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OMITTED VALUES IN DOMAINS OF NORMALITY

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ABSTRACT. It is proved that if U and V are connected components of the Fatou set of an entire function f and if $f(U) \subset V$, then $V \backslash f(U)$ contains at most one point.

Let f be an entire (or rational) function. The Fatou set F of f is the subset of the plane (or sphere) where the iterates of f form a normal family. It is easy to see that if U is a connected component of F, then $f(U)$ is contained in some connected component V of F. For rational functions, we have $f(U) = V$ (see [2, §5.4]). For transcendental entire functions, it is possible that $f(U) \neq V$. A simple example is given by $f(z) = \lambda e^z$, $0 < \lambda < e^{-1}$, where F is connected and $0 \in F \backslash f(F)$.

Theorem. *Let f be an entire transcendental function and let U and V be connected components of F satisfying $f(U) \subset V$. Then $V \backslash f(U)$ contains at most one point.*

To prove the theorem, let f, U, and V be as required and suppose that $V \backslash f(U)$ is not empty. It is easy to see that this implies that there exists a curve γ tending to ∞ in U such that $f(z)$ tends to a value in $V \backslash f(U)$ as $z \to \infty$ in γ. In particular, $f(z)$ is bounded on some curve tending to ∞. A result of Baker [1, §3] now implies that U and V are simply-connected. Hence there exist conformal maps φ and ψ from the unit disk D onto U and V. We define $g = \psi^{-1} \circ f \circ \varphi$ so that $g(D) \subset D$. Clearly, it suffices to prove that $D \backslash g(D)$ contains at most one point.

By a result of Beurling (see [4, Theorems 11.5 and 11.9]) there exists a set $A \subset [0, 2\pi]$ of capacity zero with the property that if $\theta \notin A$, then there exists $a_\theta \in \partial U \backslash \{\infty\}$ such that $\varphi(re^{i\theta}) \to a_\theta$ as $r \to 1$. It follows that $f(\varphi(re^{i\theta})) \to f(a_\theta) \in \partial V \backslash \{\infty\}$ and hence that $|g(re^{i\theta})| \to 1$ as $r \to 1$, provided $\theta \notin A$. A result of Lohwater (see [3, Theorem 5.14]) now implies that $D \backslash g(D)$ contains at most one point. This completes the proof of the theorem.

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