A PALEY-WIENER THEOREM FOR FRAMES

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Abstract. We prove a stability theorem for frames. Our result is a generalization of a classical result of Paley and Wiener about Riesz bases; it is also related to the Perturbation Theorem of Kato.

The classical Paley-Wiener Theorem states the following: Let \{f_i\}_{i=1}^{\infty} be a basis for the Banach space \(B\), and let \(\{g_i\}_{i=1}^{\infty}\) be a family of vectors in \(B\). If there exists a constant \(\lambda \in [0; 1]\) such that

\[
\left\| \sum_{i=1}^{n} c_i (f_i - g_i) \right\| \leq \lambda \cdot \left\| \sum_{i=1}^{n} c_i f_i \right\|
\]

for all scalars \(c_1, \ldots, c_n\) \((n = 1, 2, \ldots)\), then \(\{g_i\}_{i=1}^{\infty}\) is a basis for \(B\).

Intuitively, the statement is that any family which is sufficiently close to a basis (in the sense above) is a basis. The proof is not difficult. The conditions imply that there exists a bounded invertible operator \(T\) such that \(Tf_i = g_i\).

The above formulation is due to Boas (cf. [Y]). The Paley-Wiener Theorem is useful in order to show that a family \(\{g_i\}_{i=1}^{\infty}\) is a Riesz basis for a Hilbert space, so the result is sometimes used in wavelet analysis ([B], [S]). But in many cases the wavelet experts prefer to work with frames instead of Riesz bases; our aim here is to show that a similar result holds for frames, however with a completely different proof.

The needed facts about frames can be found in the paper [C].

**Theorem 1.** Let \(\mathcal{H}\) be a Hilbert space and \(\{f_i\}_{i=1}^{\infty}\) a frame for \(\mathcal{H}\) with bounds \(A\) and \(B\). Let \(\{g_i\}_{i=1}^{\infty}\) be a family of elements in \(\mathcal{H}\), and suppose that

\[
\exists \lambda, \mu \geq 0 : \lambda + \frac{\mu}{\sqrt{A}} < 1 \quad \text{and} \quad \left\| \sum_{i=1}^{n} c_i (f_i - g_i) \right\| \leq \lambda \cdot \left\| \sum_{i=1}^{n} c_i f_i \right\| + \mu \cdot \left[ \sum_{i=1}^{n} |c_i|^2 \right]^{1/2}
\]

for all \(c_1, \ldots, c_n\) \((n = 1, 2, \ldots)\). Then \(\{g_i\}_{i=1}^{\infty}\) is a frame with bounds \(A(1 - (\lambda + \mu/\sqrt{A}))^2\), \(B(1 + \lambda + \mu/\sqrt{B})^2\).

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Proof. Consider the operator

\[ U: l^2(\mathbb{N}) \to \mathcal{H}, \quad U\{c_i\} := \sum_{i=1}^{\infty} c_i f_i. \]

Frame theory says that \( U \) is well defined, bounded, and that \( \|U\| \leq \sqrt{B} \). The assumptions imply that we can define an operator

\[ T: l^2(\mathbb{N}) \to \mathcal{H}, \quad T\{c_i\} := \sum_{i=1}^{\infty} c_i g_i \]

and that

\[ \|T\{c_i\} - T\{c_i\}\| \leq \lambda \cdot \|U\{c_i\}\| + \mu \cdot \|\{c_i\}\|, \quad \forall \{c_i\} \in l^2(\mathbb{N}). \]

Therefore

\[ \|T\{c_i\}\| \leq [(1 + \lambda)\sqrt{B} + \mu] \cdot \|\{c_i\}\|, \quad \forall \{c_i\} \in l^2(\mathbb{N}). \]

That is, \( \{g_i\}_{i=1}^{\infty} \) is a Bessel sequence with upper bound \( \sqrt{B}(\lambda + 1) + \mu^2 = B(1 + \lambda + \mu/\sqrt{B})^2 \).

Now we verify the existence of the lower frame bound for \( \{g_i\}_{i=1}^{\infty} \). Observe that \( UU^* \) is the frame operator for \( \{f_i\}_{i=1}^{\infty} \) and therefore invertible. Let us consider the operator

\[ U^\dagger: \mathcal{H} \to l^2(\mathbb{N}), \quad U^\dagger f := U^*(UU^*)^{-1} f = \{(f, (UU^*)^{-1} f_i)\}. \]

\( \{(UU^*)^{-1} f_i\}_{i=1}^{\infty} \) is a frame with upper bound \( 1/A \), so

\[ \|U^\dagger f\|^2 = \sum_{i=1}^{\infty} \left| \left( f, (UU^*)^{-1} f_i \right) \right|^2 \leq \frac{1}{A} \cdot |f|^2, \quad \forall f \in \mathcal{H}. \]

Using (1) with \( \{c_i\} = U^\dagger f \) we get

\[ \|f - TU^\dagger f\| \leq \left( \lambda + \frac{\mu}{\sqrt{A}} \right) \cdot \|f\|, \quad \forall f \in \mathcal{H}. \]

Therefore \( TU^\dagger \) is invertible, and

\[ \|TU^\dagger\| \leq 1 + \lambda + \frac{\mu}{\sqrt{A}}, \quad \|(TU^\dagger)^{-1}\| \leq 1/(1 - (\lambda + \mu/\sqrt{A})). \]

Any \( f \in \mathcal{H} \) can be written as

\[ f = TU^\dagger(TU^\dagger)^{-1} f = \sum_{i=1}^{\infty} \left( (TU^\dagger)^{-1} f, (UU^*)^{-1} f_i \right) g_i ; \]

thus

\[ \|f\|^4 = (f, f)^2 = \left| \sum_{i=1}^{\infty} \left( (TU^\dagger)^{-1} f, (UU^*)^{-1} f_i \right) (g_i, f) \right|^2 \leq \sum_{i=1}^{\infty} \left| (TU^\dagger)^{-1} f, (UU^*)^{-1} f_i \right|^2 \cdot \sum_{i=1}^{\infty} \left| (g_i, f) \right|^2 \leq \frac{1}{A} \cdot \|TU^\dagger\|^2 \cdot \sum_{i=1}^{\infty} \left| (g_i, f) \right|^2 \leq \frac{1}{A(1 - (\lambda + \mu/\sqrt{A}))^2} \cdot |f|^2 \cdot \sum_{i=1}^{\infty} \left| (g_i, f) \right|^2, \quad \forall f \in \mathcal{H}. \]
Remarks. (1) Suppose that \( \{f_i\}_{i=1}^{\infty} \) is a frame with bounds \( A, B \) and that \( \{g_i\}_{i=1}^{\infty} \) is any family such that \( R := \sum_{i=1}^{\infty} \|f_i - g_i\|^2 < A \). Then the condition in Theorem 1 is satisfied with \( \lambda = 0 \) and \( \mu = \sqrt{R} \). Thus Theorem 1 generalizes Theorem 1 in [C].

(2) The condition in Theorem 1 implies that \( \lambda < 1 \). This is essential. Let \( \{f_i\}_{i=1}^{\infty} \) be an orthonormal basis for \( H \), and define \( g_i := f_i + f_{i+1} \). Then \( \text{span}\{g_i\} = H \), but \( \{g_i\} \) is not a frame;
\[
\sum_{i=1}^{n} |\langle f, g_i \rangle|^2 = 1/n \cdot \|f\|^2
\]
with \( f := \sum_{i=1}^{n} (-1)^{j-1} e_j \), so the lower frame condition is not satisfied. Since
\[
\left\| \sum_{i=1}^{n} c_i (f_i - g_i) \right\| = \left\| \sum_{i=1}^{n} c_i f_{i+1} \right\| = \left\| \sum_{i=1}^{n} c_i f_i \right\|,
\]
the example corresponds to \( \lambda = 1, \mu = 0 \).

(3) Our result is connected with the work of Kato (e.g., [K], p. 190). Consider the operator \( T \) as a perturbation of \( U \); in the terminology of Kato, the condition in Theorem 1 implies that the “perturbation operator” \( T - U \) is \( U \)-bounded with \( U \)-bound smaller than 1.

(4) In the classical Paley-Wiener Theorem, the conditions imply that \( \sum_{i=1}^{n} c_i g_i = 0 \) if and only if \( \sum_{i=1}^{n} c_i f_i = 0 \). So the sets \( \{f_i\} \) and \( \{g_i\} \) must have the same linear dependence. This is automatically satisfied if \( \{f_i\} \) and \( \{g_i\} \) are bases, but in general it is a strong condition. We have avoided this obstacle in Theorem 1; from this point of view the introduction of \( \mu \) plays an important role.

(5) Theorem 1 has applications to the important coherent frames, shortly discussed in [C]. For example, the proof of Theorem 5 in [S] uses the Paley-Wiener Theorem; it can be expected that a similar result can be proved for frames, using Theorem 1. Also, our result is applicable to the important problem of perturbation of the mother wavelet \( f \) in a coherent frame \( \{\pi(x_i) f\}_{i=1}^{\infty} \). We refer to the paper [FZ], where the reader also finds other applications of Theorem 1.

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References


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