A NEW DUALITY THEOREM FOR SEMISIMPLE MODULES AND CHARACTERIZATION OF VILLAMAYOR RINGS

CARL FAITH AND PERE MENAL

(Communicated by Lance W. Small)

Abstract. We prove the theorem: If \( R \) is a ring whose right ideals satisfy the double annihilator condition with respect to a semisimple right \( R \)-module \( W \), then every right ideal is an intersection of maximal right ideals, consequently \( R \) is a right \( V \) (for Villamayor) ring, and \( W \) is then necessarily a cogenerator of \( \text{mod-} R \). (The converse is well known.) We use this to give a new proof of a theorem of ours on right Johns rings.

Introduction

A ring \( R \) is a right \( V \) ring provided that \( R \) satisfies the f.e.c.'s:

(V1) Every simple right \( R \)-module is injective.
(V2) Every right ideal \( I \) of \( R \) is an intersection of maximal right ideals, equivalently, \( R/I \) has zero Jacobson radical, i.e., \( \text{rad}(R/I) = 0 \).
(V3) Every right \( R \)-module \( M \) has zero Jacobson radical, i.e., \( \text{rad} M = 0 \).

Proof. See [F1], p. 356, Proposition 7.32A. \( \square \)

If \( W \) is a right \( R \)-module, we say that \( W \) satisfies the double annihilator condition (= d.a.c.) with respect to right ideals provided that

\[
I = \text{ann}_R \text{ann}_W I
\]

where \( \text{ann}_W S \) denotes the annihilator of a subset \( S \) of \( R \) in \( W \), and dually for \( \text{ann}_R S \) for a subset \( S \) of \( W \).

A characterization of \( V \)-rings

We now state the new characterization of \( V \)-rings.

\( V \)-Ring Theorem. A ring \( R \) is a right \( V \)-ring iff some semisimple right \( R \)-module satisfies the d.a.c. with respect to right ideals.

Proof. Sufficiency. It suffices to prove that every right ideal of \( I \) of \( R \) is the intersection of maximal right ideals. Let \( M = \text{ann}_W I \). Then by the d.a.c.

\[
I = \bigcap_{m \in M} \text{ann}_R m.
\]
Since \( mR \) is a semisimple \( R \)-submodule of finite length, then

\[ mR = v_1R \oplus \cdots \oplus v_nR \]

where \( v_i \in W \) and \( v_iR \) is simple, \( i = 1, \ldots, n \). Write

\[ m = v_1a_1 + \cdots + v_na_n \]

where \( a_i \in R \), \( i = 1, \ldots, n \); and let

\[ H = \bigcap_{i=1}^n \text{ann}_R v_i a_i. \]

We assert that

\[ H = \text{ann}_R m \]

is the intersection of maximal right ideals. For if \( v_ia_i \neq 0 \), then \( v_ia_iR = v_iR \)

is simple, and since

\[ v_iR \cong R/(\text{ann}_R v_i a_i), \]

then \( \text{ann}_R v_ia_i \) is a maximal right ideal, hence \( H \) is the intersection of maximal right ideals.

Next we show that (5) holds. Obviously, \( \text{ann}_R m \supseteq H \). To prove the reverse inclusion, note that if \( r \in \text{ann}_R m \), then

\[ v_1a_1r + \cdots + v_na_nr = 0. \]

By (2), then \( v_ia_ir = 0 \), hence \( r \in \text{ann}_R v_ia_i \) for all \( i \), that is, \( r \in H \), so (5) holds.

Since (4) is the intersection of maximal right ideals, then so is \( \text{ann}_R m \), whence \( I \) by (1).

\textbf{Necessity.} If \( R \) is a right \( V \)-ring, then the direct sum \( W \) of a complete isomorphic set of simple right \( R \)-modules is a minimal cogenerator of \( \text{mod-R} \).

(See, e.g., [F1], p. 167, Proposition 3.55. There is a misprint in this proposition; cf. [F2].) Furthermore, every cogenerator \( W \) satisfies the d.a.c. with respect to right ideals. (Hint: if \( I \) is a right ideal, then \( R/I \) embeds in a direct product \( W^a \) of copies of \( W \), say \( h : R/I \hookrightarrow W^a \). If \( h(1+I) = (w_i) \in W^a \), then \( I = \text{ann}_R \{w_i\}_{i \in a} \).)

\textbf{Corollary.} If a ring \( R \) satisfies the d.a.c. with respect to a semisimple right \( R \)-module \( W \), then \( W \) is a right cogenerator of \( R \).

\textit{Proof.} \( R \) is a right \( V \)-ring, so every simple right \( R \)-module \( V \) is injective, so it suffices to show that \( W \) contains a copy of each such \( V \). But \( V \cong R/M \), where \( M \subset R \), and by the d.a.c., \( M = \text{ann}_R w \) for some \( w \in W \). Since \( wR \cong V \), we have \( V \hookrightarrow W \) as needed.

\textbf{F-M Theorem 2.3 ([F-M1]).} If \( R \) is a right Johns ring (= right Noetherian and every right ideal is a right annihilator), then \( R/J \) is a right \( V \)-ring, where \( J \) is the Jacobson radical.

\textit{Proof.} Let \( W = \text{soc} R \). Then

\[ J = \overset{\perp}{W} = \overset{\perp}{W} \]

is nilpotent and

\[ W = J^\perp = \overset{\perp}{J} \]
where "⊥" denotes an annihilator in $R$ on the appropriate side. (See [F-M], Lemma 2.2.)

By the fact that $R_R$ satisfies the d.a.c. with respect to right ideals, and using (*) and (**), one sees that right ideals of $R$ containing $J$, hence right ideals of $R/J$, satisfy the d.a.c. with respect to the semisimple module $W$. Then by the $V$-Ring Theorem, we see that $R/J$ is a right $V$-ring.

**Corollary.** If $R$ is such that $W = \perp J$ is a semisimple right $R$-module and every right ideal containing $J$ is a right annihilator, then $R/J$ is a right $V$-ring.

**Acknowledgment**

We wish to acknowledge P. M. Cohn for raising a question about the proof of Theorem 2.3, in connection with his reviewing [F-M1], namely, given a finitely embedded (= f.e.) module $M$ (= has finite essential socle) that occurs in the proof, and given an embedding of $M$ in a direct product $S^n$ of copies of a module ($S = \text{soc } R_R$ in the proof), how does one conclude that $M$ embeds in a finite product $S^n$ of copies of $S$? This can be resolved as follows. Let $\{p_i\}_{i \in \alpha}$ denote the set of projections $S^\alpha \to S$ of the product $S^\alpha$. Since $\bigcap_{i \in \alpha} \ker p_i = 0$, and since $M$ is f.e., then for some finite subset $p_i, \ldots, p_{n_i}$ of the induced maps $p_i : M \to S$ we have $\bigcap_{i=1}^n \ker p_i = 0$. But then the direct sum of the $\{p_i\}_{i=1}^n$ is an embedding $M \to S^n$.

**References**


**Department of Mathematics, Rutgers University New Brunswick, New Jersey 08903**

**Permanent address:** 199 Longview Drive, Princeton, New Jersey 08540