ON POLYNOMICALLY BOUNDED OPERATORS
WITH RICH SPECTRUM

RADU GADIDOV

(Communicated by Palle E. T. Jorgensen)

Abstract. D. Westood (J. Funct. Anal. 66 (1986), 96–104) proved that \( C_{00} \)-contractions with dominating spectrum are in \( A_{W_0} \). We generalize this result to polynomially bounded operators.

1. Introduction

Let \( \mathcal{H} \) be a complex, separable, infinite dimensional Hilbert space, and let \( \mathcal{B}(\mathcal{H}) \) be the algebra of all bounded, linear operators on \( \mathcal{H} \). Recall that an operator \( T \in \mathcal{B}(\mathcal{H}) \) is called polynomially bounded (notation \( T \in (PB)(\mathcal{H}) \)) if there exists a constant \( K \geq 1 \) such that for every polynomial \( p \),

\[
\|p(T)\| \leq K \sup\{|p(z)| : |z| = 1\}.
\]

Of course, all contraction operators in \( \mathcal{B}(\mathcal{H}) \) are polynomially bounded, and in the past fifteen years the theory of dual algebras generated by a single contraction operator has been used very succesfully to obtain information about the structure of such operators (see for example [1], [2], [5], [6]). More recently (cf. [11], [12], [13], [15], etc.), researchers have begun to use the theory of dual algebras generated by an arbitrary polynomially bounded operator to extract structural information about such operators. As was pointed out in [11], however, many parts of the theory for contraction operators do not readily generalize to the case of polynomially bounded operators. The purpose of this note is to make a modest contribution to this theory, by proving a generalization (Theorem 2 below) of the main result in [16] and one of the results in [11]. Before stating Theorem 2, we recall some notation and definitions from this theory.

If \( T \) is in \( \mathcal{B}(\mathcal{H}) \) and \( \mathcal{M} \) is a (closed) subspace of \( \mathcal{H} \), then \( T_{\mathcal{M}} \) denotes the compression of \( T \) to \( \mathcal{M} \), i.e., \( T_{\mathcal{M}} = P_{\mathcal{M}} T P_{\mathcal{M}} \), where \( P_{\mathcal{M}} \) denotes the orthogonal projection from \( \mathcal{H} \) onto \( \mathcal{M} \). Also the spectrum of \( T \), the point spectrum of \( T \) and the essential spectrum of \( T \) will be denoted by \( \sigma(T) \), \( \sigma_p(T) \) and \( \sigma_e(T) \), respectively. Moreover, \( C_{00}(\mathcal{H}) \) is the set of all operators \( T \) in \( \mathcal{B}(\mathcal{H}) \) such that the sequences \( \{T^n\}_{n=1}^{\infty} \), \( \{T^* n\}_{n=1}^{\infty} \) converge to 0 in the strong operator topology on \( \mathcal{B}(\mathcal{H}) \).

Received by the editors August 23, 1993 and, in revised form, October 27, 1993.

1991 Mathematics Subject Classification. Primary 47A15; Secondary 47A60.

©1995 American Mathematical Society
It is well known (cf. [9]) that $\mathcal{B}(\mathcal{H})$ is the dual space of the Banach space $\mathcal{C}(\mathcal{H})$ of trace-class operators on $\mathcal{H}$ equipped with the trace-norm $\| \|_1$, and the duality is implemented by the bilinear form $\langle T, L \rangle = \text{trace}(TL)$, $T \in \mathcal{B}(\mathcal{H})$, $L \in \mathcal{C}(\mathcal{H})$. If $T$ is an operator in $\mathcal{B}(\mathcal{H})$, $\mathcal{A}_T$ will denote the dual algebra generated by $T$ (i.e., the smallest weak*-closed algebra containing $T$ and the identity operator on $\mathcal{H}$), $\mathcal{E}_T (= \mathcal{C}_1 / \mathcal{A}_T$) the natural predual of $\mathcal{A}_T$. For any $L \in \mathcal{C}(\mathcal{H})$ the corresponding element in $\mathcal{E}_T$ will be denoted by $[L]_T$. In particular, for any vectors $x$ and $y$ in $\mathcal{H}$, $[x \otimes y]_T$ is the image in $\mathcal{E}_T$ of $x \otimes y$, where $x \otimes y$ denotes the usual rank one operator in $\mathcal{B}(\mathcal{H})$.

As usual $D$ denotes the open unit disc in $\mathbb{C}$, and $T = \partial D$. If $E$ is a measurable subset of $T$ (with respect to normalized Lebesgue measure $m$ on $T$), a set $\Lambda \subset D$ is said to be dominating for $E$ if almost every point of $E$ is a nontangential limit of a sequence of points from $\Lambda$, and the set of all nontangential limits of $\Lambda$ on $T$ will be denoted by $\text{NTL}(\Lambda)$. The spaces $\mathcal{L}^1 := \mathcal{L}^1(T)$, $\mathcal{H}^1 := \mathcal{H}^1(T)$ and $\mathcal{H}^\infty := \mathcal{H}^\infty(T)$ are the usual Lebesgue and Hardy function spaces on $T$, relative to the measure $m$. It is easy to see that if $T \in (PB)(\mathcal{H})$, there exists a smallest number $M$ such that (1) is valid for every polynomial $p$, and we denote the set of all $T \in (PB)(\mathcal{H})$ for which $M$ is the smallest such number by $(PB)^M(\mathcal{H})$ (cf. [11]). If $T \in (PB)^M(\mathcal{H})$, it is easy to see that for any pair of vectors $x$ and $y$ in $\mathcal{H}$ there exists a measure $\mu_{x,y}$ on $T$ such that for every polynomial $p$,

\begin{equation}
\langle p(T)x, y \rangle = \int_T pd\mu_{x,y},
\end{equation}

and the operator $T$ is called absolutely continuous (notation $T \in (ACPB)^M(\mathcal{H})$) if for every pair $x$, $y$ in $\mathcal{H}$ there exists an absolutely continuous measure $\mu_{x,y}$ satisfying (2) (with respect to $m$).

For absolutely continuous polynomially bounded operators it is well known (cf. [11]) that there exists a unique unital, norm continuous algebra homomorphism

$$\Phi_T : \mathcal{H}^\infty \to \mathcal{A}_T$$

onto a weak* dense subalgebra of $\mathcal{A}_T$ such that $\Phi_T$ extends the Riesz-Dunford functional calculus, $\Phi_T$ is continuous if both $\mathcal{H}^\infty$ and $\mathcal{A}_T$ are given their weak* topologies, and $\Phi_T$ is the adjoint of a bounded, linear, one to one map

$$\phi_T : \mathcal{E}_T \to \mathcal{L}^1 / \mathcal{H}^1_0.$$

Let us also recall (cf. [11]) that the class $\mathcal{A}^M(\mathcal{H})$ is the set of all $T \in (ACPB)^M(\mathcal{H})$ for which $\Phi_T$ is bounded below. In this case $\Phi_T$ is also a weak* homeomorphism between $\mathcal{H}^\infty$ onto $\mathcal{A}_T$, when $\mathcal{H}^\infty$ and $\mathcal{A}_T$ are given their weak* topologies, and $\phi_T$ is onto.

For any $f$ in $\mathcal{L}^1$, $[f]_{\mathcal{L}^1 / \mathcal{H}^1_0}$ denotes the image of $f$ in $\mathcal{L}^1 / \mathcal{H}^1_0$ under the canonical projection from $\mathcal{L}^1$ onto $\mathcal{L}^1 / \mathcal{H}^1_0$. If $\lambda \in D$ and $P_\lambda$ is the associated Poisson kernel on $T$ (i.e., $P_\lambda(t) := \frac{(1-|\lambda|^2)}{|1-\lambda e^{it}|^2}$), we write

$$[C_\lambda]_T = \phi_T^{-1}([P_\lambda]_{\mathcal{L}^1 / \mathcal{H}^1_0}),$$

and it is easy to check that for any function $h$ in $\mathcal{H}^\infty$,

$$\langle \Phi_T(h), [C_\lambda]_T \rangle = h(\lambda).$$
If \( T \in \mathcal{A}^M(\mathcal{H}) \), then, as is customary, \( \mathcal{E}_0(\mathcal{A}_T) \) denotes the set of all \( [L]_T \) in \( \mathcal{E}_T \) for which there exist sequences \( \{x_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty} \) in the unit ball of \( \mathcal{H} \) such that

\[
\begin{align*}
&\text{(i) } \lim_{n \to \infty} \|[L]_T - [x_n \otimes y_n]_T\| = 0, \text{ and} \\
&\text{(ii) } \lim_{n \to \infty} (\|[x_n \otimes w]_T\| + \|[w \otimes y_n]_T\|) = 0 \text{ for any } w \in \mathcal{H},
\end{align*}
\]

and \( \mathcal{A}^M(\mathcal{H}) \) has property \( \mathcal{E}_{0,\theta} \) (\( \theta \in (0, 1] \)) if \( \mathcal{E}_0(\mathcal{A}_T) \) (which is (cf. [4]) absolutely convex and norm closed) contains the closed ball in \( \mathcal{E}_T \) centered at 0 with radius \( \theta \).

The following result comes from [11], and will be needed in the sequel.

**Lemma 1.** Let \( T \in \mathcal{A}^M(\mathcal{H}) \cap C_{00}(\mathcal{H}) \).

(i) If \( \{x_n\}_{n=1}^{\infty} \) is a sequence of vectors converging weakly to 0, then for any vector \( z \in \mathcal{H} \),

\[
\lim_{n \to \infty} (\|[x_n \otimes z]_T\| + \|[z \otimes x_n]_T\|) = 0.
\]

(ii) If \( \lambda \in \sigma_e(T) \cap D \), then \( [C\lambda]_T \in \mathcal{E}_0(\mathcal{A}_T) \).

Finally, we write, as is customary, \( \mathcal{A}^M_{N_0}(\mathcal{H}) \) for the set of those operators \( T \) in \( \mathcal{A}^M(\mathcal{H}) \) such that for any doubly indexed sequence \( \{[L_{ij}]_T\}_{i,j=1}^{\infty} \) of elements of \( \mathcal{E}_T \), there exist sequences \( \{x_i\}_{i=1}^{\infty} \) and \( \{y_j\}_{j=1}^{\infty} \) of vectors in \( \mathcal{H} \) such that

\[
[L_{ij}]_T = [x_i \otimes y_j]_T, \quad 1 \leq i, \ 1 \leq j.
\]

Now we may state the main result of this note.

**Theorem 2.** Let \( T \in (PB)^M(\mathcal{H}) \cap C_{00}(\mathcal{H}) \) be such that \( \sigma(T) \cap D \) dominates \( T \). Then \( T \in \mathcal{A}^M_{N_0}(\mathcal{H}) \).

2. The details

In this section we prove Theorem 2.

Since for any function \( h \in \mathcal{H}^\infty \), \( h(\sigma(T) \cap D) \subset \sigma(\Phi_T(h)) \), it follows that \( \Phi_T \) is bounded below, so \( T \in \mathcal{A}^M(\mathcal{H}) \). Thus by Theorem 3.7 of [2] it is sufficient to show that \( \mathcal{A}_T \) has property \( \mathcal{E}_{0,\theta} \) for some \( \theta \in (0, 1] \). The following lemma is the main ingredient in showing this.

**Lemma 3.** Suppose \( \epsilon, \delta \) are positive numbers, \( f \) is a nonnegative function in \( L^1 \), and \( \{y_j\}_{j=1}^{p} \) is a finite sequence of vectors in \( \mathcal{H} \). Then there exists \( x \in \mathcal{H} \) such that

\[
\begin{align*}
&\text{(i) } \|\phi_T^{-1}([f]_{L^1}^{H^0}) - [x \otimes x]_T\| < \epsilon, \\
&\text{(ii) } \|x\| \leq 2 \|f\|_1^{1/2}, \quad \|[x \otimes y_j]_T\| + \|[y_j \otimes x]_T\| < \delta, \quad j = 1, \ldots, p.
\end{align*}
\]

**Proof.** Define \( \Gamma_1 = (\sigma_p(T) \setminus \sigma_e(T)) \cap D \), \( \Gamma_2 = (\sigma(T) \setminus (\sigma_p(T) \cup \sigma_e(T))) \cap D \), \( \Gamma_3 = NTL(\sigma_e(T)) \), and \( \tilde{\Gamma}_1 = NTL(\Gamma_1) \), \( \tilde{\Gamma}_2 = NTL(\Gamma_2) \), and \( \tilde{\Gamma}_3 = NTL(\sigma_e(T) \cap D) \). First we consider \( f\chi_{\Gamma_1} \). By Lemma 1.2 of [3], there exist a finite sequence of positive numbers \( \{\alpha_j\}_{j=1}^{n_1} \) and a finite sequence \( \{\lambda_j\}_{j=1}^{n_1} \) of distinct points in \( \Gamma_1 \), such that

\[
\sum_{j=1}^{n_1} \alpha_j^{(1)} \leq \|f\chi_{\Gamma_1}\|_1
\]
and

\[ \|f_{X_{1}} - \sum_{j=1}^{n_1} \alpha_j^{(1)} P_{\alpha_j^{(1)}}\|_1 < \varepsilon/5. \]

For each \( j \) choose a vector of norm one \( x_j^{(1)} \in \ker(\lambda_j^{(1)} - T) \), and define
\[ \mathcal{H}_1 = \text{span}\{x_j^{(1)}\}_{j=1}^{n_1}. \]
Then \( \mathcal{H}_1 \in \text{Lat}(T) \), and by the choice of the sequence \( \{\lambda_j^{(1)}\}_{j=1}^{n_1} \), the set \( \{x_j^{(1)}\}_{j=1}^{n_1} \) is linearly independent. So \( \dim \mathcal{H}_1 = n_1 \), and \( T_{\mathcal{H}_1} \) has the eigenvectors \( \{x_j^{(1)}\}_{j=1}^{n_1} \) corresponding to the distinct eigenvalues \( \{\lambda_j^{(1)}\}_{j=1}^{n_1} \). Therefore by Theorem 2.2 of [16] there exists \( x^{(1)} \) in \( \mathcal{H}_1 \) with

\[ \|x^{(1)}\| \leq \|f_{X_{1}}\|^{1/2} \]

such that

\[ [x^{(1)} \otimes x^{(1)}]_T = \sum_{j=1}^{n_1} \alpha_j^{(1)} [C_{\lambda_j^{(1)}}]_{T}. \]

Hence by (3),

\[ \|\phi^{-1}_T([f_{X_{1}}]_{L^1(H)}) - [x^{(1)} \otimes x^{(1)}]_T\| < \varepsilon/5. \]

Since \( \mathcal{H}_1 \) is finite dimensional and invariant for \( T \),

\[ T_{\mathcal{H}_1} \in (ACPB)^{M}(\mathcal{H} \otimes \mathcal{H}_1), \]

\( \sigma(T_{\mathcal{H}_1}) \cap \mathcal{D} \) dominates \( T \), \( \Gamma_1 \setminus \{\lambda_j^{(1)}\}_{j=1}^{n_1} \subset (\sigma_{p}(T_{\mathcal{H}_1}) \setminus \sigma_{r}(T_{\mathcal{H}_1})) \), and \( NTL(\Gamma_1 \setminus \{\lambda_j^{(1)}\}_{j=1}^{n_1}) = \tilde{\Gamma}_1 \). By the same argument as above (applied to \( T_{\mathcal{H}_1} \)), one can find a finite sequence of positive numbers \( \{\alpha_j^{(2)}\}_{j=1}^{n_2} \), a finite sequence \( \{\lambda_j^{(2)}\}_{j=1}^{n_2} \) of distinct points in \( \Gamma_1 \setminus \{\lambda_j^{(1)}\}_{j=1}^{n_1} \), and a vector \( x^{(2)} \) in \( \mathcal{H} \otimes \mathcal{H}_1 \) such that

\[ \|f_{X_{1}} - \sum_{j=1}^{n_2} \alpha_j^{(2)} P_{\lambda_j^{(2)}}\|_1 < \varepsilon/5, \]

\[ [x^{(2)} \otimes x^{(2)}]_{T_{\mathcal{H}_1}} = \sum_{j=1}^{n_2} \alpha_j^{(2)} [C_{\lambda_j^{(2)}}]_{T_{\mathcal{H}_1}}, \]

and

\[ \|x^{(2)}\| \leq \|f_{X_{1}}\|^{1/2}. \]

Since \( \mathcal{H}_1 \) is invariant for \( T \), by (5) it follows that

\[ [x^{(2)} \otimes x^{(2)}]_T = \sum_{j=1}^{n_2} \alpha_j^{(2)} [C_{\lambda_j^{(2)}}]_{T}, \]

and taking into account (4), we obtain

\[ \|\phi^{-1}_T([f_{X_{1}}]_{L^1(H)}) - [x^{(2)} \otimes x^{(2)}]_T\| < \varepsilon/5. \]

One can thus find by induction an orthogonal sequence \( \{x^{(n)}\}_{n=1}^{\infty} \) such that for any positive integer \( n \),

\[ \|x^{(n)}\| \leq \|f_{X_{1}}\|^{1/2}. \]
ON POLYNOMIALLY BOUNDED OPERATORS WITH RICH SPECTRUM

and
\[ \|\phi_T^{-1}([f x_{\Gamma_1} L y_{H_0}] - [x^{(n)} \otimes x^{(n)}])\| < \epsilon/5. \]

If M is large enough, \( y^{(1)} := x^{(M)} \) satisfies the inequalities
\[ \|\phi_T^{-1}([f x_{\Gamma_1} L y_{H_0}] - [y^{(1)} \otimes y^{(1)}])\| < \epsilon/5, \]
\[ \|y^{(1)}\| \leq \|f x_{\Gamma_1}\|^{1/2}, \|([y^{(1)} \otimes y^{(1)})_T]\| + \|y_j \otimes y^{(1)}\| < \delta/4, \quad j = 1, \ldots, p. \]

By a similar argument as above (applied to \( f x_{\Gamma_2 \setminus \Gamma_1} \)) one can obtain a vector \( y^{(2)} \) such that
\[ \|\phi_T^{-1}([f x_{\Gamma_2 \setminus \Gamma_1} L y_{H_0}] - [y^{(2)} \otimes y^{(2)}])\| < \epsilon/5, \]
\[ \|y^{(2)}\| \leq \|f x_{\Gamma_2 \setminus \Gamma_1}\|^{1/2}, \]
\[ \|([y^{(1)} \otimes y^{(2)})_T]\| + \|y^{(2)} \otimes y^{(1)}\| < \epsilon/5, \]
and
\[ \|y_j \otimes y^{(1)}\| + \|y^{(1)} \otimes y^{(2)}\| < \delta/4, \quad j = 1, \ldots, p. \]

Putting together (6)-(11), we get
\[ \|y^{(1)} + y^{(2)}\| \leq 2^{1/2} \|f x_{\Gamma_1 \cup \Gamma_2}\|^{1/2}, \]
\[ \|([y^{(1)} + y^{(2)})_T]\| + \|y_j \otimes (y^{(1)} + y^{(2)})\| < \delta/2, \quad j = 1, \ldots, p, \]
and
\[ \|\phi_T^{-1}([f x_{\Gamma_1 \cup \Gamma_2} L y_{H_0}] - [(y^{(1)} + y^{(2)}) \otimes (y^{(1)} + y^{(2)})]\| < 4\epsilon/5. \]

Now we concentrate on \( f x_{T \setminus (\Gamma_1 \cup \Gamma_2)} \). Since \( \sigma_e(T) \cap D \) dominates \( T \setminus (\Gamma_1 \cup \Gamma_2) \), again by Lemma 1.2 of [3] one can find a finite sequence of positive numbers \( \alpha_k \) \( k = 1, \ldots, L \), and a finite sequence \( \lambda_k \) \( k = 1, \ldots, L \), such that
\[ \sum_{k=1}^L \alpha_k \leq \|f x_{T \setminus (\Gamma_1 \cup \Gamma_2)}\|, \]
and
\[ \|\phi_T^{-1}([f x_{T \setminus (\Gamma_1 \cup \Gamma_2)} L y_{H_0}] - \sum_{k=1}^L \alpha_k [C_{\lambda_k}]_T\| < \epsilon/20. \]

For each \( k \in \{1, \ldots, L\} \) let \( \{x_n^{(k)}\}_{n=1}^{\infty} \) be a sequence of vectors in the unit ball of \( \mathcal{H} \), converging weakly to \( 0 \) such that
\[ \lim_{n \to \infty} \|([C_{\lambda_k}]_T - [x_n^{(k)} \otimes x_n^{(k)}])_T\| = 0. \]

By a standard argument (since \( \lim_{n \to \infty} \|([x_n^{(k)} \otimes u]_T + \|u \otimes x_n^{(k)}\||) = 0 \) for any \( u \) in \( \mathcal{H} \), \( k = 1, \ldots, L \)) one can choose inductively positive integers \( \{n_k\}_{k=1}^L \) such that
\[ y^{(3)} := \sum_{k=1}^L \alpha_k^{1/2} x_n^{(k)} \]
satisfies
\begin{align}
\|\phi_T^{-1}(\|fX_T(\widetilde{r}_1 \cup \widetilde{r}_2)\|_{L/H_0}) - \|y(3) \otimes y(3)\|_T\| < \varepsilon / 10, \\
\|y(3)\| \leq 2^{1/2} \|fX_T(\widetilde{r}_1 \cup \widetilde{r}_2)\|^{1/2}, \\
\|[(y(1) + y(2)) \otimes y(3)]_T\| + \|[y(3) \otimes (y(1) + y(2)))]_T\| < \varepsilon / 20, \\
\text{and} \\
\|\|y(3) \otimes y_j\|_T\| + \|[y_j \otimes y(3)]_T\| < \delta / 2, \quad j = 1, \ldots, L.
\end{align}

By (12)–(18) it follows easily that the vector \( x := y(1) + y(2) + y(3) \) satisfies (i) and (ii) above, and the lemma is proved.

The next step in the proof of Theorem 2 is to show that if \( f \) is a nonnegative function in \( L^1 \) such that \( \|f\|_1 \leq 1/2 \), then \( \phi_T^{-1}([f]_{L/H_0}) \in \mathbb{B}(\mathcal{A}_T) \). Once this has been shown, it will follow that if \( f \in L^1 \) is such that \( \|f\|_1 \leq 1/8 \), then \( \phi_T^{-1}([f]_{L/H_0}) \in \mathbb{B}(\mathcal{A}_T) \). Thus taking into account the facts that \( \phi_T \) is invertible and \( \|[f]_{L/H_0}\| \leq M \|\phi_T^{-1}([f]_{L/H_0})\| \) for any \( f \in L^1 \), it will follow that \( \mathcal{A}_T \) has property \( \mathbb{B}_{0,1/8} \), so we are done. To see that for any \( f \in L^1 \) such that \( \|f\|_1 \leq 1/8 \), \( \phi_T^{-1}([f]_{L/H_0}) \in \mathbb{B}(\mathcal{A}_T) \), pick two sequences of positive numbers \( \{\epsilon_n\}_{n=1}^\infty \) and \( \{\delta_n\}_{n=1}^\infty \) decreasing to 0, and a dense, countable subset \( \{z_n\}_{n=1}^\infty \) in \( \mathcal{H} \). By Lemma 2, one can find a sequence \( \{x(n)\}_{n=1}^\infty \) of vectors in the unit ball of \( \mathcal{H} \) such that for every \( n \),
\[ \|\phi_T^{-1}([f]_{L/H_0}) - [x(n) \otimes x(n)]_T\| < \epsilon_n, \]
and
\[ \|[x(n) \otimes z_k]_T\| + \|[z_k \otimes x(n)]_T\| < \delta_n, \quad k = 1, \ldots, n, \]
so the sequence \( \{x(n)\}_{n=1}^\infty \) converges weakly to 0. Hence by Lemma 1, \( \phi_T^{-1}([f]_{L/H_0}) \in \mathbb{B}(\mathcal{A}_T) \), and the proof of the theorem is complete.

Remarks. This paper constitutes part of the author's Ph.D. thesis written at Texas A&M University under the direction of Carl Pearcy. The referee has kindly pointed out that Jörg Eschmeier obtained a similar result in [10].

References


10. J. Eschmeier, *Representations of $H^\infty(G)$ and invariant subspaces*, preprint.


*Department of Mathematics, Texas A&M University, College Station, Texas 77843*

*Current address*: Department of Mathematics, Case Western Reserve University, Cleveland, Ohio 44106/7058

*E-mail address*: rxg38@po.cwru.edu