SKEW POLYNOMIAL EXTENSIONS
OF COMMUTATIVE NOETHERIAN JACOBSON RINGS

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Abstract. The Jacobson condition (i.e., that all prime ideals are semiprimitive) is proved to pass from a commutative noetherian ring \( R \) to a skew polynomial ring \( R\{y; \tau, \delta\} \), assuming only that \( \tau \) is an automorphism.

1. Introduction

This note is concerned with the prime ideal structure of a skew polynomial ring \( S = R\{y; \tau, \delta\} \) over a noetherian ring \( R \) with respect to an automorphism \( \tau \) and a (left) \( \tau \)-derivation \( \delta \) (cf. [7]). An unanswered question in this setting is whether \( S \) must satisfy the Jacobson condition (i.e., every prime ideal is an intersection of primitive ideals) when \( R \) satisfies the same property. Some positive answers are known even for non-noetherian coefficient rings: Watters [15] proved that \( K\{y\} \) is Jacobson for any Jacobson ring \( K \), and Irving [9] showed that an iterated skew polynomial extension \( T \) of a commutative Jacobson ring \( K \) is Jacobson if \( K \) is central in \( T \) (see also [12]). On the other hand, examples have been constructed of non-noetherian commutative Jacobson rings \( K \) with skew polynomial extensions \( K\{y; \tau, \delta\} \) that are not Jacobson; see Pearson and Stephenson [14] for an example in which \( \delta = 0 \), and see Bergen, Montgomery, and Passman [1] or Ferrero and Kishimoto [3] for examples in which \( \tau = 1 \). Within the noetherian context, affirmative answers to the problem were given by Goldie and Michler [4] when \( \delta \) is trivial, and by Jordan [10] when \( \tau \) is the identity.

The aim of this note is to provide an affirmative answer to the above question when \( R \) is commutative noetherian but no restrictions are placed upon \( \tau \) or \( \delta \). Such a result has remained unavailable despite the thorough analyses of the commutative case by Irving [8] and the first author [5]. Our methods rely in part on the techniques introduced in [6] as well as on the results in [5]. Moreover, it is not assumed that \( R \) be filtered, graded, or affine.

We impose the blanket hypotheses throughout that \( R \) is a commutative noetherian ring, that \( S = R\{y; \tau, \delta\} \), and that \( \tau \) is an automorphism of \( R \). However, commutativity of \( R \) is not needed for (2.2) and (3.1).
2. INDUCED VS. NONINDUCED PRIME IDEALS

Throughout this section we let \( P \) denote an arbitrary prime ideal of \( S \). If \( A \) is a ring and \( I \) is an ideal of \( A \), then \( N(I) \) denotes the intersection of all the prime ideals containing \( I \) and \( J(I) \) the intersection of all the right primitive ideals containing \( I \). The reader is referred to [7, 13] for further explanations of undefined terms.

2.1. By [6, 5.3, 5.5], we may fix a prime ideal \( Q \) of \( R \), minimal over \( PnP \), that satisfies the following property: If \( A \) denotes the Goldie quotient ring of \( R/Q \), then \( P \) is the right annihilator in \( S \) of a nonzero \( A-S \)-bimodule factor \( M \) of \( A \otimes_R S \) such that \( M_{S/P} \) is torsionfree. Next, set \( U = S/QS \), and let \( e \) denote the coset \( 1+QS \). Observe that we may identify \( RU_S \) with \( (R/Q) \otimes_R S \) by an isomorphism that sends \( e \) to \( 1 \otimes 1 \), and under this identification we may view \( (R/Q)U \) as a free left \( (R/Q) \)-module with basis

\[ \{1 \otimes 1, 1 \otimes y, 1 \otimes y^2, \ldots \}. \]

Also, observe that as a left \( R \)-module, \( A \otimes_R S \) is isomorphic to an Ore localization of \( RU \).

It follows from the above choice of \( Q \) that \( \text{ann} \, U_S \subseteq P \), since \( \text{ann} \, U_S = \text{ann}(A \otimes_R S)_S \). Our analysis divides into the two cases determined by whether or not \( P = \text{ann} \, U_S \), and we begin with an incomparability result.

2.2. Lemma. (Here \( R \) need not be commutative.) Suppose that \( J \) is an ideal of \( S \) properly containing \( P \). If \( P \not\subseteq \text{ann} \, U_S \), then \( J \cap P \subseteq Q \).

Proof. By [6, 4.6], the set \( \mathcal{C} \) of regular elements of \( R/(P \cap R) \) forms an Ore set (of regular elements of \( S/P \)) in both \( R/(P \cap R) \) and \( S/P \), and the ring \( E = (R/(P \cap R))^{\mathcal{C}^{-1}} \) is artinian. Letting \( F = (S/P)^{\mathcal{C}^{-1}} \), we see that the canonical embedding of \( R/(P \cap R) \) into \( S/P \) extends uniquely to an embedding of \( E \) into \( F \). Now choose an ideal \( I \) of \( S \) that contains \( P \) and is maximal among those ideals of \( S \) whose intersection with \( R \) lies within \( Q \). Standard arguments reveal that \( I \) is a prime ideal of \( S \) disjoint from \( \mathcal{C} \). Consequently, if \( I \) strictly contains \( P \), then \( I \) extends to a proper nonzero ideal of \( F \) (e.g., [7, 9.22]). Next, it follows from [6, 5.7, 5.8] that \( F_E \) is finitely generated when \( P \not\subseteq \text{ann} \, U_S \). However, if \( F \) has finite length as a right \( E \)-module, then \( F \) is a simple artinian ring. Therefore, \( I = P \) and the lemma follows. \( \square \)

2.3. Lemma. \((P + QS) \cap R = Q \).

Proof. We may assume without loss of generality that \( P \cap R \neq Q \), and it therefore follows from the minimality of \( Q \) that \( P \cap R \) is not prime. Moreover, it suffices to prove that \((P + QS) \cap R \subseteq Q \). Next, by [5, 3.1], either \( P \cap R \) is semiprime or \( R/(P \cap R) \) has a unique associated prime. We first consider the case where \( R/(P \cap R) \) is semiprime, and we let \( Q, Q_2, \ldots, Q_n \) be the distinct prime ideals of \( R \) minimal over \( P \cap R \). Note that \( n \geq 2 \) and \( Q_nQ_{n-1} \cdots Q_2Q \subseteq P \cap R \). Hence,

\[ Q_nQ_{n-1} \cdots Q_2 [(P + QS) \cap R] \subseteq P \cap R \subseteq Q. \]

Since \( Q_nQ_{n-1} \cdots Q_2 \not\subseteq Q \), it follows that \((P + QS) \cap R \subseteq Q \) in this case.
Now assume that \( R/(P \cap R) \) has a unique associated prime. Consequently, \( Q \) is the unique prime ideal of \( R \) minimal over \( P \cap R \) and \( \mathcal{E}_R(Q) \subseteq \mathcal{E}_R(P \cap R) \). Therefore, \( \mathcal{E}_R(Q) \subseteq \mathcal{E}_S(P) \) by [6, 4.6]. Hence, if there exists an element \( c \in (P + QS) \cap (R \setminus Q) \), then \( c \in \mathcal{E}_S(P) \). Next observe that there exists a positive integer \( n \) such that \( Q^n \subseteq P \cap R \) while \( Q^{n-1} \not\subseteq P \cap R \). However, it now follows that \( Q^{n-1}c \subseteq Q^{n-1}(P + QS) \subseteq P \), in contradiction to the regularity of \( c \) modulo \( P \). Therefore, \((P + QS) \cap R \subseteq Q \) and the lemma follows. □

The proof of the following proposition is adapted from [4, 10].

2.4. **Proposition.** If \( P \neq \text{ann} \ U_S \) and \( Q \) is semiprimitive, then \( P \) is semiprimitive.

**Proof.** For \( t = 0, 1, \ldots \) set
\[
K_t = \{ a \in R \mid e.(ay^t + a_{t-1}y^{t-1} + \cdots + a_0) \in UP \text{ for some } a_0, \ldots, a_{t-1} \in R \}
\]
\[
= \{ a \in R \mid ay^t + a_{t-1}y^{t-1} + \cdots + a_0 \in P + QS \text{ for some } a_0, \ldots, a_{t-1} \in R \}.
\]
Then let \( K = K_n \), where \( n \) is the minimum value for \( t \) such that
\[
0 \neq e.(ay^t + a_{t-1}y^{t-1} + \cdots + a_0) \in UP
\]
for some \( a_0, \ldots, a_t \in R \). (The existence of \( n \) follows from the assumption that \( P \neq \text{ann} \ U_S \).) Note, since \( \tau \) is an automorphism, that \( K \) is an ideal of \( R \) containing \( Q \), and observe, for \( a \in K \), that \( a \not\subseteq Q \) if and only if \( 0 \neq e.(ay^n + a_{n-1}y^{n-1} + \cdots + a_0) \in UP \) for some \( a_0, \ldots, a_{n-1} \in R \). In particular, \( K \) properly contains \( Q \). Moreover, since \((P + QS) \cap R \subseteq Q \) by (2.3), it follows that \( n \geq 1 \).

Now let \( M \) be a maximal ideal of \( R \) that contains \( Q \). We claim that either \( J(P) \cap R \subseteq M \) or \( K \subseteq M \). To prove this claim, assume that \( J(P) \cap R \not\subseteq M \). Choose \( j \in J(P) \cap R \) such that \( j \not\subseteq M \). Then there exist \( m \in M \) and \( b \in R \) such that \( 1 = m + jb \). Since \( jb \in J(P) \), there exists a polynomial \( f = cy^k + c_{k-1}y^{k-1} + \cdots + c_0 \in S \), with \( c, c_0, \ldots, c_{k-1} \in R \) and \( c \neq 0 \), such that \( (1 - jb)f = mf = 1 \) (mod \( P \)). Hence, \( e.mf \equiv e \) (mod \( UP \)). Now choose \( a \in K \setminus Q \). Then there exists a polynomial \( p = ay^n + a_{n-1}y^{n-1} + \cdots + a_0 \in S \), with \( a_0, \ldots, a_{n-1} \in R \), for which \( 0 \neq e.p \in UP \). Assume for the moment that \( \ell \geq n \), and observe that
\[
a.f - p\tau^{-n}(c)y^{\ell-n}
\]
has degree less than \( \ell \). It now follows from a straightforward induction that \( e.a^k f \equiv e.r \) (mod \( UP \)) for some nonnegative integer \( k \) and some polynomial \( r \in S \) with degree \( d < n \). Hence, we have
\[
e.a^k mf = e.m.e.a^k f \equiv m.e.r = e.mr \pmod{UP},
\]
and since \( a^k mf \equiv a^k \) (mod \( P \)), we see that \( e.a^k \equiv e.mr \) (mod \( UP \)). Consequently, \( e.(a^k - mr) \in UP \). However, \( a^k - mr \) has degree strictly less than \( n \). Therefore, it follows from the choice of \( n \) that \( e.(a^k - mr) = 0 \). Hence, \( a^k - mr_0 \in Q \), where \( r_0 \) is the constant term of \( r \). But this last statement implies that \( a^k \in M \), because \( Q \subseteq M \). Thus \( a \in M \), and it therefore follows from the choice of \( a \) that \( K \subseteq M \). This verifies the claim. Furthermore, it follows from the claim that \( J(P) \cap K \subseteq M \). Because \( M \) was an arbitrary maximal ideal of \( R \) containing \( Q \), we now see that \( J(P) \cap K \subseteq J(Q) = Q \).
But this inclusion means that \( J(P) \cap R \subseteq Q \), since \( K \not\subseteq Q \). Thus by (2.2), \( J(P) = P \), and the lemma is proved. □

2.5. **Lemma.** Assume that \( P = \text{ann} U_S \). Then \( P \cap R \) is \((\tau, \delta)\)-prime, and \( P = (P \cap R)S = S(P \cap R) \). Consequently, if \( \tau \) and \( \delta \) also denote their induced actions on \( R/(P \cap R) \), and \( y \) also denotes its image in \( S/P \), then \( S/P = (R/(P \cap R))[y; \tau, \delta] \).

**Proof.** Set \( I = P \cap R \). It follows from [6, 5.9ii] that there exists an \( n \in \mathbb{N} \) such that \( \tau^n(Q) = Q \) and such that \( \{Q, \tau(Q), \ldots, \tau^{n-1}(Q)\} \) is the set of prime ideals of \( R \) minimal over \( P \cap R \). In particular, \( N(I) \) is \( \tau \)-stable. Now suppose that \( I = Q \). Then \( I \) is \( \tau \)-stable and therefore \( (\tau, \delta) \)-stable (e.g., [6, 2.1v]). Hence, \( IS = SI \), and \( P = \text{ann}(S/IS)_S = IS \). Further, it is a triviality that \( I \) is \((\tau, \delta)\)-prime. Next, assume that \( I \neq Q \). Consequently, \( I \) is not a prime ideal, and so \( I \) is a \((\tau, \delta)\)-prime ideal by [5, 3.1]. It therefore follows from [5, 3.3] that \( P_0 = IS = SI \) is a prime ideal of \( S \). Moreover, \( P_0 \subseteq P \) and \( P_0 \cap R = P \cap R = I \).

Because \( Q \) is minimal over \( I \), and \( R \) is commutative, it follows that \( Q \) is an annihilator prime of \( (R/I)_R \). In particular, \( Q \) is an annihilator prime of \( (S/P_0)_R \). Hence, by [6, 5.5], \( P_0 \supseteq \text{ann} U_S = P \). The lemma follows. □

2.6. **Lemma.** Suppose that \( Q \) is a maximal ideal of \( R \) and that \( S/P \) is artinian. Then \( S/P \) has finite length as a right \( R \)-module.

**Proof.** First, it follows from [6, 4.4] that every prime ideal of \( R \) minimal over \( P \cap R \) is maximal, and so \( R/(P \cap R) \) is artinian. Therefore, if \( P \neq \text{ann} U_S \), the desired conclusion follows from [6, 5.9i]. Now assume that \( P = \text{ann} U_S \). Therefore, by (2.5), we may assume without loss of generality that \( P = 0 \). But then \( y \) is a regular noninvertible element of \( S \), a contradiction to the fact that \( S \) is artinian (e.g., [13, 3.1.1]). □

3. **Induced bimodules**

Chapter 5 of [6] contains an extensive analysis of the prime ideals of \( S \) that occur as annihilators of factors of bimodules of the form \( A \otimes_R S \) where \( A \) is the Goldie quotient ring of a prime factor ring of \( R \). We shall need one element of the corresponding analysis of bimodule subfactors of \( A \otimes_R S \), as follows. In the case of a bimodule factor, this lemma is a consequence of [6, 5.4, 5.5].

3.1. **Lemma.** (Here \( R \) need not be commutative.) Let \( P \) be a prime ideal of \( S \) and \( Q \) a prime ideal of \( R \), and let \( A \) denote the Goldie quotient ring of \( R/Q \). Further assume that \( P \) is the right annihilator in \( S \) of an \( A \)-\( S \)-bimodule subfactor \( K \) of \( A \otimes_R S \) that is torsionfree as a right \((S/P)\)-module. Then every prime ideal in \( R \) minimal over \( P \cap R \) belongs to the \( \tau \)-orbit of \( Q \).

**Proof.** Choose a nonzero element \( \ell \in K \) and let \( L = A.\ell \cdot R \). It follows from [6, 4.6] that \( R/(P \cap R) \) has an artinian quotient ring and that every regular element of \( R/(P \cap R) \) is regular in \( S/P \). Hence, \( L \) is torsionfree as a right \((R/(P \cap R))\)-module, and by Small's Theorem (e.g., [7, 10.10]) and [7, 6.3], it follows that every annihilator prime of \( L \) is minimal over \( P \cap R \). We leave to the reader the verification that \( L \) has finite length as a left \( A \)-module. Now choose a simple \( A \)-\( R \)-sub-bimodule \( M \) of \( L \). The right annihilator in \( R \) of \( M \)
is a prime ideal, say $Q'$, and we have just seen that $Q'$ must be minimal over $P \cap R$. However, it follows from the proof in [6, 4.4] that $\mathcal{M}$ is isomorphic to $A^\ell$ as an $A$-$P$-bimodule, for some positive integer $\ell$. (As a left $A$-module, $A^\ell$ has the same structure as $A$, but the right $R$-module structure is defined by the operation $a \cdot r = a \tau^\ell(r)$, for every $r \in R$ and $a \in A$.) It therefore follows that $Q' = \tau^{-\ell}(Q)$, and the desired conclusion now follows from [6, 4.4]. □

3.2. Proposition. Let $M$ be a maximal ideal of $R$. Then the right annihilator in $S$ of $S/MS$ is prime.

Proof. Set $V = S/MS = (R/M) \otimes_R S$, and let $P$ denote a maximal annihilator prime of $V_S$. It follows from [6, 5.6iv] that $V$ is uniform as an $R$-$S$-bimodule, and is therefore easy to verify that every annihilator prime of $V_S$ is contained in $P$. If $P = \text{ann } V_S$, then there is nothing to prove, and so we suppose otherwise. Next, let $0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = V$ be an affiliated series for $V$ (see, e.g., [7, p. 33]), where $n > 1$, and set $P_i = \text{ann}(V_i/V_{i-1})_S$ for $1 \leq i \leq n$. (Note that $P_1 = P$.) If $i > 1$, it follows from [6, 5.6iii] that $V_i/V_{i-1}$ has finite length as a left $R$-module. It therefore can be deduced from Lenagan's Theorem (e.g., [7, 7.10]) that $(V_i/V_{i-1})_S$ has finite length for $i > 1$. However, it now follows from [7, 7.2] that $S/P_i$ is an artinian ring. In particular, each $V_i/V_{i-1}$ is torsionfree as a right $(S/P_i)$-module, and in view of (3.1), the prime ideals of $R$ minimal over $P \cap R$ are therefore maximal ideals. We may now conclude from (2.6) that each $S/P_i$ has finite length as a right $P$-module for $i > 1$.

We next prove that $S/P = S/P_1$ is artinian and has finite length as a right $R$-module. If $P_2$ is an annihilator prime of $V_S$, then $P_2 \subseteq P$ and there is nothing to prove. So we may assume otherwise. It then follows from [11, 1.2] that there is a series of links (e.g., [7, p. 178]) from $P_2$ to some annihilator prime $P'$ of $V_S$. However, it now follows from [7, 7.2, 7.10] that $P'$ is coartinian. Hence $P = P'$ is coartinian, because $P' \subseteq P$. Next, it follows from (3.1) that every prime ideal of $R$ minimal over $P \cap R$ is a maximal ideal. Thus $S/P$ has finite length as a right $R$-module by (2.6).

To conclude, it now follows that $V_i/V_{i-1}$ has finite length as a right $R$-module for all $1 \leq i \leq n$. But we are now forced to conclude that $V_R$ has finite length, an absurdity. The lemma follows. □

4. Ascendancy of the Jacobson condition

4.1. Lemma. Assume that $R$ is artinian and $(\tau, \delta)$-prime. Then $S$ is a Jacobson ring.

Proof. First, it follows from [5, 2.3] and [4, 5*] that $R$ is $(\tau, \delta)$-simple. Also, $R$ is a Jacobson ring, and so by [10, 3.5] we may assume that $\tau$ is not the identity. Now assume that $R$ is $\tau$-prime. Then it follows from [5, 3.7] that $\delta$ is inner, and so the desired conclusion follows from [4, 1.11*] and, for example, [5, 1.5c]. It remains to consider the case that $R$ is not $\tau$-prime. Therefore, by [5, 2.6], $R$ is $\delta$-prime and has a unique maximal ideal $M$. From [5, 2.6, 4.6] it follows that $S$ contains a subring $A = (R/M)[y'; \delta']$, where $y' \in S$ and $\delta'$ is a derivation of $R/M$, and it follows from [10, 3.5] that $A$ is a Jacobson ring. It is proved in [5, 4.6] that $S$ is finitely generated as a left $A$-module. Therefore, $S$ is a Jacobson ring by [2, Theorem 1]. □
Recall that a prime ideal \( P \) of \( S \) is said to \textit{lie over} a prime ideal \( Q \) of \( R \) when \( Q \) is minimal over \( P \cap R \).

4.2. \textbf{Lemma.} Assume that there exists a maximal ideal \( M \) of \( R \) such that the module \( V = (S/MS)_S \) is faithful. Then \( S \) is semiprimitive.

\textit{Proof.} First suppose that \( M \) is minimal. By (3.2), \( S \) is prime, and so by [6, 5.12], the minimal prime ideals of \( R \) are all contained within a single \( \tau \)-orbit. Therefore, all minimal prime ideals of \( R \) are maximal, and so \( R \) is artinian. Moreover, because \( S \) is prime, and because nonzero \((\tau, \delta)\)-ideals of \( R \) induce to nonzero ideals of \( S \), it follows that \( R \) is \((\tau, \delta)\)-prime. Hence, by (4.1), \( S \) is semiprimitive. Thus we may assume that \( M \) is not minimal.

Next, suppose that \( \tau(M) = M \). Since \( V_S \) is faithful, \( MS \) cannot be an ideal of \( S \), and so \( M \) is not \( \delta \)-stable. Thus no ideal of \( S \) contracts to \( M \); see [6, 2.1v]. Now suppose that \( N \) is a prime ideal of \( S \) lying over \( M \). From the preceding observation it follows that \( N \cap R \neq M \), and so \( I = N \cap R \) must be a \((\tau, \delta)\)-prime ideal of \( R \) by [5, 3.1]. Moreover, our assumption that \( M \) not be a minimal prime ideal of \( R \) guarantees that \( I \neq 0 \). Hence \( IS \) is a nonzero ideal of \( S \) contained in \( MS \), a contradiction to the faithfulness of \( V_S \). Thus, no prime ideal of \( S \) lies over \( M \). It therefore follows from [6, 5.7] that there exist no proper simple \( R\)-\( S \)-bimodule factors of \( V \), and so \( gV_S \) is a simple \( R \)-module. It is now straightforward to prove as follows that \( S \) is right primitive: Let \( K \) be a maximal right \( S \)-submodule of \( V \), and let \( J = \text{ann}(V/K)_S \). Then \( VJ \) is a \( R\)-\( S \)-sub-bimodule of \( V \) that is not equal to \( V \). Hence \( VJ = 0 \), and so \( J = 0 \) by the faithfulness of \( V_S \). Therefore \( V/K \) is a faithful simple right \( S \)-module.

Finally, assume that \( \tau(M) \neq M \). Let \( L = \bigcap_{i \in \mathbb{Z}} \tau^i(M) \), and note that \( L \) is a semiprime, \( \tau \)-prime ideal. By [5, 3.1], for each \( i \in \mathbb{Z} \) there exists a prime ideal of \( S \) contracting to \( \tau^i(M) \). Hence, there exists an ideal of \( S \) contracting to \( L \), and it follows, for example, from [6, 2.1v] that \( L \) is \((\tau, \delta)\)-stable. Therefore, \( LS = SL \) is an ideal of \( S \) contained within \( MS \), and so \( LS = 0 \) because \( V_S \) is faithful. Consequently, \( L = 0 \), and hence \( R \) is a semiprime, \( \tau \)-prime ring.

To conclude, let \( J = J(S) \), and suppose that \( J \neq 0 \). Note that the set of leading coefficients of elements of \( J \), together with 0, namely the set \[
\{ a \in R \mid ay^t + a_{t-1}y^{t-1} + \cdots + a_0 \in J \text{ for some } a_0, \ldots, a_{t-1} \in R \},
\]
is a nonzero \( \tau \)-ideal of \( R \). This ideal must contain a regular element since \( R \) is \( \tau \)-prime, and therefore there exists a polynomial \( f \in J \) with positive degree and regular leading coefficient. Since \( 1 + f \) is a unit, there exists another polynomial \( g \) such that \((1 + f)g = 1 \). But the degree of \((1 + f)g \) is certainly greater than zero, by the regularity of the leading coefficient of \( f \), and we thus obtain a contradiction. Hence, \( J = 0 \), and the lemma follows. \( \square \)

4.3. \textbf{Theorem.} Assume that \( R \) is a commutative noetherian Jacobson ring. Then the skew polynomial ring \( S = R[y; \tau, \delta] \) is a Jacobson ring.

\textit{Proof.} Suppose that the theorem is false, and let \( P \) denote a maximally chosen nonsemiprimitive prime ideal of \( S \). As in (2.1), we may select a prime ideal \( Q \) of \( R \) such that \( Q \) is minimal over \( P \cap R \) and such that \( P \) is the annihilator in \( S \) of an \( A\)-\( S \)-bimodule factor of \( A \otimes_R S \), where \( A \) is the field of fractions for \( R/Q \). If \( P \neq \text{ann}(S/QS)_S \), then \( P \) is semiprimitive, by (2.4). Therefore,
by (2.5), we may assume without loss of generality that $P = 0$. Furthermore, $Q$ is equal to the intersection of those maximal ideals of $R$ that contain it. In particular,

$$QS = \bigcap \{ MS \mid M \in \text{max } R \text{ and } M \supseteq Q \}.$$ 

Therefore,

$$0 = \text{ann}(S/QS)_S = \bigcap \{ \text{ann}(S/MS)_S \mid M \in \text{max } R \text{ and } M \supseteq Q \}.$$ 

Next, it follows from the above equalities and (3.2) that if there exists no maximal ideal $M$ in $S$ such that $M \supseteq Q$ and $(S/MS)_S$ is faithful, then some intersection of nonzero prime ideals in $S$ is equal to zero, a contradiction to the fact that each nonzero prime ideal of $S$ is semiprimitive. Thus, there exists a maximal ideal $M$ in $R$ such that $(S/MS)_S$ is faithful. Therefore, it follows from (4.2) that $S$ is semiprimitive, a contradiction to our hypothesis. The theorem follows. □

4.4. **A question of Small.** A possible generalization of the preceding theorem would include the replacement of the commutativity hypothesis with the assumption that $R$ satisfy a polynomial identity. L. W. Small has informed us of his unpublished proof that if $R$ is an affine PI algebra over a $(\tau, \delta)$-constant field $k$, then $S[u, v] = R[y; \tau, \delta][u][v]$ is generically flat over $k[u]$, and consequently, $S$ is a Jacobson ring (cf. [13, 9.3.13]). Small further raises the following question: If $T$ is a filtered noetherian ring such that gr $T$ is Jacobson, must $T$ also be Jacobson? (We thank L. W. Small for the remarks discussed here.)

**Note added in proof (December 1994)**

A. D. Bell has communicated two counterexamples to Small's question; however, in one example the filtration is a $\mathbb{Z}$-filtration, while in the other, gr $T$ is not noetherian. The following modification of Small's question remains open: If $T$ is a nonnegatively filtered noetherian ring such that gr $T$ is Jacobson and noetherian, must $T$ be noetherian?

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