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A PALEY-WIENER THEOREM FOR FRAMES

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ABSTRACT. We prove a stability theorem for frames. Our result is a generalization of a classical result of Paley and Wiener about Riesz bases; it is also related to the Perturbation Theorem of Kato.

The classical Paley-Wiener Theorem states the following: Let $\{f_i\}_{i=1}^{\infty}$ be a basis for the Banach space B, and let $\{g_i\}_{i=1}^{\infty}$ be a family of vectors in B. If there exists a constant $\lambda \in [0; 1[$ such that

$$\left\|\sum_{i=1}^{n} c_i(f_i - g_i)\right\| \leq \lambda \cdot \left\|\sum_{i=1}^{n} c_i f_i\right\|$$

for all scalars $c_1, \dots, c_n$ $(n = 1, 2 \dots)$, then $\{g_i\}_{i=1}^{\infty}$ is a basis for B.

Intuitively, the statement is that any family which is sufficiently close to a basis (in the sense above) is a basis. The proof is not difficult. The conditions imply that there exists a bounded invertible operator T such that $Tf_i = g_i$.

The above formulation is due to Boas (cf. [Y]). The Paley-Wiener Theorem is useful in order to show that a family $\{g_i\}_{i=1}^{\infty}$ is a Riesz basis for a Hilbert space, so the result is sometimes used in wavelet analysis ([B], [S]). But in many cases the wavelet experts prefer to work with frames instead of Riesz bases; our aim here is to show that a similar result holds for frames, however with a completely different proof.

The needed facts about frames can be found in the paper [C].

Theorem 1. *Let $\mathscr{H}$ be a Hilbert space and $\{f_i\}_{i=1}^{\infty}$ a frame for $\mathscr{H}$ with bounds A and B. Let $\{g_i\}_{i=1}^{\infty}$ be a family of elements in $\mathscr{H}$, and suppose that*

$$\exists \lambda, \mu \geq 0 : \lambda + \frac{\mu}{\sqrt{A}} < 1 \quad \text{and} \quad \left\|\sum_{i=1}^{n} c_i(f_i - g_i)\right\| \leq \lambda \cdot \left\|\sum_{i=1}^{n} c_i f_i\right\| + \mu \cdot \left[\sum_{i=1}^{n} |c_i|^2\right]^{1/2}$$

for all $c_1, \dots, c_n$ $(n = 1, 2, \dots)$. Then $\{g_i\}_{i=1}^{\infty}$ is a frame with bounds $A(1 - (\lambda + \mu/\sqrt{A}))^2$, $B(1 + \lambda + \mu/\sqrt{B})^2$.

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Proof. Consider the operator

$$U:\ l^2(\mathbf{N}) \to \mathscr{H}, \qquad U\{c_i\} := \sum_{i=1}^{\infty} c_i f_i.$$

Frame theory says that U is well defined, bounded, and that $\|U\| \le \sqrt{B}$. The assumptions imply that we can define an operator

$$T:\ l^2(\mathbf{N}) \to \mathscr{H}, \qquad T\{c_i\} := \sum_{i=1}^{\infty} c_i g_i$$

and that

(1) $$\|U\{c_i\} - T\{c_i\}\| \le \lambda \cdot \|U\{c_i\}\| + \mu \cdot \|\{c_i\}\|, \qquad \forall \{c_i\} \in l^2(\mathbf{N}).$$

Therefore

$$\|T\{c_i\}\| \le [(1+\lambda)\sqrt{B} + \mu] \cdot \|\{c_i\}\|, \qquad \forall \{c_i\} \in l^2(\mathbf{N}).$$

That is, $\{g_i\}_{i=1}^{\infty}$ is a Bessel sequence with upper bound $[\sqrt{B}(\lambda+1)+\mu]^2 = B(1+\lambda+\mu/\sqrt{B})^2$.

Now we verify the existence of the lower frame bound for $\{g_i\}_{i=1}^{\infty}$. Observe that UU^* is the frame operator for $\{f_i\}_{i=1}^{\infty}$ and therefore invertible. Let us consider the operator

$$U^\dagger:\ \mathscr{H} \to l^2(\mathbf{N}), \qquad U^\dagger f := U^*(UU^*)^{-1} f = \{\langle f, (UU^*)^{-1} f_i\rangle\}.$$

$\{(UU^*)^{-1} f_i\}_{i=1}^{\infty}$ is a frame with upper bound $1/A$, so

$$\|U^\dagger f\|^2 = \sum_{i=1}^{\infty} \left|\left\langle f, (UU^*)^{-1} f_i \right\rangle\right|^2 \le \frac{1}{A} \cdot \|f\|^2, \quad \forall f \in \mathscr{H}.$$

Using (1) with $\{c_i\} = U^\dagger f$ we get

$$\|f - TU^\dagger f\| \le \left(\lambda + \frac{u}{\sqrt{A}}\right) \cdot \|f\|, \quad \forall f \in \mathscr{H}.$$

Therefore $TU^\dagger$ is invertible, and

$$\|TU^\dagger\| \le 1 + \lambda + \frac{\mu}{\sqrt{A}}, \qquad \|(TU^\dagger)^{-1}\| \le 1/(1 - (\lambda + \mu/\sqrt{A})).$$

Any $f \in \mathscr{H}$ can be written as

$$f = TU^\dagger (TU^\dagger)^{-1} f = \sum_{i=1}^{\infty} \langle (TU^\dagger)^{-1} f, (UU^*)^{-1} f_i\rangle g_i;$$

thus

$$\begin{aligned}
\|f\|^4 = \langle f, f\rangle^2 &= \left|\sum_{i=1}^{\infty} \langle (TU^\dagger)^{-1} f, (UU^*)^{-1} f_i\rangle \langle g_i, f\rangle\right|^2 \\
&\le \sum_{i=1}^{\infty} |(TU^\dagger)^{-1} f, (UU^*)^{-1} f_i\rangle|^2 \cdot \sum_{i=1}^{\infty} |\langle g_i, f\rangle|^2 \\
&\le \frac{1}{A} \cdot \|(TU^\dagger)^{-1} f\|^2 \cdot \sum_{i=1}^{\infty} |\langle g_i, f\rangle|^2 \\
&\le \frac{1}{A(1 - (\lambda + \frac{\mu}{\sqrt{A}}))^2} \cdot \|f\|^2 \cdot \sum_{i=1}^{\infty} |\langle g_i, f\rangle|^2, \quad \forall f \in \mathscr{H}.
\end{aligned}$$

Remarks. (1) Suppose that $\{f_i\}_{i=1}^{\infty}$ is a frame with bounds A, B and that $\{g_i\}_{i=1}^{\infty}$ is any family such that $R := \sum_{i=1}^{\infty} \|f_i - g_i\|^2 < A$. Then the condition in Theorem 1 is satisfied with $\lambda = 0$ and $\mu = \sqrt{R}$. Thus Theorem 1 generalizes Theorem 1 in [C].

(2) The condition in Theorem 1 implies that $\lambda < 1$. This is essential. Let $\{f_i\}_{i=1}^{\infty}$ be an orthonormal basis for $\mathscr{H}$, and define $g_i := f_i + f_{i+1}$. Then $\overline{\text{span}}\{g_i\} = \mathscr{H}$, but $\{g_i\}$ is not a frame;

$$\sum |\langle f, g_i \rangle|^2 = 1/n \cdot \|f\|^2$$

with $f := \sum_{i=1}^{n} (-1)^{j-1} e_j$, so the lower frame condition is not satisfied. Since

$$\left\| \sum_{i=1}^{n} c_i (f_i - g_i) \right\| = \left\| \sum_{i=1}^{n} c_i f_{i+1} \right\| = \left\| \sum_{i=1}^{n} c_i f_i \right\|,$$

the example corresponds to $\lambda = 1$, $\mu = 0$.

(3) Our result is connected with the work of Kato (e.g., [K], p. 190). Consider the operator T as a perturbation of U; in the terminology of Kato, the condition in Theorem 1 implies that the "perturbation operator" $T - U$ is U-bounded with U-bound smaller than 1.

(4) In the classical Paley-Wiener Theorem, the conditions imply that $\sum_{i=1}^{n} c_i g_i = 0$ if and only if $\sum_{i=1}^{n} c_i f_i = 0$. So the sets $\{f_i\}$ and $\{g_i\}$ must have the same linear dependence. This is automatically satisfied if $\{f_i\}$ and $\{g_i\}$ are bases, but in general it is a strong condition. We have avoided this obstacle in Theorem 1; from this point of view the introduction of μ plays an important role.

(5) Theorem 1 has applications to the important coherent frames, shortly discussed in [C]. For example, the proof of Theorem 5 in [S] uses the Paley-Wiener Theorem; it can be expected that a similar result can be proved for frames, using Theorem 1. Also, our result is applicable to the important problem of perturbation of the mother wavelet f in a coherent frame $\{\pi(x_i)f\}_{i=1}^{\infty}$. We refer to the paper [FZ], where the reader also finds other applications of Theorem 1.

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