CARLEMAN INEQUALITIES FOR THE DIRAC OPERATOR
AND STRONG UNIQUE CONTINUATION

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Abstract. Using a Carleman inequality, we prove a strong unique continuation
theorem for the Schrödinger operator \( D + V \), where \( D \) is the Dirac operator
and \( V \) is a potential function in some \( L^p \) space.

1. Introduction

Let \( U \) be a nonempty connected open subset of \( \mathbb{R}^n \), and \( u \) be a solution of
the differential equation

\[
(D + V)u = 0.
\]

Here \( D \) is the Dirac operator and \( V \in L^s(\mathbb{R}^n) \) for some suitable \( s \). The main
theorem says if \( u \) vanishes to infinite order at a point, then \( u = 0 \) identically.
This is called a unique continuation theorem because it says that the behavior
of a solution at a point determines the behavior in a neighborhood. In 1939
Carleman [2] proved this theorem when \( n = 2 \) and \( V \) is bounded, and all
subsequent work follows his basic idea. The main step is to prove Carleman
inequalities. We need the following type of inequality:

\[
\|e^{t\phi} \nabla f\|_{L^p(U\setminus\{0\}), dx} \leq C\|e^{t\phi} \Delta f\|_{L^p(U\setminus\{0\}), dx}, \quad f \in C_0^\infty(U\setminus\{0\}), \quad \frac{1}{p} \frac{1}{q_1} = \frac{1}{r},
\]

for \( C \) independent of \( t \) as \( t \to \infty \) and \( U \) an open neighborhood of the origin,
where \( \phi \) is a suitable weight function which is radial and decreasing. Once
this inequality is proved, a straightforward argument due to Carleman yields
uniqueness. The key feature that distinguishes these inequalities from ordinary
Sobolev inequalities is that the constant \( C \) is independent of the parameter \( t \).
Our main contribution is to improve an earlier unique continuation theorem
due to Hörmander [3] or Jerison [4] which required that \( u \) vanish on an open
set rather than at a single point. In particular Hörmander proved inequalities
of type (1) in the special case in which the function \( f \) vanishes not only at
the origin, but also in a ball \( B \) about the origin of fixed positive radius. There
was a great deal of work going on and references can be found in [3, 10]. In
order to obtain optimal inequalities of type (1) with a radial decreasing weight

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function, we have to choose \( \phi \) carefully. We will use the weight function \( \phi \) defined implicitly by \( \phi(x) = \psi(y) = -\psi(y) + e^{-\psi(y)} \) when \( y = \log |x| < 0 \). The idea is from Alinhac-Baouendi [1]. Then \( e^{t\phi} \sim |x|^{-t} \). This is an algebraic blowup but still can be handled since \( u \) vanishes to infinite order at the origin. This is better than \( |x|^{-t} \) because of convexity: \( (\partial \psi / \partial y)^2 \geq e^{\psi} \).

2. Statements of results

The Dirac operator is a first-order constant coefficient operator on \( R^n \) of the form \( D = \sum_{j=1}^n \alpha_j \partial / \partial x_j \), where \( \alpha_1, \ldots, \alpha_n \) are skew hermitian matrices satisfying the Clifford relations: \( \alpha_j \alpha_k + \alpha_k \alpha_j = -2\delta_{jk}, j, k = 1, \ldots, n \). Also \( D^2 = -\Delta \) and Carleman estimates for \( D \) imply estimates of type (1).

Let \( \phi(x) = \psi(y) \) be defined as above.

**Theorem 1.** Let \( n \geq 3 \). There is a constant \( C \) depending only on \( n \) such that for all \( t \in R \) and all \( h \in C_0^\infty((-\infty, 0) \times S) \)

\[
(1') \quad \sqrt{t} \| e^{t\phi} h \|_{L^2((-\infty, 0) \times S, dy \, dw)} \leq C \| e^{t\phi} Dh \|_{L^2((-\infty, 0) \times S, dy \, dw)}.
\]

**Corollary.** Let \( U \ni 0 \) be a connected, open subset of \( R^n \).

Suppose we have a solution of a Schrödinger operator \( (D + V)u = 0 \) in \( U \), \( V \in L^\infty \) and \( \int_{|x|<e} |u(x)|^2 \, dx = 0(e^N) \) for any \( N \). Then \( u \equiv 0 \) on \( U \).

**Theorem 2.** Let \( n \geq 3 \), \( p = (6n - 4)/(3n + 2) \), i.e., \( 1/p - 1/2 = 1/y \), with \( y = (3n - 2)/2 \). There is a constant \( C \) depending only on \( n \) such that for all \( t \in R \)

\[
(1'') \quad \| e^{t\phi} f \|_{L^2((-\infty, 0) \times S, dy \, dw)} \leq C \| e^{t\phi} Df \|_{L^p((-\infty, 0) \times S, dy \, dw)}
\]

for all \( f \in C_0^\infty((-\infty, 0) \times S, C^m) \).

Moreover,

\[
(2) \quad \| e^{t\phi} \nabla f \|_{L^2((-\infty, 0) \times S, dy \, dw)} \leq C \| e^{t\phi} \Delta f \|_{L^p((-\infty, 0) \times S, dy \, dw)}
\]

for \( f \in C_0^\infty((-\infty, 0) \times S) \).

**Corollary 2.** Let \( \Omega \) be a connected, open subset of \( R^n \), \( n \geq 3 \). If \( V \in L^\gamma(\Omega; M(m, C)) \) and \( u \) satisfies \( Du \in L^2(\Omega; C^m) \), \( (D + V)u = 0 \) in \( \Omega \). If \( \int_{|x|<e} |u(x)|^2 \, dx = 0(e^N) \) for any \( N \), then \( u \) is identically zero in \( \Omega \).

First, we want to set up some notation and elementary results, following Jerison [4].

2.1. Polar coordinates. Let \( S \) denote the unit sphere in \( R^n \). For \( y \in R \) and \( w \in S \), \( x = e^y w \) gives polar coordinates on \( R^n \), i.e., \( y = \log |x| \) and \( w = x/|x| \). The operator

\[
L = \sum_{j<k} \alpha_j \alpha_k \left( x_j \frac{\partial}{\partial x_k} - x_k \frac{\partial}{\partial x_j} \right)
\]

acts only in the \( w \)-variables \( -[L, \partial / \partial y] = 0 \). We will view \( L \) as an operator on the sphere \( S \). Let

\[
\hat{\alpha} = \sum_{j=1}^n \alpha_j x_j / |x|.
\]
Then
\[ \hat{\alpha}D = e^{-y} \left[ \frac{\partial}{\partial y} - L \right] ; \]
and since \( \hat{\alpha}^2 = -1 \),
\[ (3) \quad e^{y}D = \hat{\alpha} \left( \frac{\partial}{\partial y} - L \right). \]
Note that \( \hat{\alpha} \) is unitary and \( L^* = L \). If we recall that
\[ (4) \quad \Delta = e^{-2y} \left( \frac{\partial^2}{\partial y^2} + (n - 2) \frac{\partial}{\partial y} + \Delta_S \right), \]
where \( \Delta_S \) denotes the Laplace-Beltrami operator of the sphere, then it follows from \( D^* = D \), \( D^2 = -\Delta \) that
\[ (5) \quad L(L + n - 2) = -\Delta_S. \]
In general if \( \psi \in C^{\infty}(R) \), then (3) implies that in polar coordinates \( x = e^{y}w \),
\[ (6) \quad e^{y}\psi(y) h = \hat{\alpha}A_{y}h \]
where \( A_{y} = \partial / \partial y - (t\psi'(y) + L) \).

Proof of Theorem 1. To prove the inequality, it suffices to show \( A_{y}A_{y} \geq ct\psi''(y) \), since this implies
\[ \|A_{y}f\|_{L_2}^2 = \langle A_{y}A_{y}h, h \rangle \geq \langle t\psi''(y) h, h \rangle = t\|\sqrt{\psi''(y)}h\|_{L_2(dx)}^2. \]
But
\[ A_{y}A_{y} = \left( -\frac{\partial}{\partial y} - n - (t\psi'(y) + L) \right) \left( \frac{\partial}{\partial y} - (t\psi'(y) + L) \right) \geq t\psi''(y). \]
So the claim is true. We had the relation \( y = -\psi(y) + e^{-\psi(y)} \). From this, we get
\[ \psi'(y) = -1/1 + e^{-\psi(y)} < 0. \]
We also find
\[ \psi''(y) = e^2 e^{-\psi(y)}/(1 + e^{-\psi(y)})^3 \geq ce^{\psi(y)}. \]

Now we want to prove Theorem 2.

Proof of Theorem 2. We will prove the following inequality first and prove the dual version later:
\[ \|f\|_{L^2(e^{\psi}dydw, R^{-\times}S)} \leq C\|A_{y}f\|_{L^2(e^{\psi}dydw, R^{-\times}S)} \quad \text{for} \quad f \in C_0^{\infty}(U). \]
We can rewrite
\[ A_{y}f = \sum_{k} \left( \frac{\partial}{\partial y} - (t\psi'(y) + k) \right) \pi_{k}f. \]
If we have an operator of type \( \partial/\partial y - ay + b \) for \( a, b \) constant coefficients and \( a > 0 \), then we can find a left inverse operator for \( \partial/\partial y - ay + b \) easily. So first consider an operator
\[ \Omega = d/dy - y. \]
Jerrson [4] exhibited the following exact formula for the symbol of a left inverse
of $\Omega$: there is a unique operator $B$ on $R$ satisfying $B\Omega = I$ and $(Be^{-y^2/2}) = 0$ given by
$$Bf(y) = (1/2\pi) \int F_0(y, \eta)e^{iy\eta} \hat{f}(\eta) d\eta,$$
where
$$F_0(y, \eta) = \sqrt{2} \int_0^\infty e^{-s^2 - 2sy} ds e^{-i\eta(y^2 + \eta^2)/2} - \int_0^\infty e^{-s^2 - s(y - \eta)} ds.$$  

Now if we have an operator $\partial/\partial y - ay + b$, then
$$\sigma(y, \eta; a, b) = \frac{1}{\sqrt{a}} F_0 \left( \sqrt{ay} \delta - \frac{b}{\sqrt{a}}, \frac{\eta}{\sqrt{a}} \right)$$
is the symbol of the left inverse of $\partial/\partial y - ay + b$.

Then by the method of freezing coefficients, we get an approximate symbol for the inverse of $\partial/\partial y - (t\psi'(y) + k)$. Namely,
$$F(y, \eta) = \sigma(y, \eta; t\psi''(y), -t\psi'(y) + t\psi''(y)y - k).$$

Also the following symbol estimate is true:

$$\left| \left( \frac{\partial}{\partial y} \right)^j \left( \frac{\partial}{\partial \eta} \right)^l F_0(y, \eta) \right| \leq C_{j,l}(1 + |y + i\eta|)^{-1-j-l}, \quad j, l = 0, 1, \ldots$$

From (8) we have similar estimates for our symbol $F(y, \eta)$

$$\left| \left( \frac{\partial}{\partial y} \right)^j \left( \frac{\partial}{\partial \eta} \right)^l F(y, \eta) \right| 
\leq C_{j,l}(\sqrt{a} + |t\psi'(y) + k - i\eta|)^{-1-j-l}(a + |t\psi'(y) + k|)^l.$$  

The main tool in the proof is the spherical restriction theorem of Sogge [8].

**Theorem.** Let $\xi_k$ denote the projection operator from $L^2(S)$ to the space of spherical harmonics of degree $k$. Then there is a constant $c$ such that
$$||\xi_k g||_{L^{p'}(S)} \leq c k^{-2/|s|} ||g||_{L^p(S)},$$
where $p = 2n/(n + 2)$, $p' = 2n/(n - 2)$. Formula (5) implies that
$$(L + (n - 2)/2)^2 = -\Delta_s + (n - 2)^2/4.$$  
Hence
$$T = \text{sgn}(L + (n - 2)/2)(L + (n - 2)/2)(-\Delta_s + (n - 2)^2/4)^{-1/2}$$
is a classical pseudodifferential operator on $S$. Thus $T$ is bounded from $L^q(S; C^m)$ to $L^q(S; C^m)$ for all $q$, $1 < q < \infty$. Moreover,

$$\pi_k = \frac{1}{2} (1 + T)\xi_k, \quad k = 0, 1, 2, \ldots,$$
$$\pi_k = \frac{1}{2} (1 - T)\xi_k, \quad k = 1 - n, -n, -n - 1, \ldots.$$
Therefore, Sogge's theorem implies that
\[ \|\pi_k g\|_{L^p(S; C^m)} \leq C k^{1-2/n} \|g\|_{L^p(S; C^m)}. \]

Define \( \pi_{M, N} \) by
\[ \pi_k \pi_{M, N} g = \{ \pi_k g \text{ if } M \leq k \leq N, \hat{0} \text{ otherwise} \}. \]

The triangle inequality implies
\[ \|\pi_{M, N} g\|_{L^p(S; C^m)} \leq C N^{1-2/n} (N - M + 1) \|g\|_{L^p(S; C^m)}. \]

Next use a device due to Tomas [11]:
\[ \|\pi_{M, N} g\|_{L^2(S; C^m)} = \int_S (\pi_{M, N} g, g) \leq \|\pi_{M, N} g\|_{L^p(S; C^m)} \]
\[ \leq C N^{1-2/N} (N - M + 1) \|g\|_{L^2(S; C^m)}. \]

We conclude that
\[ \|\pi_{M, N} g\|_{L^2(S; C^m)} \leq C N^{1/p'} (N - M + 1)^{1/2} \|g\|_{L^p(S; C^m)} \]
and by duality
\[ \|\pi_{M, N} g\|_{L^{p'}(S; C^m)} \leq C N^{1/p'} (N - M + 1)^{1/2} \|g\|_{L^2(S; C^m)}. \]

If we interpolate with the trivial estimate
\[ \|\pi_{M, N} g\|_{L^q(S; C^m)} \leq \|g\|_{L^2(S; C^m)} \]
we find that
\[ \|\pi_{M, N} g\|_{L^q(S; C^m)} \leq C (N^{(n-2)/2} (N - M + 1)^{n/2})^{1/2-1/q} \|g\|_{L^2(S; C^m)} \]
for \( 2 \leq q \leq p' = 2n/n - 2. \)

Let \( N \) be the integer satisfying \( 2N - 1 \leq 10e^{x/2} t^{1/2} \leq 2N \). Consider a partition of unity \( \{\phi_\beta\}_{\beta=0}^N \) of the positive real axis satisfying
\[ \sum_{\beta=0}^N \phi_\beta(r) = 1, \quad \text{all } r > 0, \]
\[ \text{supp } \phi_\beta \subset \{r : 2^{\beta-2} \leq r \leq 2^\beta\}, \quad \beta = 1, 2, \ldots, N - 1, \]
\[ \text{supp } \phi_0 \subset \{r : r \leq 1\}, \quad \text{supp } \phi_N \subset \{r : r \geq s/400\}, \]
\[ \left| \left( \frac{d}{dr} \right)^l \phi_\beta(r) \right| \leq C l 2^{-\beta l}, \quad l = 0, 1, \ldots. \]

Define
\[ F_t f(y, w) = \sum_k \frac{1}{2\pi} \int F_t(y, \eta, k) \pi_k \tilde{f}(\eta, \cdot)(w) e^{i\eta y} d\eta. \]

Define
\[ F_t^\beta(y, \eta, k) = \phi_\beta \left( \frac{1}{\sqrt{a}} |\psi'(y) + k - i\eta| \right) F_t(y, \eta, k). \]
Then $F_t$ satisfies

$$
\left| \left( \frac{\partial}{\partial \eta} \right)^j \left( \frac{\partial}{\partial y} \right)^l F_t(y, \eta, k) \right| 
\leq C_j, l(\sqrt{a} + |t\psi'(y)| \delta + k - i\eta)^{-1-j-l} (a + |t\psi'(y) + k|)^l,
$$

From (12) and the property of $| (\partial / \partial \tau)^l \phi_\beta(r) | \leq 2^{-\beta l}$, we deduce that the following inequalities hold uniformly for $y \in I = I_t = (-l, -l + 1)$

$$
\left( \frac{\partial}{\partial \eta} \right)^j (F_t(y, \eta, k) - F_t(y, \eta, k + 1)) \leq C_j (2^\beta \sqrt{a})^{-1-j},
$$

Now if we define

$$
(F_t^\beta f)(y, w) = \sum_k \frac{1}{2\pi} \int F_t^\beta(y, \eta, k) \pi_k \hat{f}(\eta, \cdot)(w) e^{iy\eta} d\eta,
$$

then $F_t = \sum_{\beta=0}^N F_t^\beta$. We begin by estimating $F_t^N$. In the case $\beta = N$, we need different estimates. By a choice of $N$ such that $2^N \sim 10e^{j\sqrt{a}}$ we have the following.

Since $F_t^N$ is supported where

$$|t\psi'(y) + k - i\eta| \geq 2^N \sqrt{a} \sim 10t,$$

we have

$$|t\psi'(y) + k - i\eta| > c(1 + |\eta| + |k|) \quad \text{uniformly for } y < 0.$$

Hence

$$
\left( \frac{\partial}{\partial \eta} \right)^j \left( \frac{\partial}{\partial y} \right)^m F_t^N(y, \eta, k) \leq C_j, m(1 + |\eta| + |k|)^{-1-j-\delta}, \quad j = 0, 1, \ldots.
$$

It follows that $F_t^N$ is controlled by standard pseudodifferential operators and by the Sobolev inequality

$$
\|f\|_{L^p(I \times S, e^{\rho y} dy du)} \leq \|rDf\|_{L^2(I \times S, e^{\rho y} dy du)}, \quad \text{for all } f \in C_0^\infty(I \times S)
$$

and $p' = 2n/(n - 2)$.

We have

$$
\|F_t^N f\|_{L^q(I \times S, dx)} \leq C\|f\|_{L^2(I \times S, dx)} \quad \text{for all } f \in C_0^\infty(I \times S; C^m) \text{ for } 1 \leq q \leq p'. \quad \text{In particular this holds for } q = (6n - 4)/(3n - 6).
$$

Let

$$
M = [-t\psi'(y) - 2^\beta \sqrt{a}], \quad M' = [M + 2 \times 2^\beta \sqrt{a}] + 1.
$$
Denote
\[ T_\beta^i(y, \eta) g(w) = \sum_k F_\beta^i(y, \eta, k) \pi_k g(w). \]

Here \( F_\beta^i(y, \eta, k) = 0 \) unless \( M \leq k \leq M' \). Summation by parts gives
\[ T_\beta^i(y, \eta) = \sum_{M} (F_\beta^i(y, \eta, k) - F_\beta^i(y, \eta, k + 1)) \pi_{M, k} \quad \text{for} \quad M \leq k \leq M'. \]

Now (10) and (13) give
\[ \left\| \left( \frac{\partial}{\partial \eta} \right)^j T_\beta^i(y, \eta) \pi_{M, k} g \right\|_{L^q(S; C^m)} \leq C_j \left( 2^\beta \sqrt{a} \right)^{-1-j} \left( \sqrt{\pi(n-2)/2} \left( 2^\beta \sqrt{a} \right)^{n/2} \right)^{1/2-1/q} \| g \|_{L^2(S; C^m)} \]
uniformly for \( y \in I \).

Define
\[ K_\beta^i(y, z) = \frac{1}{2\pi} \int T_\beta^i(y, \eta) e^{iz\eta} d\eta \]
\[ = \frac{1}{2\pi} \int \left( \frac{\partial}{\partial \eta} \right)^j T_\beta^i(y, \eta) \frac{1}{(iz)^j} e^{iz\eta} d\eta; \]
and since the length of the interval in \( \eta \) where \( T_\beta^i \) is nonzero is less than \( 2 \times 2^\beta \sqrt{a} \),
\[ \| K_\beta^i(y, z) g \|_{L^q(S; C^m)} \leq C(1 + |2^\beta z|)^{-10} \left( \sqrt{\pi(n-2)/2} \left( 2^\beta \sqrt{a} \right)^{n/2} \right)^{1/2-1/q} \| g \|_{L^2(S; C^m)}. \]

Note that
\[ F_\beta^i f(y, w) = \int K_\beta^i(y, y-y') f(y', \cdot)(w) dy'. \]

**Lemma.** Let \( H(y, y') \) be a bounded operator from \( L^p(S) \) to \( L^q(S) \) of operator norm \( \leq h(y-y') \) for each \( y, y' \in R \). Suppose that \( h \in L^r(R) \) for \( 1/r + 1/p = 1 + 1/q \). Then
\[ T f(y, w) = \int H(y, y') f(y', \cdot)(w) dy' \]
satisfies
\[ \| T f \|_{L^q(R \times S)} \leq \| h \|_{L^r(R)} \| f \|_{L^p(R \times S)}. \]

Now we see that the lemma implies for \( \beta \leq N - 1 \),
\[ \| F_\beta^i f \|_{L^q(I_j \times S, dx)} \leq C 2^{-(n-2)\beta/2} \sqrt{e^y} \| f \|_{L^2(I_j \times S, dx)}, \quad m > -1/3. \]

If we sum the series in \( \beta \) and add the final term \( \beta = N \), we get
\[ \sum_\beta F_\beta^i f \leq C' e^{ij/3} \| f \|_{L^2(I_j \times S, e^{y \sqrt{+}} dx \ dy u)}. \]

So far we have obtained estimates only for the main term, so we will work on the remainder term from now on.
From the relation \( f(y) = F_t A_t f(y) - R_t f(y) \), we have
\[
R_t f(y) = \int_{I_1} F_t(y, \eta, k) t(y - y')^2 g(y, y') e^{i(y-y')n} f(y', \cdot) \, dy' \, d\eta k
\]
for
\[
g(y, y') = \int_0^1 (1 - s) \psi'''(y)(y + s(y' - y)) \, ds.
\]
We hope to obtain a similar type of inequality, i.e.,
\[
\|R_t f\|_{L^1(I_1 \times S, \alpha x)} \leq C t \|f\|_{L^2(I_1 \times S, d\alpha)} \quad \text{in } I_1 = (-l, -l + 1).
\]
For that we are going to adopt the same techniques as in the main step. Then using the same partition of unity, \( \{\phi_\beta\}_{\beta=0,...,N} \),
\[
R_t f(y) = \sum \int_{I_1} F_t^\beta(y, \eta, k) t(y - y')^2 e^{i\eta(y-y')} g(y, y') f(y', \cdot) \, dy' \, d\eta k.
\]
Let's denote \( \tilde{K}_t^\beta(y, y') \) as follows:
\[
\tilde{K}_t^\beta(y, y') = \frac{1}{2\pi} \int t T_t^\beta(y, \eta)(y - y')^2 e^{i\eta(y-y')} g(y, y') \, d\eta.
\]
Then the following relation holds:
\[
R_t f(y, w) = \int \tilde{K}_t^\beta(y, y') f(y', \cdot) \, dy'.
\]
Since \( a = t \psi''(y) \sim t e^{-x} \) uniformly for \( y \in I_1 \) and \( \|g\|_\infty \leq e^{-x} \) for \( y, y' \in I_1 \).

As a result, \( \|t(2^\beta \sqrt{a})^{-2} g\|_\infty \leq C' \). Then following the same steps as before we obtain
(16')
\[
\|R_t f\|_{L^1(I_1 \times S, \alpha x)} \leq C' t^{\frac{2}{3}} \|f\|_{L^2(I_1 \times S, \alpha x)} \quad \text{when } y = (3n - 2)/2.
\]
The case \( \beta = N \) works for the same reason as in the main terms: from the definition,
\[
\tilde{F}_t^N(y, y', \eta, k) = (y - y')^2 g(y, y') F_t^N.
\]
Now as in the main step this operator is controlled by standard pseudodifferential operators and we can deduce
\[
\|\tilde{F}_t^N f\|_{L^1(I_1 \times S, \alpha x)} \leq C \|f\|_{L^2(I_1 \times S, d\alpha)}.
\]
If we sum the series in \( \beta \) and add the final term \( \beta = N \), we obtain
(14')
\[
\|R_t f\|_{L^1(I_1 \times S, \alpha x)} \leq C e^{e^{ij/3}} \|f\|_{L^2(I_1 \times S, d\alpha)}.
\]
Now with the estimate on the unit anulus, i.e. \( I_1 \times S \), we try to extend it to the whole ball, in this case \( R^{-} \times S \). First, we restate Theorem 1 as
\[
\|f\|_{L^2(I_1 \times S, e^{\eta y} \, dy \, dw)} \leq C e^{e^{ij/2}} \|A_t f\|_{L^2(I_1 \times S, e^{\eta y} \, dy \, dw)}.
\]
Combining this with (14') we obtain
\[
\|R_t f\|_{L^1(I_1 \times S, e^{\eta y} \, dy \, dw)} \leq C e^{e^{ij/6}} \|A_t f\|_{L^2(I_1 \times S, e^{\eta y} \, dy \, dw)}.
\]
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\[ \norm{f}_{L^q(I \times S, e^{\eta y} \, dy \, dw)} \leq C e^{5 \epsilon j/6} \norm{A_t f}_{L^2(I \times S, e^{\eta y} \, dy \, dw)} . \]

But with the main estimates, the above implies
\[ \norm{f}_{L^q(I \times S, e^{\epsilon(q+\lambda)y} \, dy \, dw)} \leq C \norm{A_t f}_{L^2(I \times S, e^{\eta y} \, dy \, dw)} . \]

Now choose \( \{\psi_{jk}\}_{j \in \mathbb{N}, k=1,2} \) to be partitions of unity such that
\[ \psi_{j1} \in C_0^\infty(-j, -j + 3/4), \quad \psi_{j2} \in C_0^\infty(-j + 2/4, -j + 5/4). \]

Then using \( f = \sum \psi_j f \) (we will just call \( \{\psi_{jk}\}, \{\psi_j\} \)), we come to the final estimate:
\[
\begin{align*}
\norm{f}_{L^q(e^{\gamma y} \, dy \, dw, R^{-} \times S)} & \leq (1) C_0 \sum_j \norm{\psi_j f}_{L^q(e^{\gamma y} \, dy \, dw, I_j \times S)} \\
& \leq C_1 \sum_j \norm{A_t(\psi_j f)}_{L^2(e^{\gamma y} \, dy \, dw, I_j \times S)} \\
& \leq C_2 \sum_j \norm{\psi_j f}_{L^2(e^{\gamma y} \, dy \, dw, I_j \times S)} + C_2 \sum_j \norm{\psi_j A_t f}_{L^2(e^{\gamma y} \, dy \, dw, I_j \times S)} \\
& \leq (2) C' \norm{f}_{L^2(e^{\gamma y} \, dy \, dw, R^{-} \times S)} + C' \norm{A_t f}_{L^2(e^{\gamma y} \, dy \, dw, R^{-} \times S)} \\
& \leq (3) C'' \norm{A_t f}_{L^2(e^{\gamma y} \, dy \, dw, R^{-} \times S)} .
\end{align*}
\]

Inequalities (1) and (2) hold since for each \( x \in R \), only finitely many \( \psi_j \)'s overlap. Inequality (3) comes from \( L^2 \) estimates. The above estimate is equivalent to
\[ (17') \quad \norm{e^{\eta y} f}_{L^q(R^{-} \times S, e^{\eta y} \, dy \, dw)} \leq C \norm{e^{\eta y} D f}_{L^2(R^{-} \times S, e^{\eta y} \, dy \, dw)} . \]

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