MAYER-VIETORIS FORMULA
FOR THE DETERMINANT OF A LAPLACE OPERATOR
ON AN EVEN-DIMENSIONAL MANIFOLD

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Abstract. Let $\Delta$ be a Laplace operator acting on differential $p$-forms on an even-dimensional manifold $M$. Let $\Gamma$ be a submanifold of codimension 1. We show that if $B$ is a Dirichlet boundary condition and $R$ is a Dirichlet-Neumann operator on $\Gamma$, then $\text{Det}(\Delta + \lambda) = \text{Det}(\Delta + \lambda, B) \text{Det}(R + \lambda)$ and $\text{Det}^* \Delta = \frac{1}{\text{Det}(\Delta, B)} \text{Det}(\Delta, B) \text{Det}^* R$. This result was established in 1992 by Burghelea, Friedlander, and Kappeler for a 2-dimensional manifold with $p = 0$.

1. Introduction

Let $M$ be a compact oriented Riemannian manifold of dimension $d$, and let $\Gamma$ be a submanifold of $M$ with dimension $d - 1$ such that $\Gamma$ has a collared neighborhood $U$ diffeomorphic to $\Gamma \times (-1, 1)$. Let $M_\Gamma$ be the compact manifold with boundary $\Gamma \cup \Gamma$ obtained by cutting $M$ along $\Gamma$. Let $E = \Lambda^p T^* M$ be a $p$-th exterior product of the cotangent bundle $T^* M$, $i: M_\Gamma \to M$ be the identification map, and $E_\Gamma := i^* E$.

Define the Dirichlet boundary condition $(\Delta + \lambda, B)$ to be

$$(\Delta + \lambda, B): C^\infty(M_\Gamma, E_\Gamma) \to C^\infty(M_\Gamma, E_\Gamma) \oplus C^\infty(\partial M_\Gamma, E_\Gamma|_{\partial M_\Gamma}),$$

$$\omega \mapsto ((\Delta + \lambda)\omega, \omega|_{\partial M_\Gamma}).$$

Define the Poisson operator $P_B$ to be the restriction of $(\Delta + \lambda, B)^{-1}$ to $0 \oplus C^\infty(\partial M_\Gamma, E_\Gamma|_{\partial M_\Gamma})$. Let $\nu$ be a unit normal vector field along $\partial M_\Gamma$; one can extend $\nu$ to be a global vector field on $M_\Gamma$ by using a cut-off function. Define the Neumann boundary condition $C$ to be

$$(\Delta + \lambda, C): C^\infty(M_\Gamma, E_\Gamma) \to C^\infty(\partial M_\Gamma, E_\Gamma|_{\partial M_\Gamma}),$$

$$\omega \mapsto \nabla_\nu \omega|_{\partial M_\Gamma}.$$
**Definition.** For any positive real number \(\lambda > 0\), define \(R(\lambda)\) to be the composition of the following maps:

\[
\begin{align*}
C^\infty(\Gamma, E|\Gamma) &\xrightarrow{\Delta_{id}} C^\infty(\Gamma, E|\Gamma) \oplus C^\infty(\Gamma, E|\Gamma) \oplus 0 \\
&\xrightarrow{P} C^\infty(M, E|\Gamma) \\
&\subseteq C^\infty(\Gamma, E|\Gamma) \oplus C^\infty(\Gamma, E|\Gamma) \\
&\xrightarrow{\Delta_f} C^\infty(\Gamma, E|\Gamma),
\end{align*}
\]

where \(\Delta_{id}\) is the diagonal inclusion and \(\Delta_f\) is the difference map.

Then \(R(\lambda)\) is a positive definite selfadjoint elliptic operator. When \(\lambda = 0\), both the Laplacian \(\Delta\) and \(R\) have zero eigenvalues and so \(\det\Delta = \det R = 0\). In this case we define the modified determinants \(\det^* \Delta\) and \(\det^* R\) to be the determinants of \(\Delta\) and \(R\) respectively, when restricted to the orthogonal complement of the null space.

In [BFK], Burghelea, Friedlander, and Kappeler proved that on a 2-dimensional manifold and for the trivial line bundle \(E = \Lambda^0 T^* M\),

1. \(\det (\Delta + \lambda) = \det (\Delta + \lambda, B) \det R(\lambda)\) for \(\lambda > 0\),
2. \(\det^* \Delta = \frac{V}{l} \det (\Delta, B) \det^* R\),

where \(V\) is the area of the manifold and \(l\) is the length of \(\Gamma\).

Let \(\mathcal{H}_p\) be the space of harmonic \(p\)-forms equipped with the natural inner product \(\langle \phi, \psi \rangle = \int_M \phi \wedge \ast \psi = \int_M (\phi, \psi)d\text{vol}(M)\), where \((, , )\) is a metric in \(E = \Lambda^p T^* M\) induced by the Riemannian metric \(g\) on \(M\). Let \(\mathcal{H}_p|\Gamma\) be the restriction of harmonic \(p\)-forms to \(\Gamma\). Define an inner product on \(\mathcal{H}_p|\Gamma\) by \(\langle \alpha, \beta \rangle_{\Gamma} = \int_{\Gamma} (\alpha, \beta) d\mu_{\Gamma}\), where \(d\mu_{\Gamma}\) is a volume element of \(\Gamma\) coming from \(g\) restricted to \(\Gamma\).

Suppose \(k = \dim \mathcal{H}_p\), and let \(\psi_1, \ldots, \psi_k\) be an orthonormal basis of \(\mathcal{H}_p\) and \(\phi_1, \ldots, \phi_k\) be an orthonormal basis of \(\mathcal{H}_p|\Gamma\). Let \(J: \mathcal{H}_p \to \mathcal{H}_p|\Gamma\) denote the restriction map. Let \(J(\psi_i) = a_{ij}\phi_j\) and let \(A = (a_{ij})_{1 \leq i, j \leq k}\). In this paper we extend the result of Burghelea et al. to arbitrary even dimensions and arbitrary \(p\)-forms.

If \(M\) is a compact oriented Riemannian manifold of dimension \(d\) with \(d\) even and \(E = \Lambda^p T^* M\), then

**Theorem A.** \(\det (\Delta + \lambda, B) = \det (\Delta + \lambda, B) \det R(\lambda)\) for any \(\lambda > 0\).

**Theorem B.** \(\det^* \Delta = \frac{1}{(\det A)^2} \det (\Delta, B) \det^* R\).

**Remark.** If \(p = 0\), then \(E = M \times R\), and the matrix \(A\) is \((\sqrt{\lambda})\). Hence Theorem B reduces to

\[
\det^* \Delta = \frac{V}{l} \det (\Delta, B) \det^* R,
\]

as stated in [BFK].

**II. The proof of Theorem A**

In [BFK], it is shown that

\[
\det (\Delta + \lambda) = c \det (\Delta + \lambda, B) \cdot \det R(\lambda),
\]
and that \( \log \det(\Delta + \lambda), \log \det(\Delta + \lambda, B), \) and \( \log \det R(\lambda) \) admit asymptotic expansions;

\[
\log \det(\Delta + \lambda), \log \det(\Delta + \lambda, B) \sim \sum_{k=-d}^{\infty} \alpha_k |\lambda|^{-k/2} + \beta_0 \log |\lambda|, \quad \text{with} \quad \alpha_0 = 0,
\]

\[
\log \det R(\lambda) \sim \sum_{j=-d}^{\infty} \pi_j |\lambda|^{-j/2} + \sum_{j=0}^{d} q_j |\lambda|^{j/2} \log |\lambda|, \quad \text{with}
\]

\[
\pi_0 = \sum_{j} \frac{\partial}{\partial s} \frac{1}{(2\pi)^{d-1}} \int_{R^{d-1}} J_{d-1}(s, \lambda; x) \varphi_j(x)|_{s=0} \, d\text{vol}(x),
\]

where

\[
J_{d-1} = (s, \lambda; x) = \frac{1}{2\pi i} \int_{R^{d-1}} d\xi \int_{\gamma} \mu^{-i} r_{-1-(d-1)} \left( \mu, \frac{\lambda}{|\lambda|}, x, \xi \right) d\mu,
\]

\[
r_{-1} = (\mu - p_1(\lambda, x, \xi))^{-1},
\]

\[
r_{-1-j} = - (\mu - p_1(\lambda, x, \xi))^{-1}
\]

\[
\cdot \sum_{j=0}^{d-1} \sum_{\alpha=0}^{d} \frac{1}{\alpha!} \partial_{\xi}^\alpha p_{1-j}(\lambda, x, \xi) D^\alpha r_{-1-k}(\mu, \lambda, x, \xi)
\]

\[
\sigma(R(\lambda)) \sim p_1 + p_0 + p_{-1} + \cdots \quad \text{asymptotic symbol of} \ R(\lambda), \ \{\varphi_j\} \ \text{a partition of unity subordinate to coordinate charts, and} \ \gamma \ \text{is a curve on a complex plane enclosing all the eigenvalues of} \ R(\lambda) \ \text{counterclockwise.}
\]

Hence

\[
\log c = -\pi_0.
\]

The proof of Theorem A reduces to the verification of the following equation:

\[
p_{1-j}(x, -\xi, \lambda) = (-1)^j p_{1-j}(x, \xi, \lambda).
\]

Then

\[
r_{-1-j}(\mu, \frac{\lambda}{|\lambda|}, x, -\xi) = (-1)^j r_{-1-j}(\mu, \frac{\lambda}{|\lambda|}, x, \xi),
\]

so when \( d \) is even, \( r_{-1-(d-1)}(\mu, \frac{\lambda}{|\lambda|}, x, \xi) \) is odd with respect to \( \xi \). So \( J_{d-1} = 0 \) and \( \pi_0 = 0 \). Therefore we conclude

\[
\det(\Delta + \lambda) = \det(\Delta + \lambda, B) \det R(\lambda).
\]

**Definition.** Let \( U \) be a collared neighborhood of \( \Gamma \) diffeomorphic to \( \Gamma \times (-1, 1) \) with diffeomorphism \( \eta: U \to \Gamma \times (-1, 1) \). Let \( \Gamma_t = \eta^{-1}(\Gamma \times t), -1 < t < 1 \). Let \( N_t^+, N_t^- \) be Neumann operators to each side with respect to \( \Delta + \lambda \); i.e. if \( \varphi \in C^\infty(\Gamma_t, E|_{\Gamma_t}) \), define \( N_t^+(\varphi) = \nabla_t u|_{\Gamma_t} \), where \( (\Delta + \lambda) u = 0 \) in \( M - \Gamma_t \), \( u|_{\Gamma_t} = \varphi \), and \( \nu_t \) is a normal vector field along \( \Gamma_t \).

Then

\[
R(\lambda) = -(N_0^+ + N_0^-).
\]

**Lemma 1.** In a local coordinate system such that the first fundamental form looks like

\[
\begin{pmatrix}
g_{ij}(x, t) & 0 \\
0 & 1
\end{pmatrix}
\]
on $\Gamma \times (-1, 1)$, the Laplacian is $\Delta = -\frac{d^2}{dt^2} + F(x, t)\frac{d}{dt} + \Delta_t$, where $\Delta_t$ is the Laplacian on $\Gamma_t$ and $F(x, t)$ is a $C^\infty$ function valued $\left(\frac{d}{dx}\right) \times \left(\frac{d}{dx}\right)$ matrix. Then

$$\frac{d N_t^+}{dt} = -(N_t^+)^2 + F(x, t)N_t^+ + (\Delta_t + \lambda),$$

$$\frac{d N_t^-}{dt} = (N_t^-)^2 + F(x, t)N_t^- - (\Delta_t + \lambda).$$

**Remark.** The idea to consider the Neumann operator as a solution of operator-valued differential equations goes back to I. M. Gel’fand.

**Proof.** It is enough to show that the first statement is true. Let $\phi \in C^\infty(\Gamma_t, E|_{\Gamma_t})$. Choose $u(x, t) \in C^\infty(M_{\Gamma_t}, E_{\Gamma_t})$ such that $(\Delta + \lambda)u(x, t) = 0$ on $M - \Gamma_t$ and $u(x, t)|_{\Gamma_t} = \phi$. Then

$$\frac{d}{dt}u(x, t) = N_t^+(u(x, t)),$$

$$\frac{d^2}{dt^2}u(x, t) = \frac{d}{dt}(N_t^+(u(x, t))) = \frac{dN_t^+}{dt}(u(x, t)) + N_t^+ \left(\frac{du}{dt}\right) = \left(\frac{dN_t^+}{dt} + (N_t^+)^2\right)u(x, t),$$

and

$$\frac{d^2}{dt^2}u(x, t) = F(x, t)\frac{du}{dt} + (\Delta_t + \lambda)u(x, t) = (F(x, t)N_t^+ + \Delta_t + \lambda)u(x, t).$$

Hence $\frac{dN_t^+}{dt} + (N_t^+)^2 = F(x, t)N_t^+ + (\Delta_t + \lambda)$, so

$$\frac{dN_t^+}{dt} = -(N_t^+)^2 + F(x, t)N_t^+ + (\Delta_t + \lambda).$$

Let

$$\sigma(N_t^+) \sim \alpha_1 + \alpha_0 + \cdots + \alpha_{1-i} + \cdots,$$

$$\sigma(N_t^-) \sim \beta_1 + \beta_0 + \cdots + \beta_{1-i} + \cdots,$$

$$\sigma(\Delta + \lambda) \sim (\sigma_2 + \lambda) + \sigma_1 + \sigma_0.$$

Note that

$$\sigma_2 + \lambda = \left(\sum_{i,j=1}^{d-1} g^{ij}\xi_i\xi_j + \lambda\right) I_d,$$

$$\sigma((N_t^+)^2) \sim \sum_{k=0}^{\infty} \sum_{i,j \geq 0} \frac{1}{\omega_k} D_{\xi}^{\omega_k} \alpha_{1-i} \alpha_{1-j},$$

where $\omega$ is a multi-index and $D_x = \frac{1}{\xi} \frac{d}{d\xi}$.

Since $\frac{dN_t^+}{dt}$, $\frac{dN_t^-}{dt}$ are first order operators, $-\alpha_1^2 + (\sigma_2 + \lambda) = 0$ and $\beta_1^2 - (\sigma_2 + \lambda) = 0$. So

$$\alpha_1 = \beta_1 = \sqrt{\sum_{i,j=1}^{d-1} g^{ij}\xi_i\xi_j + \lambda I_d}$$

and

$$\alpha_1 + \beta_1 = 2 \sqrt{\sum_{i,j=1}^{d-1} g^{ij}\xi_i\xi_j + \lambda I_d}.$$
which is even with respect to $\xi$. Note that $\frac{d\alpha_1}{dt} = -(2\alpha_0\alpha_1 + d\xi_1 \cdot D_\xi \alpha_1) + F\alpha_1 + \sigma_1$ and $\frac{d\beta_1}{dt} = (2\beta_0\beta_1 + d\xi_1 \cdot D_\xi \beta_1) + F\beta_1 - \sigma_1$. Hence

$$\alpha_0 = \frac{1}{2} \alpha_1^{-1} \left( \frac{d\alpha_1}{dt} - d\xi_1 \cdot D_\xi \alpha_1 + F\alpha_1 + \sigma_1 \right),$$

$$\beta_0 = \frac{1}{2} \beta_1^{-1} \left( \frac{d\beta_1}{dt} - d\xi_1 \cdot D_\xi \beta_1 - F\beta_1 + \sigma_1 \right).$$

Since $\alpha_1 = \beta_1$, it follows that $\alpha_0 + \beta_0 = \alpha_1^{-1}(d\xi_1 \cdot D_\xi \alpha_1 + \sigma_1)$, which is odd with respect to $\xi$.

**Theorem.** If $\sigma(R(\lambda)) \sim p_1 + p_0 + \cdots + p_{1-j} + \cdots$, then $p_{1-k}$, which is equal to $-\alpha_{1-k} - \beta_{1-k}$, is even (odd) with respect to $\xi$ when $k$ is even (odd).

**Proof.** Note that one has

$$\alpha_{1-k} = \frac{1}{2} \alpha_1^{-1} \left\{ \frac{d\alpha_1}{dt} - \sum_{|i+j| = k} \frac{1}{\alpha_1^{-1}} \left( \frac{d\alpha_1(x)}{dt} q_r^{k-r} \right) \right\},$$

$$\beta_{1-k} = \frac{1}{2} \beta_1^{-1} \left\{ \frac{d\beta_1}{dt} - \sum_{|i+j| = k} \frac{1}{\beta_1^{-1}} \left( \frac{d\beta_1(x)}{dt} s_r^{k-r} \right) \right\}.$$
and so \( p_{1-k} \) is even if \( k \) is even, and \( p_{1-k} \) is odd if \( k \) is odd, with respect to \( \xi \).

III. The proof of Theorem B

Lemma 2. \( R(e)^{-1} = J \circ (\Delta + e)^{-1} \circ (\varphi \otimes \delta \gamma) \), where \( J \) is the restriction map to \( \Gamma \) and \( \delta \gamma \) is the Dirac \( \delta \)-function along \( \Gamma \).

Proof. For \( \phi \in C^\infty(\Gamma, E|_\Gamma) \) choose \( u \) such that \((\Delta + e)u = 0\) in \( M - \Gamma \) and \( u|_\Gamma = \varphi \). Then

\[
\frac{du}{dt} = \begin{cases} \nabla_{\nu_+} u = N^+_i(u(x, t)) & \text{for } t > 0, \\ -\nabla_{\nu_-} u = -N^-_i(u(x, t)) & \text{for } t < 0. \end{cases}
\]

Now \( R(e)\phi = -N^+_0(\phi) - N^-_0(\phi) \). So

\[
\frac{du}{dt} = \begin{cases} -R(e)\phi + N^+_i(u(x, t)) + R(e)\phi, & t \geq 0, \\ -N^-_i(u(x, t)), & t < 0. \end{cases}
\]

Let

\[
v(x, t) = \begin{cases} N^+_i(u(x, t)) + R(e)\phi, & t \geq 0, \\ -N^-_i(u(x, t)), & t < 0. \end{cases}
\]

Then

\[
\frac{dv}{dt} = -R(e)\otimes H(t) + v(x, t).
\]

For \( t \geq 0 \),

\[
\frac{dv}{dt}(x, t) = \frac{d}{dt} N^+_i(u(x, t)) = \left\{ \frac{dN^+_i}{dt} + (N^+_i)^2 \right\} u(x, t) = (F(x, t)N^+_i + \Delta_t + e)u(x, t)
\]

by Lemma 1. In the same way for \( t < 0 \), \( \frac{dv}{dt} = (-F(x, t)N^-_i + \Delta_t + e)u(x, t) \).

Hence

\[
\frac{d^2u}{dt^2} = -R(\phi) \otimes \delta \gamma + \frac{dv}{dt}(x, t)
\]

\[
= -R(\phi) \otimes \delta \gamma + (F(x, t)N^+_i + \Delta_t + e)u(x, t),
\]

\[
- \frac{d^2u}{dt^2} + (F(x, t)N^+_i + \Delta_t + e)u(x, t) = R(\phi) \otimes \delta \gamma,
\]

\[
(\Delta + e)u = R(\phi) \otimes \delta \gamma.
\]

Hence

\[
R(e)^{-1}(\phi) = J \circ (\Delta + e)^{-1} \circ (\varphi \otimes \delta \gamma).
\]

Theorem B. \( \text{Det}^*(\Delta) = \frac{1}{(\det A)^2} \text{Det}(\Delta, B) \cdot \text{Det}^* R \).

Proof. Let \( k = \dim \mathcal{H}_\gamma \). Then

(1) \[ \log \text{Det}(\Delta + e) = k \log e + \log \text{Det}^*(\Delta) + o(e). \]

Denote by \( \mu_j = \mu_j(e) \) (\( j \geq 1 \)) the eigenvalues of \( R(e) \) with \( 0 < \mu_1(e) \leq \cdots \leq \mu_k(e) < \mu_{k+1}(e) \leq \cdots \). It is clear that \( \lim_{e \to 0} \mu_j(e) = 0 \) for \( 1 \leq j \leq k \). Then

\[
\log \text{Det} R(e) = \log \mu_1(e) \cdots \mu_k(e) + \log \text{Det}^* R + o(e).
\]
Now we want to calculate $\mu_1(e) \cdots \mu_k(e)$. Let $\{\psi_j\}_{j \geq 1}$ be the complete orthonormal system of eigenforms of $A$ with eigenvalue $\lambda_j$ in $L^2(M, E)$. For any $\varphi \in C^\infty(\Gamma, E|_\Gamma)$, $\varphi \otimes \delta_\Gamma \in H^{-1}(M, E)$ and $(A + \varepsilon)^{-1}(\varphi \otimes \delta_\Gamma) \in L^2(M, E)$.

$$
((A + \varepsilon)^{-1}(\varphi \otimes \delta_\Gamma), \psi_j) = (\varphi \otimes \delta_\Gamma, (A + \varepsilon)^{-1}\psi_j) = (\varphi \otimes \delta_\Gamma, \frac{1}{\lambda_j + \varepsilon}\psi_j)
$$

where $d\mu_\Gamma$ is a volume element in $\Gamma$. Hence

$$(A + \varepsilon)^{-1}(\varphi \otimes \delta_\Gamma) = \sum_{j=1}^\infty \frac{1}{\lambda_j + \varepsilon} \int_\Gamma (\varphi, \psi_j) d\mu_\Gamma \cdot \psi_j.$$

Let $\psi_1, \ldots, \psi_k$ be harmonic forms and $\lambda_1 = \cdots = \lambda_k = 0$. Then

$$(2) \quad R(e)^{-1}\varphi = \frac{1}{\varepsilon} \sum_{i=1}^k \int_\Gamma (\varphi, \psi_i) d\mu_\Gamma \cdot \psi_i|\Gamma + \sum_{j=k+1}^{\infty} \frac{1}{\lambda_j + \varepsilon} \int_\Gamma (\varphi, \psi_j) d\mu_\Gamma \cdot \psi_j|\Gamma.$$

From (2), one can check that $R(e)^{-1}$ is symmetric and positive definite; it follows that $R(e)$ is also symmetric and positive definite.

Let $\phi_1(e), \ldots, \phi_k(e)$ be orthonormal eigenforms of $R(e)$ corresponding to eigenvalues $\mu_1(e), \ldots, \mu_k(e)$. Then $\phi_j(e) \to \phi_j$ as $e \to 0$, where $\phi_j$ is the restriction of a harmonic form to $\Gamma$ with $\langle \phi_j, \phi_j \rangle_\Gamma = 1$. Let $a_{ij}(e) = \langle \psi_i, \phi_j(e) \rangle_\Gamma$, $1 \leq i, j \leq k$, and $A(e) = (a_{ij}(e))$. Now $\psi_i|\Gamma = a_{ij}(e)\phi_j(e) + \psi_i(e)|\Gamma$ for some $\psi_i(e)|\Gamma \in (\text{span}\{\phi_1(e), \ldots, \phi_k(e)\})^\perp$. Define

$$I: C^\infty(\Gamma, E|_\Gamma) \to C^\infty(\Gamma, E|_\Gamma)$$

by

$$\varphi \mapsto \sum_{j=1}^k \int_\Gamma (\varphi, \psi_j) d\mu_\Gamma \cdot \psi_j|\Gamma = \sum_{j=1}^k (\varphi, \psi_j)_\Gamma \cdot \psi_j|\Gamma.$$

Then

$$\langle I(\phi_i(e)), \phi_j(e) \rangle_\Gamma = \sum_{l=1}^k a_{li}(e)a_{lj}(e) = (AA)_{ij}(e).$$

Define

$$G_e: C^\infty(\Gamma, E|_\Gamma) \to C^\infty(\Gamma, E|_\Gamma)$$

by

$$\varphi \mapsto \sum_{j=k+1}^\infty \frac{1}{\lambda_j + \varepsilon} (\varphi, \psi_j)_\Gamma \cdot \psi_j|\Gamma.$$

Then $\|G_e\|_{L^2}$ converges to $\frac{1}{\lambda_{k+1}} > 0$ as $e \to 0$. Now

$$R(e)^{-1}(\varphi) = \frac{1}{\varepsilon} I(\varphi) + G_e(\varphi).$$
For $1 \leq j \leq k$,
\[
\frac{1}{\mu_j(e)} = \langle (R(e)^{-1}) \phi_j(e), \phi_j(e) \rangle \\
= \frac{1}{e} \langle (I(\phi_j(e)), \phi_j(e)) + (G_e(\phi_j(e)), \phi_j(e)) \rangle \\
= \frac{1}{e} (lAA)_{jj}(e) + N_j(e),
\]
where $N_j(e) = \langle (G_e(\phi_j(e)), \phi_j(e)) \rangle$ is bounded as $e \to 0$. For $i \neq j$, $1 \leq i, j \leq k$,
\[
0 = \langle (R(e)^{-1}) \phi_i(e), \phi_j(e) \rangle \\
= \frac{1}{e} \langle (I(\phi_i(e)), \phi_j(e)) + (G_e(\phi_i(e)), \phi_j(e)) \rangle \\
= \frac{1}{e} (lAA)_{ij}(e) + (G_e(\phi_i(e)), \phi_j(e)).
\]
Since $(lAA)_{ij}(e)$ and $(G_e(\phi_i(e)), \phi_j(e))$ are bounded, $(lAA)_{ij}(e) \to 0$ as $e \to 0$. So
\[
\frac{1}{\mu_1(e) \cdots \mu_k(e)} = \left( \frac{1}{e} (lAA)_{11} + N_1(e) \right) \cdots \left( \frac{1}{e} (lAA)_{kk} + N_k(e) \right) \\
= \frac{1}{e^k} \left( \frac{1}{e^2} \frac{(lAA)_{11}(lAA)_{22} \cdots (lAA)_{kk}}{(\det A)^2} + \frac{\tilde{N}(e)}{(\det A)^2} \right),
\]
where $\tilde{N}(e)$ is bounded as $e \to 0$. Hence
\[
\log \det R(e) = k \log e - \log(\det A)^2 + \log \det^* R + o(e).
\]
If we combine equation (1) and equation (3), we get
\[
\log \det^* \Delta = -\log(\det A)^2 + \log \det^* R + \log \det(\Delta, B).
\]

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REFERENCES


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