PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 123, Number 7, July 1995

SOME INEQUALITIES OF ALGEBRAIC POLYNOMIALS

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(Communicated by Andrew Bruckner)

Dedicated to Professor A. Sharma and Mrs. Durga Sharma

ABSTRACT. Erdös and Lorentz showed that by considering the special kind of the polynomials better bounds for the derivative are possible. Let us denote by H_n the set of all polynomials whose degree is n and whose zeros are real and lie inside $[-1, 1)$. Let $P_n \in H_n$ and $P_n(1) = 1$; then the object of Theorem 1 is to obtain the best lower bound of the expression $\int_{-1}^{1} |P_n'(x)|^p \, dx$ for $p \geq 1$ and characterize the polynomial which achieves this lower bound. Next, we say that $P_n \in S_n[0, \infty)$ if P_n is a polynomial whose degree is n and whose roots are all real and do not lie inside $[0, \infty)$. In Theorem 2, we shall prove Markov-type inequality for such a class of polynomials belonging to $S_n[0, \infty)$ in the weighted L_p norm (p integer). Here $\|P_n\|_{L_p} = (\int_0^\infty |P_n(x)|^p e^{-x} \, dx)^{1/p}$. In Theorem 3 we shall consider another analogous problem as in Theorem 2.

INTRODUCTION

Let H_n be the set of all polynomials whose degree is n and whose zeros are real and lie inside $[-1, 1)$. Concerning this class of polynomials belonging to H_n we shall prove the following theorem.

Theorem 1. *Let $P_n \in H_n$, subject to the condition $P_n(1) = 1$. Then we have (for $p \geq 1$)*

$$\int_{-1}^{1} |P_n'(x)|^p \, dx \geq \frac{n^p}{2^{p-1}((n-1)p+1)}, \tag{1.1}$$

with equality iff $P_n(x) = (\frac{1+x}{2})^n$.

The case $p = 2$ was considered in [5] and [8].

In 1964 G. Szegö [6] studied the order of magnitude of $\|P_n'\|_{L_\infty} / \|P_n\|_{L_\infty}$ for unrestricted polynomials P_n of degree $\leq n$ for the norm

$$\|P_n\| = \sup_{x \geq 0} |P_n(x) e^{-x}|$$

on $(0, \infty)$. More precisely, he proved the following

Received by the editors April 30, 1993 and, in revised form, July 6, 1993 and August 10, 1993.
1991 *Mathematics Subject Classification*. Primary 26D15.
The author is deceased (December 8, 1994).

Theorem A. *Let $P_n(x)$ be a polynomial of fixed degree n and not vanishing identically. Then we have*

$$\|P_n'\| < cn\|P_n\|, \quad n = 2, 3, \dots .$$

In 1968, G. G. Lorentz [4] considered the problem of G. Szegö for the special polynomials with positive coefficients in x

$$P_n(x) = \sum_{k=0}^{n} a_k x^k, \quad a_k \geq 0, k = 0, 1, \dots, n$$

and where the norm of a function P_n on $(0, \infty)$ is given by

$$\|P_n\| = \sup_{x \geq 0} |P_n(x) e^{-w(x)}|$$

where $w(x)$ increases on $(0, \infty)$.

Motivated by the theorems of G. Szegö [6] and Lorentz [4] and the earlier result of the author [9] we shall consider the following problem concerning the class of polynomials $P \in S_n[0, \infty)$. Here let $S_n[0, \infty)$ be the set of all polynomials whose degree is n and whose roots are real and do not lie inside $[0, \infty)$. It is easy to see that if $P_n \in S_n[0, \infty)$, then it can be expressed in the form

$$P_n(x) = \sum_{k=0}^{n} a_k x^k, \qquad a_k \geq 0 \text{ for } k = 0, 1, \dots, n.$$

Now, we state the following.

Theorem 2. *Let*

$$P_n(x) = \sum_{k=0}^{n} a_k x^k, \qquad a_k \geq 0 \textit{ for } k = 0, 1, \dots, n.$$

Then, we have for any positive integer p

$$\frac{\int_0^\infty |P_n'(x)|^p e^{-x}\, dx}{\int_0^\infty |P_n(x)|^p e^{-x}\, dx} \leq \frac{1}{p!} \tag{1.2}$$

with equality iff $P_n(x) = \alpha x$.

It is of some interest to remark that the extreme value in the above inequality is independent of the degree of the polynomials. In view of the above theorem, we shall now prove

Theorem 3. *Let $P_n \in S_n[0, \infty)$ and r, p be positive integers. Then we have (for $r \leq p$)*

$$\int_0^\infty |P_n'(x)|^r x^{p-1} e^{-x}\, dx \leq \frac{n^r (nr + p - r - 1)!}{(nr + p - 1)!} \int_0^\infty |P_n(x)|^r x^{p-1} e^{-x}\, dx \tag{1.3}$$

with equality iff $P_n(x) = \alpha x^n$.

2. Proof of Theorem 1

Let $P_n \in H_n$, $P_n(1) = 1$. We shall denote the zeros of $P_n(x)$ by $x_n, x_{n-1}, \dots, x_2, x_1$ satisfying the inequality

$$-1 \leq x_n \leq x_{n-1} \leq \cdots \leq x_2 \leq x_1 < 1. \tag{2.1}$$

We may express $P_n(x)$ by

$$P_n(x) = c\prod_{i=1}^{n}(x - x_i), \quad P_n(1) = 1. \tag{2.2}$$

Next, we note that

$$P_n(x) \geq 0, \quad x_1 \leq x \leq 1. \tag{2.3}$$

From (2.1)–(2.3) and

$$P_n'(x) = P_n(x)\sum_{i=1}^{n}\frac{1}{x - x_i} \tag{2.4}$$

we obtain

$$P_n'(x) \geq 0, \quad x_1 \leq x \leq 1. \tag{2.5}$$

Next, we note that for $y \geq 0$ and $p \geq 1$ we have

$$y^p - 1 \geq p(y - 1) \tag{2.6}$$

with equality only for $y = 1$ or for $p = 1$. Proof of (2.6) can be given as follows. Consider $(y \geq 0, p \geq 1)$ $\varphi(y) = y^p - 1 - p(y - 1)$. Then $\varphi(1) = 0$, $\varphi'(1) = 0$, $\varphi''(y) = p(p-1)y^{p-2} \geq 0$. Therefore, by using Taylor's Theorem, we have

$$\begin{aligned}\varphi(y) &= \varphi(1) + (y - 1)\varphi'(1) + \varphi''(\xi)\frac{(y-1)^2}{2!}\\ &= \varphi''(\xi)\frac{(y-1)^2}{2!} = \frac{p(p-1)\xi^{p-2}(y-1)^2}{2} \geq 0\end{aligned}$$

(ξ being between y and 1).

From this (2.6) follows. Next, we put

$$y = \frac{P_n'(x)}{\frac{nP_n(x)}{x - x_n}}$$

in (2.6). Then we have $(x_1 \leq x \leq 1)$ after some simplification

$$\begin{aligned}(P_n'(x))^p \geq{}& \frac{pn^{p-1}P_n'(x)(P_n(x))^{p-1}}{(x - x_n)^{p-1}}\\ &- (p-1)n^p\frac{(P_n(x))^p}{(x - x_n)^p}.\end{aligned}$$

Clearly, then

$$\begin{aligned}\int_{x_1}^{1}|P_n'(x)|^p\,dx \geq{}& pn^{p-1}\int_{x_1}^{1}\frac{P_n'(x)(P_n(x))^{p-1}}{(x - x_n)^{p-1}}\,dx\\ &- (p-1)n^p\int_{x_1}^{1}\frac{(P_n(x))^p}{(x - x_n)^p}\,dx.\end{aligned} \tag{2.7}$$

Next, we note that

$$\begin{aligned}p\int_{x_1}^{1}\frac{P_n'(x)(P_n(x))^{p-1}}{(x - x_n)^{p-1}}\,dx &= \int_{x_1}^{1}\left(\frac{d}{dx}(P_n(x))^p\right)\frac{1}{(x - x_n)^{p-1}}\,dx\\ &= \frac{(P_n(1))^p}{(1 - x_n)^{p-1}} + (p-1)\int_{x_1}^{1}\frac{(P_n(x))^p}{(x - x_n)^p}\,dx.\end{aligned}$$

Therefore, from (2.7) and above we obtain

$$\begin{aligned}\int_{x_1}^{1} |P_n'(x)|^p\,dx &\geq n^{p-1}\left\{\frac{1}{(1-x_n)^{p-1}} + (p-1)\int_{x_1}^{1}\frac{(P_n(x))^p}{(x-x_n)^p}\,dx\right\}\\ &\quad - (p-1)n^p\int_{x_1}^{1}\frac{(P_n(x))^p}{(x-x_n)^p}\,dx \qquad (2.8)\\ &\geq \frac{n^{p-1}}{(1-x_n)^{p-1}} - n^{p-1}(n-1)(p-1)\int_{x_1}^{1}\frac{(P_n(x))^p}{(x-x_n)^p}\,dx.\end{aligned}$$

Since

$$0 \leq x - x_k \leq x - x_n, \quad k = 1, 2, \dots, n, \quad x_1 \leq x \leq 1,$$

we have

$$\begin{aligned}(P_n'(x))^p &= (P_n(x))^p\left(\sum_{k=1}^{n}\frac{1}{x-x_k}\right)^p\\ &\geq \frac{n^p(P_n(x))^p}{(x-x_n)^p}, \qquad x_1 \leq x \leq 1.\end{aligned}$$

From above and (2.8) we obtain

$$n^p\int_{x_1}^{1}\frac{(P_n(x))^p}{(x-x_n)^p}\,dx \leq \int_{x_1}^{1}(P_n'(x))^p\,dx \tag{2.9}$$

and

$$\int_{x_1}^{1}(P_n'(x))^p\,dx \geq \frac{n^{p-1}}{(1-x_n)^{p-1}} - \frac{(n-1)(p-1)}{n}\int_{x_1}^{1}(P_n'(x))^p\,dx$$

with equality iff $P_n'(x) = \frac{nP_n(x)}{x-x_n}$ and $x_n = -1$, $p \geq 1$.

From the above (1.1) follows. Thus, we have proved Theorem 1.

3. Proof of Theorem 2

We set

$$P_n(x) = \sum_{k=0}^{n} a_k x^k, \qquad a_k \geq 0,\ k = 0, 1, \dots, n,$$

and note that $P_n^{(r)}(x)$ is a polynomial of degree $\leq n-r$ in x with nonnegative coefficients. If we denote

$$P_n(x) = a_0 + r_n(x), \qquad a_0 \geq 0,\ r_n(x) = \sum_{k=1}^{n} a_k x^k, \quad a_k \geq 0,$$

then we notice that

$$\frac{\int_0^\infty (P_n'(x))^p e^{-x}\,dx}{\int_0^\infty (P_n(x))^p e^{-x}\,dx} \leq \frac{\int_0^\infty (r_n'(x))^p e^{-x}\,dx}{\int_0^\infty (r_n(x))^p e^{-x}\,dx}. \tag{3.1}$$

Therefore, in order to prove Theorem 2 it is enough to consider the class of all polynomials $P_n(x)$ of degree $\leq n$ in x with nonnegative coefficients

subject to the condition that $P_n(0) = 0$. Next, we note that $P_n^{(r)}(x) \geq 0$, for $0 \leq x < \infty$, and

$$\begin{aligned}
&\int_0^\infty (P_n'(x))^{p-r}(P_n(x))^r e^{-x}\,dx \\
&\quad = \int_0^\infty P_n'(x)(P_n'(x))^{p-r-1}(P_n(x))^r e^{-x}\,dx \\
&\quad = -\int_0^\infty P_n(x)[-(P_n'(x))^{p-r-1}(P_n(x))^r e^{-x} + r(P_n(x))^{r-1}(P_n'(x))^{p-r}e^{-x} \\
&\qquad\qquad + (p-r-1)(P_n'(x))^{p-r-2}P_n''(x)(P_n(x))^r e^{-x}]\,dx.
\end{aligned}$$

From above, we may conclude that

$$\begin{aligned}
&(r+1)\int_0^\infty (P_n'(x))^{p-r}(P_n(x))^r e^{-x}\,dx \\
&\quad = \int_0^\infty (P_n'(x))^{p-r-1}(P_n(x))^{r+1}e^{-x}\,dx \\
&\qquad + (r+1-p)\int_0^\infty (P_n'(x))^{p-r-2}P_n''(x)(P_n(x))^r e^{-x}\,dx \\
&\quad \leq \int_0^\infty (P_n'(x))^{p-r-1}(P_n(x))^{r+1}e^{-x}\,dx \quad (p \geq r+1)
\end{aligned}$$

with equality iff $P_n(0) = 0$ and $P_n''(x) = 0$.

Putting $r = 0, 1, \dots, p-1$ we obtain

$$\int_0^\infty (P_n'(x))^p e^{-x}\,dx \leq \frac{1}{p!}\int_0^\infty (P_n(x))^p e^{-x}\,dx$$

with equality iff $P_n''(x) = 0$ and $P_n(0) = 0$.

From this the proof of Theorem 2 is complete.

4. Proof of Theorem 3

Let $x_1, x_2, \dots, x_n$ be any real zero of $P_n \in S_n[0, \infty)$. Then $x_k \leq 0$, $k = 1, 2, \dots, n$. Also, using Turán's identity [7] we have

$$(P_n'(x))^2 - P_n(x)P_n''(x) = (P_n(x))^2 \sum_{k=1}^n \frac{1}{(x-x_k)^2}. \tag{4.1}$$

Therefore we obtain

$$\begin{aligned}
x[(P_n'(x))^2 - P_n(x)P_n''(x)] &= P_n^2(x)\sum_{k=1}^n \frac{x - x_k + x_k}{(x-x_k)^2} \\
&\leq P_n^2(x)\sum_{k=1}^n \frac{1}{x-x_k} = P_n(x)P_n'(x).
\end{aligned} \tag{4.2}$$

Since $P_n \in S_n[0, \infty)$, it follows that $P_n^{(r)}(x) \geq 0$ for $0 \leq x < \infty$.

We now claim that $(j+1 \leq r \leq p)$

$$\begin{aligned}
&\int_0^\infty (P_n'(x))^{r-j}(P_n(x))^j x^{p-1}e^{-x}\,dx \\
&\quad \leq \frac{n}{(n-1)r+p+j}\int_0^\infty (P_n'(x))^{r-j-1}(P_n(x))^{j+1}x^{p-1}e^{-x}\,dx.
\end{aligned} \tag{4.3}$$

First we note that for $j+1 < r \le p$ we have

$$\begin{aligned} I_{r,j} &= \int_0^\infty (P_n'(x))^{r-j}(P_n(x))^j x^{p-1}e^{-x}\,dx \\ &= \int_0^\infty (P_n'(x))^{r-j-2}(P_n(x))^j x^{p-1}e^{-x}((P_n'(x))^2 - P_n(x)P_n''(x))\,dx \\ &\quad + \int_0^\infty (P_n'(x))^{r-j-2}(P_n(x))^{j+1}x^{p-1}e^{-x}P_n''(x)\,dx\,; \end{aligned}$$

using (4.2), we have

$$\begin{aligned} I_{r,j} \le \int_0^\infty &(P_n'(x))^{r-j-1}(P_n(x))^{j+1}x^{p-2}e^{-x}\,dx \\ &+ \int_0^\infty (P_n'(x))^{r-j-2}(P_n(x))^{j+1}x^{p-1}e^{-x}P_n''(x)\,dx\,. \end{aligned} \tag{4.4}$$

Next, we observe that

$$\begin{aligned} &\int_0^\infty P_n''(x)(P_n'(x))^{r-j-2}(P_n(x))^{j+1}x^{p-1}e^{-x}\,dx \\ &\quad = \frac{1}{r-j-1}\int_0^\infty \frac{d}{dx}(P_n'(x))^{r-j-1}(P_n(x))^{j+1}x^{p-1}e^{-x}\,dx \\ &\quad = -\frac{1}{r-j-1}\int_0^\infty (P_n'(x))^{r-j-1} \\ &\qquad\qquad \times \{-e^{-x}x^{p-1}(P_n(x))^{j+1} + (p-1)x^{p-2}e^{-x}(P_n(x))^{j+1} \\ &\qquad\qquad\qquad + (j+1)(P_n(x))^j P_n'(x)x^{p-1}e^{-x}\}\,dx\,. \end{aligned} \tag{4.5}$$

From (4.4) and (4.5) we obtain

$$\begin{aligned} I_{r,j} \le &\int_0^\infty (P_n'(x))^{r-j-1}(P_n(x))^{j+1}x^{p-2}e^{-x}\,dx \\ &+ \frac{1}{r-j-1}\int_0^\infty (P_n'(x))^{r-j-1}(P_n(x))^{j+1}e^{-x}x^{p-1}\,dx \\ &- \frac{p-1}{r-j-1}\int_0^\infty (P_n'(x))^{r-j-1}(P_n(x))^{j+1}x^{p-2}e^{-x}\,dx \\ &- \frac{j+1}{r-j-1}I_{r,j}\,. \end{aligned}$$

From above, we obtain

$$\begin{aligned} rI_{r,j} \le (r-p-j)&\int_0^\infty (P_n'(x))^{r-j-1}(P_n(x))^{j+1}x^{p-2}e^{-x}\,dx \\ &+ \int_0^\infty (P_n'(x))^{r-j-1}(P_n(x))^{j+1}x^{p-1}e^{-x}\,dx\,. \end{aligned} \tag{4.6}$$

Next, we note that

$$(P_n'(x))^{r-j-1}(P_n(x))^{j+1} = \sum_{k=1}^{(n-1)r+j+1} b_k x^k\,, \qquad b_k \ge 0\,. \tag{4.7}$$

Therefore,

$$\begin{aligned}\int_0^\infty (P_n'(x))^{r-j-1}(P_n(x))^{j+1}x^{p-1}e^{-x}\,dx &= \sum_{k=1}^{(n-1)r+j+1} b_k \int_0^\infty x^{k+p-1}e^{-x}\,dx \\ &= \sum_{k=1}^{(n-1)r+j+1} b_k(k+p-1)!\end{aligned}$$

and

$$\int_0^\infty (P_n'(x))^{r-j-1}(P_n(x))^{j+1}x^{p-2}e^{-x}\,dx = \sum_{k=1}^{(n-1)r+j+1} b_k(k+p-2)!.$$

From these two relations, it follows that

$$\begin{aligned}&\int_0^\infty (P_n'(x))^{r-j-1}(P_n(x))^{j+1}x^{p-1}e^{-x}\,dx \\ &\qquad \le ((n-1)r+j+p)\int_0^\infty (P_n'(x))^{r-j-1}(P_n(x))^{j+1}x^{p-2}e^{-x}\,dx.\end{aligned} \tag{4.8}$$

Therefore, using (4.8) and (4.6) we obtain

$$\begin{aligned}rI_{r,j} &\le \left(1+\frac{r-p-j}{(n-1)r+j+p}\right)\int_0^\infty (P_n'(x))^{r-j-1}(P_n(x))^{j+1}x^{p-1}e^{-x}\,dx \\ &= \frac{nr}{(n-1)r+j+p}\int_0^\infty (P_n'(x))^{r-j-1}(P_n(x))^{j+1}x^{p-1}e^{-x}\,dx.\end{aligned}$$

From above, (4.3) follows, for $j+1<r\le p$.

The proof of (4.3) for $j+1=r$ is as follows. From (4.8) we have

$$\int_0^\infty (P_n(x))^r x^{p-1}e^{-x}\,dx \le (nr+p-1)\int_0^\infty (P_n(x))^r x^{p-2}e^{-x}\,dx.$$

Also

$$\begin{aligned}&\int_0^\infty (P_n'(x))^{r-j}(P_n(x))^{j}x^{p-1}e^{-x}\,dx \\ &\quad = \int_0^\infty P_n'(x)(P_n(x))^{r-1}x^{p-1}e^{-x}\,dx \\ &\quad = \frac{1}{r}\int_0^\infty \frac{d}{dx}(P_n(x))^r x^{p-1}e^{-x}\,dx \\ &\quad = -\frac{1}{r}\int_0^\infty (P_n(x))^r(-e^{-x}x^{p-1}+(p-1)x^{p-2}e^{-x})\,dx \\ &\quad \le \left(\frac{1}{r}-\frac{p-1}{nr+p-1}\right)\int_0^\infty (P_n(x))^r x^{p-1}e^{-x}\,dx \\ &\quad = \frac{n}{nr+p-1}\int_0^\infty (P_n(x))^r x^{p-1}e^{-x}\,dx.\end{aligned}$$

From (4.3), we have $(j+1 \le r \le p)$

$$\begin{aligned} &\frac{\int_0^\infty (P_n'(x))^r x^{p-1} e^{-x}\,dx}{\int_0^\infty (P_n(x)^r x^{p-1} e^{-x}\,dx} \\ &\le \frac{n^r}{[(n-1)r+p][(n-1)r+p+1]\cdots[(n-1)r+p+r-1]} \\ &\le \frac{n^r(nr+p-r-1)!}{(nr+p-1)!} \end{aligned} \tag{4.9}$$

((4.9) become an equality for $P_n(x) = \alpha x^n$).

This proves Theorem 3 as well.

Acknowledgment

The author wishes to thank the referee for many valuable suggestions.

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