MAYER-VIETORIS FORMULA
FOR THE DETERMINANT OF A LAPLACE OPERATOR
ON AN EVEN-DIMENSIONAL MANIFOLD

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Abstract. Let $\Delta$ be a Laplace operator acting on differential $p$-forms on an even-dimensional manifold $M$. Let $\Gamma$ be a submanifold of codimension 1. We show that if $B$ is a Dirichlet boundary condition and $R$ is a Dirichlet-Neumann operator on $\Gamma$, then $\det(\Delta + \lambda) = \det(\Delta + \lambda, B) \det(R + \lambda)$ and $\det^* \Delta = \frac{1}{\det(A)} \det(\Delta, B) \det^* R$. This result was established in 1992 by Burghelea, Friedlander, and Kappeler for a 2-dimensional manifold with $p = 0$.

1. Introduction

Let $M$ be a compact oriented Riemannian manifold of dimension $d$, and let $\Gamma$ be a submanifold of $M$ with dimension $d-1$ such that $\Gamma$ has a collared neighborhood $U$ diffeomorphic to $\Gamma \times (-1, 1)$. Let $M_{\Gamma}$ be the compact manifold with boundary $\Gamma \cup \Gamma$ obtained by cutting $M$ along $\Gamma$. Let $E = \Lambda^p T^* M$ be a $p$-th exterior product of the cotangent bundle $T^* M$, $i: M_{\Gamma} \to M$ be the identification map, and $E_{\Gamma} := i^* E$.

Define the Dirichlet boundary condition $(\Delta + \lambda, B)$ to be

$$(\Delta + \lambda, B): C^{\infty}(M_{\Gamma}, E_{\Gamma}) \to C^{\infty}(M_{\Gamma}, E_{\Gamma}) \oplus C^{\infty}(\partial M_{\Gamma}, E_{\Gamma}|_{\partial M_{\Gamma}}),$$

$\omega \mapsto (\Delta + \lambda)\omega, \omega|_{\partial M_{\Gamma}}$.

Define the Poisson operator $P_B$ to be the restriction of $(\Delta + \lambda, B)^{-1}$ to $0 \oplus C^{\infty}(\partial M_{\Gamma}, E_{\Gamma}|_{\partial M_{\Gamma}})$. Let $\nu$ be a unit normal vector field along $\partial M_{\Gamma}$; one can extend $\nu$ to be a global vector field on $M_{\Gamma}$ by using a cut-off function. Define the Neumann boundary condition $C$ to be

$C: C^{\infty}(M_{\Gamma}, E_{\Gamma}) \to C^{\infty}(\partial M_{\Gamma}, E_{\Gamma}|_{\partial M_{\Gamma}})$,

$\omega \mapsto \nabla_\nu \omega|_{\partial M_{\Gamma}}$.

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**Definition.** For any positive real number $\lambda > 0$, define $R(\lambda)$ to be the composition of the following maps:

$$C^\infty(\Gamma, E|_\Gamma) \xrightarrow{\Delta_{ia}} C^\infty(\Gamma, E|_\Gamma) \oplus C^\infty(\Gamma, E|_\Gamma) \oplus 0$$

$$\xrightarrow{\Delta_{ij}} C^\infty(M|_\Gamma, E|_\Gamma) \subset C^\infty(\Gamma, E|_\Gamma) \oplus C^\infty(\Gamma, E|_\Gamma)$$

$$\xrightarrow{\Delta_j} C^\infty(\Gamma, E|_\Gamma),$$

where $\Delta_{ia}$ is the diagonal inclusion and $\Delta_{ij}$ is the difference map.

Then $R(\lambda)$ is a positive definite selfadjoint elliptic operator. When $\lambda = 0$, both the Laplacian $\Delta$ and $R$ have zero eigenvalues and so $\det \Delta = \det R = 0$. In this case we define the modified determinants $\det^* \Delta$ and $\det^* R$ to be the determinants of $\Delta$ and $R$ respectively, when restricted to the orthogonal complement of the null space.

In [BFK], Burghelea, Friedlander, and Kappeler proved that on a 2-dimensional manifold and for the trivial line bundle $E = \Lambda^0 T^* M$,

1. $\det(\Delta + \lambda) = \det(\Delta + \lambda, B) \det R(\lambda)$ for $\lambda > 0$,
2. $\det^* \Delta = \frac{V}{l} \det(\Delta, B) \det^* R$,

where $V$ is the area of the manifold and $l$ is the length of $\Gamma$.

Let $\mathbb{H}_p$ be the space of harmonic $p$-forms equipped with the natural inner product $\langle \varphi, \psi \rangle = \int_M \varphi \wedge^* \psi = \int_M \langle \varphi, \psi \rangle d\text{vol}(M)$, where $(\ , \ )$ is a metric in $E = \Lambda^p T^* M$ induced by the Riemannian metric $g$ on $M$. Let $\mathbb{H}_p|_\Gamma$ be the restriction of harmonic $p$-forms to $\Gamma$. Define an inner product on $\mathbb{H}_p|_\Gamma$ by $\langle \alpha, \beta \rangle_\Gamma = \int_\Gamma (\alpha, \beta) d\mu_\Gamma$, where $d\mu_\Gamma$ is a volume element of $\Gamma$ coming from $g$ restricted to $\Gamma$.

Suppose $k = \dim \mathbb{H}_p$, and let $\psi_1, \ldots, \psi_k$ be an orthonormal basis of $\mathbb{H}_p$ and $\phi_1, \ldots, \phi_k$ be an orthonormal basis of $\mathbb{H}_p|_\Gamma$. Let $J : \mathbb{H}_p \to \mathbb{H}_p|_\Gamma$ denote the restriction map. Let $J(\psi_j) = a_{ij} \phi_j$ and let $A = (a_{ij})_{1 \leq i, j \leq k}$. In this paper we extend the result of Burghelea et al. to arbitrary even dimensions and arbitrary $p$-forms.

If $M$ is a compact oriented Riemannian manifold of dimension $d$ with $d$ even and $E = \Lambda^p T^* M$, then

**Theorem A.** $\det(\Delta + \lambda, B) = \det(\Delta + \lambda, B) \det R(\lambda)$ for any $\lambda > 0$.

**Theorem B.** $\det^* \Delta = \frac{1}{(\det A)^2} \det(\Delta, B) \det^* R$.

**Remark.** If $p = 0$, then $E = M \times R$, and the matrix $A$ is $(\sqrt{V}/\sqrt{l})$. Hence Theorem B reduces to

$$\det^* \Delta = \frac{V}{l} \det(\Delta, B) \det^* R,$$

as stated in [BFK].

**II. The proof of Theorem A**

In [BFK], it is shown that

$$\det(\Delta + \lambda) = c \det(\Delta + \lambda, B) \cdot \det R(\lambda),$$
and that $\log \det(\Delta + \lambda)$, $\log \det(\Delta + \lambda, B)$, and $\log \det R(\lambda)$ admit asymptotic expansions;

$$\log \det(\Delta + \lambda) \sim \sum_{k=-d}^{\infty} \alpha_k |\lambda|^{-k/2} + \beta_0 \log |\lambda|, \quad \text{with} \quad \alpha_0 = 0,$$

$$\log \det R(\lambda) \sim \sum_{j=-d}^{\infty} \pi_j |\lambda|^{-j/2} + \sum_{j=0}^{d} q_j |\lambda|^{j/2} \log |\lambda|, \quad \text{with}$$

$$\pi_0 = \sum_{j} \frac{1}{\partial_s (2\pi)^{d-1}} \int_{R^{d-1}} J_{d-1}(s, \lambda; x) \varphi_j(x)|_{s=0} \, d\text{vol}(x),$$

where

$$J_{d-1} = \frac{1}{2\pi i} \int_{R^{d-1}} d\xi \int_{\gamma} \mu^{-s} r_{-1-(d-1)} \left( \mu, \frac{\lambda}{|\lambda|}, x, \xi \right) \, d\mu,$$

$$r_{-1} = (\mu - p_1(\lambda, x, \xi))^{-1},$$

$$r_{-1-j} = - (\mu - p_1(\lambda, x, \xi))^{-1} \cdot \sum_{k=0}^{j-1} \sum_{|\alpha|+l=j-k} \frac{1}{\alpha!} \frac{\partial^\alpha}{\partial^\alpha} p_{i-l}(\lambda, x, \xi) D^\alpha r_{-1-k}(\mu, \lambda, x, \xi)$$

$$\sigma(R(\lambda)) \sim p_1 + p_0 + p_{-1} + \cdots \quad \text{asymptotic symbol of} \quad R(\lambda), \quad \{\varphi_j\} \quad \text{a partition of unity subordinate to coordinate charts, and} \quad \gamma \quad \text{is a curve on a complex plane enclosing all the eigenvalues of} \quad R(\lambda) \quad \text{counterclockwise.}$$

Hence

$$\log c = -\pi_0.$$

The proof of Theorem A reduces to the verification of the following equation:

$$p_{1-j}(x, -\xi, \lambda) = (-1)^j p_{1-j}(x, \xi, \lambda).$$

Then $r_{-1-j}(\mu, \frac{\lambda}{|\lambda|}, x, -\xi) = (-1)^j r_{-1-j}(\mu, \frac{\lambda}{|\lambda|}, x, \xi)$, so when $d$ is even, $r_{-1-(d-1)}(\mu, \frac{\lambda}{|\lambda|}, x, \xi)$ is odd with respect to $\xi$. So $J_{d-1} = 0$ and $\pi_0 = 0$. Therefore we conclude

$$\det(\Delta + \lambda) = \det(\Delta + \lambda, B) \det R(\lambda).$$

**Definition.** Let $U$ be a collared neighborhood of $\Gamma$ diffeomorphic to $\Gamma \times (-1, 1)$ with diffeomorphism $\eta: U \rightarrow \Gamma \times (-1, 1)$. Let $\Gamma_t = \eta^{-1}(\Gamma \times t)$, $-1 < t < 1$. Let $N^+_t$, $N^-_t$ be Neumann operators to each side with respect to $\Delta + \lambda$; i.e. if $\varphi \in C^\infty(\Gamma_t, E|_{\Gamma_t})$, define $N^+_t(\varphi) = \nabla_{\nu_t} u|_{\Gamma_t}$, where $(\Delta + \lambda) u = 0$ in $M - \Gamma_t$, $u|_{\Gamma_t} = \varphi$, and $\nu_t$ is a normal vector field along $\Gamma_t$.

Then

$$R(\lambda) = -(N^+_0 + N^-_0).$$

**Lemma 1.** In a local coordinate system such that the first fundamental form looks like

$$\begin{pmatrix} g_{ij}(x, t) & 0 \\ 0 & 1 \end{pmatrix}$$
on $\Gamma \times (-1, 1)$, the Laplacian is $\Delta = -\frac{d^2}{dt^2} + F(x, t)\frac{dt}{dt} + \Delta_t$, where $\Delta_t$ is the Laplacian on $\Gamma_t$ and $F(x, t)$ is a $C^\infty$-function valued $(d) \times (d)$ matrix. Then

$$\frac{dN_t^+}{dt} = -(N_t^+)^2 + F(x, t)N_t^+ + (\Delta_t + \lambda),$$

$$\frac{dN_t^-}{dt} = (N_t^-)^2 + F(x, t)N_t^- - (\Delta_t + \lambda).$$

**Remark.** The idea to consider the Neumann operator as a solution of operator-valued differential equations goes back to I. M. Gel’fand.

**Proof.** It is enough to show that the first statement is true. Let $\phi \in C^\infty(\Gamma_t, E|\Gamma_t)$. Choose $u(x, t) \in C^\infty(M_{\Gamma_t}, E_{\Gamma_t})$ such that $(\Delta + \lambda)u(x, t) = 0$ on $M - \Gamma_t$ and $u(x, t)|_{\Gamma_t} = \phi$. Then

$$\frac{d}{dt}u(x, t) = N_+^+(u(x, t)),$$

$$\frac{d^2}{dt^2}u(x, t) = \frac{d}{dt}(N_+^+(u(x, t))) = \frac{dN_+^+}{dt}(u(x, t)) + N_+^+ \left( \frac{du}{dt} \right) = \left( \frac{dN_+^+}{dt} + (N_+^+)^2 \right)u(x, t), \quad \text{and}$$

$$\frac{d^2}{dt^2}u(x, t) = F(x, t)\frac{du}{dt} + (\Delta_t + \lambda)u(x, t) = (F(x, t)N_t^+ + \Delta_t + \lambda)u(x, t).$$

Hence $\frac{dN_t^+}{dt} + (N_t^+)^2 = F(x, t)N_t^+ + (\Delta_t + \lambda)$, so

$$\frac{dN_t^+}{dt} = -(N_t^+)^2 + F(x, t)N_t^+ + (\Delta_t + \lambda).$$

Let

$$\sigma(N_t^+) \sim \alpha_1 + \alpha_0 + \cdots + \alpha_{1-i} + \cdots,$$

$$\sigma(N_t^-) \sim \beta_1 + \beta_0 + \cdots + \beta_{1-i} + \cdots,$$

$$\sigma(\Delta + \lambda) \sim (\sigma_2 + \lambda) + \sigma_1 + \sigma_0.$$

Note that

$$\sigma_2 + \lambda = \left( \sum_{ij=1}^{d-1} g^{ij} \xi_\omega \xi_j + \lambda \right) Id,$$

$$\sigma((N_t^+)^2) \sim \sum_{k=0}^{\infty} \sum_{i,j \geq 0} \frac{1}{\omega_i \omega_j} d^{\omega_i} \alpha_{1-i} d^{\omega_j} \alpha_{1-j},$$

where $\omega$ is a multi-index and $D_x = \frac{1}{d} \frac{d}{dx}$.

Since $\frac{dN_t^+}{dt}, \frac{dN_t^-}{dt}$ are first order operators, $-\alpha_1^2 + (\sigma_2 + \lambda) = 0$ and $\beta_1^2 - (\sigma_2 + \lambda) = 0$. So

$$\alpha_1 = \beta_1 = \sqrt{\sum_{ij=1}^{d-1} g^{ij} \xi_i \xi_j} + \lambda Id \quad \text{and} \quad \alpha_1 + \beta_1 = 2 \sqrt{\sum_{ij=1}^{d-1} g^{ij} \xi_i \xi_j} + \lambda Id.$$
which is even with respect to \( \xi \). Note that \( \frac{d\alpha}{dt} = -(2\alpha_0 \alpha_1 + d\xi \alpha_1 \cdot D_x \alpha_1) + F \alpha_1 + \sigma_1 \) and \( \frac{d\beta}{dt} = (2\beta_0 \beta_1 + d\xi \beta_1 \cdot D_x \beta_1) + F \beta_1 - \sigma_1 \). Hence

\[
\alpha_0 = \frac{1}{2} \alpha_1^{-1} \left( \frac{d\alpha_1}{dt} - d\xi \alpha_1 \cdot D_x \alpha_1 + F \alpha_1 + \sigma_1 \right),
\]

\[
\beta_0 = \frac{1}{2} \beta_1^{-1} \left( \frac{d\beta_1}{dt} - d\xi \beta_1 \cdot D_x \beta_1 - F \beta_1 + \sigma_1 \right).
\]

Since \( \alpha_1 = \beta_1 \), it follows that \( \alpha_0 + \beta_0 = \alpha_1^{-1}(d\xi \alpha_1 \cdot D_x \alpha_1 + \sigma_1) \), which is odd with respect to \( \xi \).

**Theorem.** If \( \sigma(R(\lambda)) \sim p_1 + p_0 + \cdots + p_{1-j} + \cdots \), then \( p_{1-k} \), which is equal to \(-\alpha_1 - \beta_1\), is even (odd) with respect to \( \xi \) when \( k \) is even (odd).

**Proof.** Note that one has

\[
\alpha_{1-k} = \frac{1}{2} \alpha_1^{-1} \left\{ \frac{d\alpha_1}{dt} - \sum_{0 \leq i, j \leq 1} \frac{1}{(i+j+1)!(i+j)} \alpha_1^{-1} D_x^{i+j} \alpha_1^{-1} + F \alpha_1^{-1} \right\},
\]

\[
\beta_{1-k} = \frac{1}{2} \beta_1^{-1} \left\{ \frac{d\beta_1}{dt} - \sum_{0 \leq i, j \leq 1} \frac{1}{(i+j+1)!(i+j)} \beta_1^{-1} D_x^{i+j} \beta_1^{-1} - F \beta_1^{-1} \right\}.
\]

Since \( \alpha_1 = \beta_1 = \sqrt{\sum_{i,j=1}^{d} g_{ij} \xi_i \xi_j + \lambda I} \), we can use (*) for each \( \alpha_{1-i}, \beta_{1-j} \) to express \( \alpha_{1-k} \) and \( \beta_{1-k} \) in terms of \( \alpha_1, \sigma_1 \), and \( \sigma_0 \). In fact,

\[
\alpha_{1-k} = \sum_r (-1)^r \frac{1}{2} \alpha_1^{-1} \frac{d}{dt} \left\{ \frac{1}{2} \alpha_1^{-1} \left\{ \frac{d}{dt} \cdots \frac{1}{2} \alpha_1^{-1} \left( \frac{d}{dt} q_r^{k-r} \right) \cdots \right\} \right\}
\]

\[
+ \sum_s (-1)^s \frac{1}{2} \alpha_1^{-1} F \left\{ \frac{1}{2} \alpha_1^{-1} \left\{ \alpha_1^{-1} \left( F q_s^{k-s} \right) \cdots \right\} \right\} + P_k
\]

and

\[
\beta_{1-k} = \sum_r (-1)^r \frac{1}{2} \alpha_1^{-1} \frac{d}{dt} \left\{ \frac{1}{2} \alpha_1^{-1} \left\{ \frac{d}{dt} \cdots \frac{1}{2} \alpha_1^{-1} \left( \frac{d}{dt} q_r^{k-r} \right) \cdots \right\} \right\}
\]

\[
+ \sum_s (-1)^s \frac{1}{2} \alpha_1^{-1} F \left\{ \frac{1}{2} \alpha_1^{-1} \left\{ \alpha_1^{-1} \left( F q_s^{k-s} \right) \cdots \right\} \right\} + P_k,
\]

where \( \frac{d}{dt} \) appears \( r \) times and \( F \) appears \( s \) times, respectively, and \( q_r^{k-r}, q_s^{k-s} \), \( P_k \) are functions consisting of some jets of \( \alpha_1, \alpha_1^{-1}, \sigma_1 \), and \( \sigma_0 \) satisfying

\[
q_r^{k-r}(x, -\xi) = (-1)^{k-r} q_r^{k-r}(x, \xi),
\]

\[
q_s^{k-s}(x, -\xi) = (-1)^{k-s} q_s^{k-s}(x, \xi),
\]

\[
P_k(x, -\xi) = (-1)^k P_k(x, \xi).
\]

Hence

\[
-p_{1-k} = \alpha_{1-k} + \beta_{1-k}
\]

\[
= 2 \sum_{r: \text{even}} \frac{1}{2} \alpha_1^{-1} \frac{d}{dt} \left\{ \frac{1}{2} \alpha_1^{-1} \left\{ \frac{d}{dt} \cdots \frac{1}{2} \alpha_1^{-1} \left( \frac{d}{dt} q_r^{k-r} \right) \cdots \right\} \right\}
\]

\[
+ 2 \sum_{s: \text{even}} \frac{1}{2} \alpha_1^{-1} F \left\{ \frac{1}{2} \alpha_1^{-1} \left\{ \alpha_1^{-1} \left( F q_s^{k-s} \right) \cdots \right\} \right\} + 2P_k,
\]
and so \( p_{1-k} \) is even if \( k \) is even, and \( p_{1-k} \) is odd if \( k \) is odd, with respect to \( \xi \).

III. The proof of Theorem B

Lemma 2. \( R(e)^{-1} = J \circ (\Delta + e)^{-1} \circ (\varphi \otimes \delta_T) \), where \( J \) is the restriction map to \( \Gamma \) and \( \delta_T \) is the Dirac \( \delta \)-function along \( \Gamma \).

Proof. For \( \varphi \in C^\infty(\Gamma, E|_\Gamma) \) choose \( u \) such that \((\Delta + e)u = 0 \) in \( M - \Gamma \) and \( u|_\Gamma = \varphi \). Then

\[
\frac{du}{dt} = \begin{cases} \nabla_{\nu_i} u = N^+_i(u(x,t)) & \text{for } t > 0, \\ -\nabla_{-\nu_i} u = -N^-_i(u(x,t)) & \text{for } t < 0. \end{cases}
\]

Now \( R(e)\varphi = -N^+_0(\varphi) - N^-_0(\varphi) \). So

\[
\frac{du}{dt} = \begin{cases} -R(e)\varphi + N^+_i(u(x,t)) + R(e)\varphi, & t \geq 0, \\ -N^-_i(u(x,t)), & t < 0. \end{cases}
\]

Let

\[
v(x,t) = \begin{cases} N^+_i(u(x,t)) + R(e)\varphi, & t \geq 0, \\ -N^-_i(u(x,t)), & t < 0. \end{cases}
\]

Then

\[
\frac{du}{dt} = -R(e)(\varphi) \otimes H(t) + v(x,t).
\]

For \( t \geq 0 \),

\[
\frac{dv}{dt}(x,t) = \frac{d}{dt}N^+_i(u(x,t)) = \left\{ \frac{dN^+_i}{dt} + (N^+_i)^2 \right\} u(x,t)
\]

\[
= (F(x,t)N^+_i + \Delta_t + e)u(x,t)
\]

by Lemma 1. In the same way for \( t < 0 \), \( \frac{dv}{dt} = -(F(x,t)N^-_i + \Delta_t + e)u(x,t) \).

Hence

\[
\frac{d^2u}{dt^2} = -R(\varphi) \otimes \delta_T + \frac{dv}{dt}(x,t)
\]

\[
= -R(\varphi) \otimes \delta_T + (F(x,t)N^+_i + \Delta_t + e)u(x,t),
\]

\[
-\frac{d^2u}{dt^2} + (F(x,t)N^+_i + \Delta_t + e)u(x,t) = R(\varphi) \otimes \delta_T,
\]

\[
(\Delta + e)u = R(\varphi) \otimes \delta_T.
\]

Hence

\[
R(e)^{-1}(\varphi) = J \circ (\Delta + e)^{-1} \circ (\varphi \otimes \delta_T).
\]

Theorem B. \( \det^*(\Delta) = \frac{1}{(\det A)^*} \det(\Delta, B) \cdot \det^* R \).

Proof. Let \( k = \text{dim} \mathcal{H}^\varphi \). Then

(1)

\[
\log \det(\Delta + e) = k \log e + \log \det^*(\Delta) + o(e).
\]

Denote by \( \mu_j = \mu_j(e) \) (\( j \geq 1 \)) the eigenvalues of \( R(e) \) with \( 0 < \mu_1(e) \leq \cdots \leq \mu_k(e) < \mu_{k+1}(e) \leq \cdots \). It is clear that \( \lim_{e \to 0} \mu_j(e) = 0 \) for \( 1 \leq j \leq k \). Then

\[
\log \det R(e) = \log \mu_1(e) \cdots \mu_k(e) + \log \det^* R + o(e).
\]
Now we want to calculate $p_1(e) \cdots p_k(e)$. Let $\{\psi_j\}_{j \geq 1}$ be the complete orthonormal system of eigenforms of $\Delta$ with eigenvalue $\lambda_j$ in $L^2(M, E)$. For any $\varphi \in C^\infty(\Gamma, E|\Gamma)$, $\varphi \otimes \delta_\Gamma \in H^{-1}(M, E)$ and $(\Delta + \varepsilon)^{-1}(\varphi \otimes \delta_\Gamma) \in L^2(M, E)$.

$$(\Delta + \varepsilon)^{-1}(\varphi \otimes \delta_\Gamma), \psi_j) = (\varphi \otimes \delta_\Gamma, (\Delta + \varepsilon)^{-1}\psi_j) = (\varphi \otimes \delta_\Gamma, \frac{1}{\lambda_j + \varepsilon}\psi_j)$$

where $d\mu_\Gamma$ is a volume element in $\Gamma$. Hence

$$(\Delta + \varepsilon)^{-1}(\varphi \otimes \delta_\Gamma) = \sum_{j=1}^{\infty} \frac{1}{\lambda_j + \varepsilon} \int_\Gamma (\varphi, \psi_j) d\mu_\Gamma \cdot \psi_j.$$

Let $\psi_1, \ldots, \psi_k$ be harmonic forms and $\lambda_1 = \cdots = \lambda_k = 0$. Then

$$(2) \quad R(\varepsilon)^{-1}\varphi = \frac{1}{\varepsilon} \sum_{i=1}^{k} \int_\Gamma (\varphi, \psi_i) d\mu_\Gamma \cdot \psi_i|\Gamma + \sum_{j=k+1}^{\infty} \frac{1}{\lambda_j + \varepsilon} \int_\Gamma (\varphi, \psi_j) d\mu_\Gamma \cdot \psi_j|\Gamma.$$

From (2), one can check that $R(\varepsilon)^{-1}$ is symmetric and positive definite; it follows that $R(\varepsilon)$ is also symmetric and positive definite.

Let $\phi_1(\varepsilon), \ldots, \phi_k(\varepsilon)$ be orthonormal eigenforms of $R(\varepsilon)$ corresponding to eigenvalues $\mu_1(\varepsilon), \ldots, \mu_k(\varepsilon)$. Then $\phi_j(\varepsilon) \rightarrow \phi_j$ as $\varepsilon \rightarrow 0$, where $\phi_j$ is the restriction of a harmonic form to $\Gamma$ with $\langle \phi_j, \phi_j \rangle_\Gamma = 1$. Let $a_{ij}(\varepsilon) = \langle \psi_i, \phi_j(\varepsilon) \rangle_\Gamma$, $1 \leq i, j \leq k$, and $A(\varepsilon) = (a_{ij}(\varepsilon))$. Now $\psi_i|\Gamma = a_{ij}(\varepsilon)\phi_j(\varepsilon) + \psi_i(\varepsilon)|\Gamma$ for some $\psi_i(\varepsilon)|\Gamma \in (\text{span}\{\phi_1(\varepsilon), \ldots, \phi_k(\varepsilon)\})^\perp$. Define

$I: C^\infty(\Gamma, E|\Gamma) \rightarrow C^\infty(\Gamma, E|\Gamma)$

by

$$\varphi \mapsto \sum_{j=1}^{k} \int_\Gamma (\varphi, \psi_j) d\mu_\Gamma \cdot \psi_j|\Gamma = \sum_{j=1}^{k} (\varphi, \psi_j|\Gamma) \cdot \psi_j|\Gamma.$$

Then

$$\langle I(\phi_i(\varepsilon)), \phi_j(\varepsilon) \rangle_\Gamma = \sum_{l=1}^{k} a_{il}(\varepsilon)a_{lj}(\varepsilon) = (A^A)_{ij}(\varepsilon).$$

Define

$G_\varepsilon: C^\infty(\Gamma, E|\Gamma) \rightarrow C^\infty(\Gamma, E|\Gamma)$

by

$$\varphi \mapsto \sum_{j=k+1}^{\infty} \frac{1}{\lambda_j + \varepsilon} (\varphi, \psi_j|\Gamma) \cdot \psi_j|\Gamma.$$

Then $\|G_\varepsilon\|_{L^2}$ converges to $\frac{1}{\lambda_{k+1}} > 0$ as $\varepsilon \rightarrow 0$. Now

$$R(\varepsilon)^{-1}(\varphi) = \frac{1}{\varepsilon} I(\varphi) + G_\varepsilon(\varphi).$$
For $1 \leq j \leq k$,
\[
\frac{1}{\mu_j(\varepsilon)} = \langle R(\varepsilon)^{-1} \phi_j(\varepsilon), \phi_j(\varepsilon) \rangle \\
= \frac{1}{\varepsilon} \langle I(\phi_j(\varepsilon)), \phi_j(\varepsilon) \rangle + \langle G_\varepsilon(\phi_j(\varepsilon)), \phi_j(\varepsilon) \rangle \\
= \frac{1}{\varepsilon} \langle \langle AA \rangle_{jj}(\varepsilon) + N_j(\varepsilon) \rangle,
\]
where $N_j(\varepsilon) = \langle G_\varepsilon(\phi_j(\varepsilon)), \phi_j(\varepsilon) \rangle \Gamma$ is bounded as $\varepsilon \to 0$. For $i \neq j$, $1 \leq i, j \leq k$,
\[
0 = \langle R(\varepsilon)^{-1} \phi_i(\varepsilon), \phi_j(\varepsilon) \rangle \\
= \frac{1}{\varepsilon} \langle I(\phi_i(\varepsilon)), \phi_j(\varepsilon) \rangle + \langle G_\varepsilon(\phi_i(\varepsilon)), \phi_j(\varepsilon) \rangle \\
= \frac{1}{\varepsilon} \langle \langle AA \rangle_{ij}(\varepsilon) + \langle G_\varepsilon(\phi_i(\varepsilon)), \phi_j(\varepsilon) \rangle \rangle.
\]
Since $\langle \langle AA \rangle_{ij}(\varepsilon) \rangle$ and $\langle G_\varepsilon(\phi_i(\varepsilon)), \phi_j(\varepsilon) \rangle$ are bounded, $\langle AA \rangle_{ij}(\varepsilon) \to 0$ as $\varepsilon \to 0$. So
\[
\frac{1}{\mu_1(\varepsilon) \cdots \mu_k(\varepsilon)} = \left( \frac{1}{\varepsilon} \langle \langle AA \rangle_{11} + N_1(\varepsilon) \rangle \cdots \left( \frac{1}{\varepsilon} \langle \langle AA \rangle_{kk} + N_k(\varepsilon) \rangle \right) \right) \\
= \frac{1}{\varepsilon^k (\det A)^2} \left( \langle \langle AA \rangle_{11} \langle AA \rangle_{22} \cdots \langle AA \rangle_{kk} \rangle + \varepsilon \cdot \tilde{N}(\varepsilon) \right),
\]
where $\tilde{N}(\varepsilon)$ is bounded as $\varepsilon \to 0$. Hence
\[
(3) \quad \log \det R(\varepsilon) = k \log \varepsilon - \log(\det A)^2 + \log \det^* R + o(\varepsilon).
\]
If we combine equation (1) and equation (3), we get
\[
\log \det^* \Delta = -\log(\det A)^2 + \log \det^* R + \log \det(\Delta, B).
\]

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REFERENCES