NORMALIZING ELEMENTS IN $PI$ RINGS

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Abstract. This paper explores the question: If $R$ is a prime $PI$ ring and $a$ an element such that $aR \supseteq Ra$, is it true that $aR = Ra$?

1. Introduction

In [M] Susan Montgomery raised the following question, which arose in connection with Picard groups:

Question. Let $R$ be a prime $PI$ ring and $a$ an element of $R$ such that $aR$ is a (two-sided) ideal. Is it true that $a$ is a normalizing element; i.e., that $aR = Ra$?

Indeed [M, Lemma 2] shows, without the $PI$ hypothesis, that this holds provided that $a$ is right regular and $aR$ is an invertible $R$-ideal in the Martindale quotient ring of $R$.

In this paper we present some results, both positive and negative, related to this question.

2. Examples

First we make the elementary observation that the restriction to the case of a prime ring is essential. For if we let $R$ be the tiled ring

$$R = \left( \begin{array}{cc} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{array} \right)$$

and $a = \left( \begin{array}{cc} 1 & 0 \\ 0 & 2 \end{array} \right)$,

then one checks immediately that

$$aR = \left( \begin{array}{cc} \mathbb{Z} & \mathbb{Z} \\ 0 & 2\mathbb{Z} \end{array} \right) \not\supseteq Ra = \left( \begin{array}{cc} \mathbb{Z} & 2\mathbb{Z} \\ 0 & 2\mathbb{Z} \end{array} \right).$$

More seriously, we next show that, in general, the question has a negative answer even for an affine algebra:

Theorem 1. There exists a prime $PI$ ring $R$ which is an affine (i.e., finitely generated) algebra over a field and which has an element $a$ such that $aR$ is an ideal but $a$ is not normalizing.

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Proof. Let \( k \) be any field, \( T \) be the commutative \( k \)-algebra \( T = k[x, y, y^{-1}] \), and \( S \) be the tiled \( 2 \times 2 \) matrix ring

\[
S = \begin{pmatrix} k[y] + xT & T \\ xT & T \end{pmatrix}.
\]

We will make use of the ideal \( I = \begin{pmatrix} xT & T \\ T & T \end{pmatrix} \) and the element \( u = \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix} \). The following facts will be needed:

(i) \( S \) is an affine \( k \)-algebra, since \( T \) is affine; for \( S \) is generated over \( k \) by \( e_{22} \) times each of the generators of \( T \), together with \( e_{11}, y e_{11}, e_{12} \) and \( x e_{21} \).

(ii) \( I \) is indeed an ideal of \( S \) and is finitely generated, as both a right ideal and a left ideal by \( e_{12} \) and \( e_{22} \).

(iii) \( uI = Iu = I \).

(iv) \( uS = Su = \begin{pmatrix} yk[y] + xT & T \\ xT & T \end{pmatrix} \neq S \).

Finally, let \( R = \begin{pmatrix} S & S \\ I & S \end{pmatrix} \) and \( a = \begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix} \). Since (i) and (ii) hold, one sees that \( R \) is an affine \( k \)-algebra. Also, using (iii) and (iv), one checks readily that

\[
aR = \begin{pmatrix} S & S \\ I & uS \end{pmatrix} \supsetneq Ra = \begin{pmatrix} S & uS \\ I & uS \end{pmatrix}.
\]

3. Positive results

In this section we present two positive results together.

Theorem 2. Let \( R \) be a prime \( PI \) ring which is finitely generated as a module over its center. If \( aR \) is an ideal of \( R \), then \( aR = Ra \).

Proof. We may as well assume that \( a \neq 0 \). Then \( a \) is clearly a left regular element and, since a prime \( PI \) ring is a Goldie ring, one knows that \( a \) is regular. Moreover, \( R \) has a ring of quotients, \( Q \) say, in which \( a \) has an inverse. Thus, there is an automorphism \( \mu \) of \( Q \) via \( \mu(q) = a^{-1} qa \) for each \( q \) in \( Q \). Of course, since \( aR \) is an ideal \( Ra \subseteq aR \). Hence \( a^{-1} Ra \subseteq R \) and so \( \mu \) restricts to a ring endomorphism of \( R \) which is also an endomorphism of \( R \) as a module over its center, \( Z \) say. Moreover, if \( K \) is a quotient field of \( Z \), then \( K \) is the center of \( Q \) and \( \mu \) is a \( K \)-automorphism of \( Q \); indeed \( \mu = \ell(a^{-1})r(a) \), where \( \ell() \) denotes left multiplication and \( r() \) denotes right multiplication.

Since \( R \) is a finite module over \( Z \), then \( Q \) is a finite dimensional \( K \)-algebra, say \( [Q : K] = m = n^2 \). Therefore we may view \( \mu \) as an element of \( M_m(K) \) and consider its determinant \( \text{Det} \mu \) and its characteristic polynomial \( \chi_{\mu}(x) \). Of course, \( \text{Det} \mu = \text{Det}(\ell(a^{-1})) \text{Det}(r(a)) \). If we let \( N() \) denote the reduced norm, then it is known (see, for example, [C], pp. 274-275) that \( (N(r(a)))^n = \text{Det}(r(a)) \) and \( (N(\ell(a^{-1})))^n = \text{Det}(\ell(a^{-1})) \). Hence \( \text{Det} \mu = 1 \). In particular, we learn that the constant term of \( \chi_{\mu}(x) \) is 1.

Next note that, since \( R \) is a finitely generated \( Z \)-module and \( \mu \) is a \( Z \)-endomorphism of \( R \), then \( \mu \) is integral over \( Z \). It follows that the coefficients of \( \chi_{\mu}(x) \) are all integral over \( Z \); i.e., belong to \( \overline{Z} \), the integral closure of \( Z \). [Why? Well, let \( \mu \) satisfy the polynomial \( p(x) \) in \( Z[x] \) and let \( m(x) \) denote the minimum polynomial of \( \mu \) in \( K[x] \). Then \( m[x] \) divides \( p(x) \), so all roots
of $m(x)$ are integral over $Z$. This is also true, therefore, of the roots of $\chi(x)$.

Hence the coefficients are, indeed, integral over $Z$.]

Suppose that $\chi(x) = x^n + b_{n-1}x^{n-1} + \ldots + b_1x + 1$. Then one sees that

$$\mu^{-1} = -(\mu^{n-1} + b_{n-2}\mu^{n-2} + \ldots + b_1\mu)$$

and so $\mu^{-1}$ belongs to $\mathbb{Z}[\mu]$. Hence $\mu^{-1}$ is integral over $Z$, from which it follows that $\mu^{-1}$ is in $\mathbb{Z}[\mu]$ and so is a $Z$-endomorphism of $R$. Thus $aRa^{-1} \subseteq R$ and so $aR = Ra$.

Before giving a consequence of this, we note that, if $R$ is a prime right Goldie ring with right quotient ring $Q$ and $aR$ is a nonzero ideal of $R$, then $R \subseteq aRa^{-1} \subseteq Q$.

**Theorem 3.** Let $R$ be a Noetherian prime $PI$ ring which is affine over a field $k$ (or, more generally, over a Noetherian subring of its center) and let $aR$ be an ideal of $R$. Then $aR = Ra$.

**Proof.** We make use of the trace ring $TR$ of $R$; see, for example, [McR], Chapter 13, Section 9 and, in particular, 13.9.11 which shows that $TR$ is finitely generated as an $R$-module and that $TR$ is a finite module over its center. Bearing in mind that $TR$ is generated, over $R$, by central elements, one sees that $aTR$ is an ideal of $TR$. We can, therefore, see from Theorem 2 that $aTR = TRa$, and so $a^sRa^{-s} \subseteq TR$ for all $s$.

Now note that

$$aRa^{-1} \subseteq a^2Ra^{-2} \subseteq \ldots \subseteq a^sRa^{-s} \subseteq \ldots \subseteq TR.$$

Since $aR \supseteq Ra$, one sees that each of the subsets in the chain is a left $R$-submodule of the finitely generated $R$-module $TR$. Hence $a^sRa^{-s} = a^{s+1}Ra^{-s-1}$, for some $s$, and so $aR = Ra$.

We have recently learned that A. Braun has removed “affine” from the hypothesis of Theorem 3.

**References**


