ \infty \} \) and hence that \( |g(re^{i\theta})| \to 1 \) as \( r \to 1 \), provided \( \theta \notin A \). A result of Lohwater (see [3, Theorem 5.14]) now implies that \( D \setminus g(D) \) contains at most one point. This completes the proof of the theorem.

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