CHARACTERISTIC FUNCTIONS
AND PRODUCTS OF BOUNDED DERIVATIVES

ALEKSANDER MALISZEWSKI

(Communicated by Andrew M. Bruckner)

ABSTRACT. This article is dedicated to the answer to the following question: "Which characteristic functions can be expressed as the product of two or more bounded derivatives?".

The real line $(-\infty, \infty)$ is denoted by $\mathbb{R}$, the set of integers by $\mathbb{Z}$ and the set of positive integers by $\mathbb{N}$. The only measure used is Lebesgue measure in $\mathbb{R}$ and each integral should be interpreted as the corresponding Lebesgue integral. The distance between two nonempty subsets A, B of $\mathbb{R}$ will be denoted by $\varrho(A, B)$ (i.e. $\varrho(A, B) = \inf\{|x - y|: x \in A, y \in B\}$). For each set $A \subset \mathbb{R}$, let int A denote its (Euclidean) interior, cl A its closure, fr A its boundary, $|A|$ its outer measure and χ_A its characteristic function.

All functions will be real functions of a real variable. If I is an interval (throughout this paper we deal only with nondegenerate intervals), then $\mathscr{D}(I)$ denotes the family of all derivatives defined on I (in the case of an endpoint of I that belongs to I we consider the corresponding one-sided condition). Let $\mathscr{D}$ denote $\mathscr{D}(\mathbb{R})$.

The terms "d-closed", "d-interior" (d-int), etc., refer to the Denjoy topology (density topology) on $\mathbb{R}$. (See, e.g., [2], [5].) We say that a function is approximately continuous if and only if it is continuous relative to the Denjoy topology. It is well known that approximately continuous functions are Baire one functions and bounded approximately continuous functions are derivatives (cf., e.g., [1]). We will use an old lemma of Zahorski.

Lemma 1. *Let A and B be disjoint, d-closed, G_δ sets. Then there is an approximately continuous function $g: \mathbb{R} \to \mathbb{R}$ such that $0 \le g \le 1$ on $\mathbb{R}$, $g = 1$ on A and $g = 0$ on B* [6, *Lemma* 12].

For each set $T \in \mathbb{R}$ and each interval I, let $\varphi_T(I)$ denote the measure of the greatest interval J contained in $I \backslash T$, if any such interval exists, and 0 otherwise (cf., e.g., [4]). If a and b are endpoints of I, then we will write also $\varphi_T(a, b)$ instead of $\varphi_T(I)$.

We say that the set $T \subset \mathbb{R}$ is *nonporous* at x iff

$$\lim_{t \to 0^+} \frac{\varphi_T(x - t, x + t)}{2t} = 0.$$

Also, T is called *nonporous* if it is nonporous at each of its points. We say that the set $T \subset \mathbb{R}$ is *ambiguous* iff it is both an F_σ and a G_δ set.

For the remainder of this article, let T denote a fixed ambiguous and nonporous subset of $\mathbb{R}$, and $S = \mathbb{R} \backslash T$. We will write $\varphi(I)$ instead of $\varphi_T(I)$.

For each bounded interval I and $\varepsilon \in (0, 1]$, we will denote by $\mathscr{P}(I, \varepsilon)$ the family of all pairs of derivatives (f_1, f_2) defined on $\mathrm{cl}\, I$ which satisfy the following conditions:

(a) $f_1 \cdot f_2 = 0$ on $T \cap I$,
(b) $f_1 = f_2$ and $|f_1| = 1$ on $S \cap I$,
(c) if a is an endpoint of I, then $f_1(a) = f_2(a) = \chi_S(a)$,
(d) $|f_1| < 2$, $|f_2| < 2$,
(e) $\int_I f_1 = \int_I f_2$,
(f) $|\int_I f_1 - |I|| \le \varepsilon \cdot |I|$,
(g) $|\{x \in I : f_1(x) = f_2(x) = 1\}| \ge (1 - \varepsilon) \cdot |S \cap I|$.

We will denote by $\mathscr{J}$ the family of all bounded intervals I such that $\mathscr{P}(I, \varepsilon) \ne \varnothing$ for each $\varepsilon \in (0, 1]$.

The following lemmas are proved in [4].

Lemma 2. *Let H be ambiguous and let A be a nonempty G_δ set in $\mathbb{R}$. Then there is an open interval I such that $I \cap A \ne \varnothing$ and that either $I \cap A \subset H$ or $I \cap A \subset \mathbb{R} \backslash H$ [4, Lemma 2].*

Lemma 3. *Let J be an interval such that $J \cap T \ne \varnothing$. Then $J \cap T$ contains an interval [4, Proposition 8].*

Denote by Ω the first uncountable ordinal. We will use the following well-known theorem (Cantor-Baire stationary principle).

Theorem 4. *Let $\{F_\alpha\}_{\alpha < \Omega}$ be a transfinite descending sequence of closed subsets of $\mathbb{R}$. Then there is a $\xi < \Omega$ such that $F_\alpha = F_\xi$ for $\alpha \ge \xi$ [3, Theorem 2, p. 146].*

Define a transfinite ascending sequence $\{G_\alpha\}_{\alpha < \Omega}$ of subsets of $\mathbb{R}$ as follows:

(1) $G_0 = \varnothing$,
(2) $G_{\alpha+1} = \mathrm{int}(S \cup G_\alpha)$, if $\alpha < \Omega$ is even,
(3) $G_{\alpha+1} = \mathrm{int}(T \cup G_\alpha)$, if $\alpha < \Omega$ is odd,
(4) $G_\alpha = \bigcup_{\beta < \alpha} G_\beta$, if $\alpha < \Omega$ is a limit ordinal.

Then for each $\alpha < \Omega$, the set G_α is open. Note that $G_\alpha \ne \mathbb{R}$ implies $G_\alpha \ne G_{\alpha+2}$ (cf. Lemma 2), so by Theorem 4, there exists a $\xi < \Omega$ such that $G_\xi = \mathbb{R}$.

Lemma 5. *Whenever J is a bounded open interval and $\varepsilon > 0$ there exist $n \in \mathbb{N}$ and $x_1, \dots, x_{n-1} \in T \cap J$ such that $x_1 < \cdots < x_{n-1}$ and, setting $x_0 = \inf J$, $x_n = \sup J$, we have*

$$x_i - x_{i-1} \le \varphi(x_{i-1}, x_i) + \varepsilon$$

for $i \in \{1, \dots, n\}$.

Proof. Since $T \cap J$ is a totally bounded separable space, we can find elements $x_1, \dots, x_{n-1} \in T \cap J$ such that for each $t \in T \cap J$, there is an $i \in \{1, \dots, n-1\}$

such that $|t - x_i| < \varepsilon/2$. Set x_0 and x_n as above. Then for $i \in \{1, \ldots, n\}$,

$$x_i - x_{i-1} > \varepsilon \Rightarrow [x_{i-1} + \varepsilon/2, x_i - \varepsilon/2] \subset S \Rightarrow \varphi(x_{i-1}, x_i) \geq x_i - x_{i-1} - \varepsilon. \quad \square$$

Lemma 6. *For every* $a, b \in T$ *with* $a < b$, *there exists a strictly increasing sequence* $(x_z)_{z \in \mathbb{Z}}$ *of elements of* $T \cap (a, b)$ *such that*

(1) $\lim_{z \to -\infty} x_z = a$, $\lim_{z \to \infty} x_z = b$,

(2) *for each* $z \in \mathbb{Z}$,

$$x_z - x_{z-1} \leq \varphi(x_{z-1}, x_z) + [\varrho(\{a, b\}, \{x_{z-1}, x_z\})]^2.$$

Proof. Since $a, b \in T$, there exists a strictly increasing sequence $(y_z)_{z \in \mathbb{Z}}$ of elements of $T \cap (a, b)$ with limit points a and b (cf. Lemma 3). For each $z \in \mathbb{Z}$, use Lemma 5 with $J = J_z = (y_{z-1}, y_z)$ and $\varepsilon = [\varrho(\{a, b\}, \{y_{z-1}, y_z\})]^2$, and finds points $x_{z,1}, \ldots, x_{z,n_z-1} \in T \cap J_z$ such that $x_{z,1} < \cdots < x_{z,n_z-1}$, and setting $x_{z,0} = y_{z-1}$, $x_{z,n_z} = y_z$, we have for $i \in \{1, \ldots, n_z\}$,

$$x_{z,i} - x_{z,i-1} \leq \varphi(x_{z,i-1}, x_{z,i}) + [\varrho(\{a, b\}, \{y_{z-1}, y_z\})]^2.$$

Arrange all elements of the set $\{x_{z,i} : z \in \mathbb{Z}, i \in \{1, \ldots, n_z\}\}$ in a strictly increasing sequence $(x_z)_{z \in \mathbb{Z}}$. It is easy to verify that this sequence possesses desired properties. $\square$

Lemma 7. *Whenever* $D \subset \mathbb{R}$ *is measurable and bounded, and* $\varepsilon_1, \varepsilon_2 \in (-1, 1)$, *we can find approximately continuous functions* $f_1, f_2 \colon \mathbb{R} \to \mathbb{R}$ *such that*

(1) $f_1 \cdot f_2 = 0$ *on* $\mathbb{R}$,

(2) $f_1 = f_2 = 0$ *on* $\mathbb{R} \backslash D$,

(3) $|f_1| < 2$, $|f_2| < 2$,

(4) $\int_D f_j = \varepsilon_j \cdot |D|$ $(j \in \{1, 2\})$.

Proof. If $|D| = 0$, then set $f_1 = f_2 = 0$. Otherwise find disjoint measurable sets D_1, D_2 such that $|D_1| = |D_2|$ and $D_1 \cup D_2 = D$. For $j \in \{1, 2\}$, first find a closed set $C_j \subset d\text{-int}\, D_j$ such that $|C_j| > |\varepsilon_j| \cdot |D_j|$ and then, using Lemma 1, find an approximately continuous function $g_j \colon \mathbb{R} \to \mathbb{R}$ such that $0 \leq g_j \leq 1$ on $\mathbb{R}$, $g_j(x) = 1$ for $x \in C_j$ and $g_j(x) = 0$ for $x \notin D_j$. Finally observe that for $j \in \{1, 2\}$,

$$\left| \frac{\varepsilon_j \cdot |D|}{\int_{D_j} g_j} \right| \leq \frac{2|\varepsilon_j| \cdot |D_j|}{|C_j|} < 2,$$

and set

$$f_j = g_j \cdot \frac{\varepsilon_j \cdot |D|}{\int_{D_j} g_j}.$$

It is easy to verify that the requirements of the lemma are fulfilled. $\square$

The following two lemmas are due to S. Konjagin. They have never been published.

Lemma 8. *Let* $r_1, \ldots, r_n \geq 0$. *Then there exist numbers* $t_1, \ldots, t_n \in \{-1, 1\}$ *such that for* $k, l \in \{1, \ldots, n\}$, $k \leq l$, *we have*

$$\left| \sum_{s=k}^{l} t_s r_s \right| \leq 2 \max\{r_s : s \in \{k, \ldots, l\}\}.$$

Proof. We will use induction on n.

(1) If $n = 1$, then we set $t_1 = 1$.

(2) Assume that the assertion holds for some $n \in \mathbb{N}$. Let $r_1, \ldots, r_{n+1}$ be nonnegative numbers and choose $t_1, \ldots, t_n \in \{-1, 1\}$ according to the induction assumption. Suppose that both the sequence $(t_1, \ldots, t_n, 1)$ and the sequence $(t_1, \ldots, t_n, -1)$ do not satisfy our requirements. Then there exist $k, l \in \{1, \ldots, n\}$ such that

$$\sum_{s=k}^{n} t_s r_s + r_{n+1} > 2 \max\{r_s \colon s \in \{k, \ldots, n+1\}\} \geq 2 r_{n+1}$$

and

$$\sum_{s=l}^{n} t_s r_s - r_{n+1} < -2 \max\{r_s \colon s \in \{l, \ldots, n+1\}\} \leq -2 r_{n+1}.$$

We may assume that $k \leq l$ (the opposite case is similar). Subtracting the two above inequalities we get

$$\sum_{s=k}^{l-1} t_s r_s + 2 r_{n+1} > 2 \max\{r_s \colon s \in \{k, \ldots, n+1\}\} + 2 r_{n+1},$$

contrary to the induction assumption. This completes the proof. □

Lemma 9. *Given a countable family of nonoverlapping intervals $\mathscr{K}$ and a nonnegative function $r \colon \mathscr{K} \to \mathbb{R}$ such that $\sum_{K \in \mathscr{K}} r(K) < \infty$, we can find a function $t \colon \mathscr{K} \to \{-1, 1\}$ such that for every interval $I \subset \mathbb{R}$,*

$$\left| \sum_{K \subset I, K \in \mathscr{K}} t(K) \cdot r(K) \right| \leq 2 \sup\{r(K) \colon K \subset I, K \in \mathscr{K}\}.$$

Proof. Let $K_1, K_2, \ldots$ be a sequence of all elements of $\mathscr{K}$. For each $k \in \mathbb{N}$, apply Lemma 8 and find $t_{k,1}, \ldots, t_{k,k} \in \{-1, 1\}$ such that for every interval $I \subset \mathbb{R}$,

$$(1) \qquad \left| \sum_{K_n \subset I, n \leq k} t_{k,n} \cdot r(K_n) \right| \leq 2 \max\{r(K_n) \colon K_n \subset I, n \leq k\}.$$

In this way we get a triangular matrix $[t_{k,n}]_{k \in \mathbb{N}, n \leq k}$. Next we proceed by induction. Let $t(K_1)$ be a number which appears in the first column of this matrix infinitely many times. If we have already defined $t(K_n)$ for $n < m$, then let $t(K_m)$ be a number which appears in the mth column of this matrix, in rows number k for which $t_{k,n} = t(K_n)$ for $n < m$, infinitely many times.

Let $I \subset \mathbb{R}$ be an arbitrary interval. Let $(m_s)_s$ be a strictly increasing sequence of those $m \in \mathbb{N}$, for which $K_m \subset I$. Take an $\varepsilon > 0$ and find an $l \in \mathbb{N}$ such that $\sum_{s>l} r(K_{m_s}) < \varepsilon$. There is a $k \in \mathbb{N}$ such that $t_{k,n} = t(K_n)$ for $n \leq m_l$. So by (1),

$$\left| \sum_{s=1}^{l} t(K_{m_s}) \cdot r(K_{m_s}) \right| = \left| \sum_{K_n \subset I, n \leq m_l} t_{k,n} \cdot r(K_n) \right|$$

$$\leq 2 \max\{r(K_n) \colon K_n \subset I, n \leq m_l\}$$

$$\leq 2 \sup\{r(K) \colon K \subset I, K \in \mathscr{K}\}.$$

Hence

$$\left| \sum_{K \subset I, K \in \mathscr{K}} t(K) \cdot r(K) \right| = \left| \sum_s t(K_{m_s}) \cdot r(K_{m_s}) \right|$$

$$\leq \left| \sum_{s=1}^l t(K_{m_s}) \cdot r(K_{m_s}) \right| + \sum_{s>l} |t(K_{m_s}) \cdot r(K_{m_s})|$$

$$< 2 \sup\{r(K) \colon K \subset I, K \in \mathscr{K}\} + \varepsilon.$$

Since ε was an arbitrary positive number, the assertion of the lemma holds. $\square$

Proposition 10. *Every compact interval belongs to $\mathscr{J}$.*

Proof. Since $G_\xi = \mathbb{R}$, it is enough to prove by transfinite induction on $\alpha < \Omega$ that every compact interval contained in G_α belongs to $\mathscr{J}$.

I. Let $\alpha = 0$. Then $G_\alpha = \varnothing$ and there is nothing to prove.

II. (a) Assume that the assertion holds for each compact interval contained in G_α and α is even. Let I be a compact interval contained in $G_{\alpha+1}$ and $\varepsilon \in (0, 1]$. Let $\{I_n \colon n\}$ be a family (maybe empty) of all components of $I \cap G_\alpha$ and $a_n = \inf I_n$, $b_n = \sup I_n$. For each n:

- if $a_n, b_n \in G_\alpha$, then $I \subset G_\alpha$ and so by assumption $I \in \mathscr{J}$;
- if $a_n, b_n \notin G_\alpha$, then find a strictly increasing sequence $(y_{n,z})_{z \in \mathbb{Z}}$ with limit points a_n and b_n, and such that for each $z \in \mathbb{Z}$, $|I_{n,z}| \leq [\varrho(I_{n,z}, \{a_n, b_n\})]^2$, where $I_{n,z} = [y_{n,z-1}, y_{n,z}] \subset G_\alpha$; (Note that unlike in the proof of Lemma 6, we do not require that $y_{n,z} \in T$ for $z \in \mathbb{Z}$.)
- if $a_n \in G_\alpha$ and $b_n \notin G_\alpha$, then find a strictly increasing sequence $(y_{n,z})_{z \in \mathbb{N}}$ with $y_{n,1} = a_n$ and $\lim_{z \to \infty} y_{n,z} = b_n$, and such that for each $z \in \mathbb{N} \setminus \{1\}$, $|I_{n,z}| \leq [\varrho(I_{n,z}, \{b_n\})]^2$, where $I_{n,z} = [y_{n,z-1}, y_{n,z}] \subset G_\alpha$;
- similarly if $a_n \notin G_\alpha$ and $b_n \in G_\alpha$, then find a strictly increasing sequence $(y_{n,z})_{z \in \mathbb{Z} \setminus \mathbb{N}}$ with $y_{n,0} = b_n$ and $\lim_{z \to -\infty} y_{n,z} = a_n$, and such that for each $z \in \mathbb{Z} \setminus \mathbb{N}$, $|I_{n,z}| \leq [\varrho(I_{n,z}, \{a_n\})]^2$, where $I_{n,z} = [y_{n,z-1}, y_{n,z}] \subset G_\alpha$.

For each n and z, let $(f_{n,z,1}, f_{n,z,2}) \in \mathscr{P}(I_{n,z}, \varepsilon/2^{n+|z|})$. For $j \in \{1, 2\}$, define the function $f_j \colon I \to \mathbb{R}$ by the formula

$$f_j(x) = \begin{cases} f_{n,z,j}(x) & \text{if } x \in I_{n,z}, \\ 1 & \text{if } x \in I \setminus G_\alpha. \end{cases}$$

The pair (f_1, f_2) obviously satisfies (a)–(e) and f_1, f_2 are derivatives at points of $G_\alpha \cap I$. If $x \in I \setminus G_\alpha$, then $x \in S$. Let $\tau > 0$ be arbitrary and let $k \in \mathbb{N}$ be such that $\tau > \varepsilon/2^k$. Set $\eta = \min\{\varrho(\{x\}, \bigcup_{n+|z| \leq k} I_{n,z}), \tau/6\}$. Let $y \in (0, \eta)$. We may assume that $x + y \in \operatorname{int} I_{s,m}$ for some s and m (otherwise we would drop the last two terms in the first line of the estimation below). Then for

$j \in \{1, 2\}$,

$$\left| \frac{1}{y} \cdot \int_x^{x+y} f_j - 1 \right| \leq \frac{1}{y} \left(\sum_{I_{n,z} \subset (x, x+y]} \left| \int_{I_{n,z}} f_j - |I_{n,z}| \right| + \int_{I_{s,m}} |f_j| + |I_{s,m}| \right)$$

$$\leq \frac{1}{y} \left(\sum_{I_{n,z} \subset (x, x+y]} \frac{\varepsilon \cdot |I_{n,z}|}{2^{n+|z|}} + 3[\varrho(\{x\}, I_{s,m})]^2 \right)$$

$$\leq \frac{\varepsilon}{2^{k+1}} + 3y < \tau.$$

A similar argument holds for $y \in (-\eta, 0)$. So $f_1, f_2 \in \mathscr{D}(I)$. Finally observe that

$$\left| \int_I f_1 - |I| \right| = \left| \sum_n \sum_z \left(\int_{I_{n,z}} f_1 - |I_{n,z}| \right) \right| \leq \sum_n \sum_z \frac{\varepsilon \cdot |I_{n,z}|}{2^{n+|z|}} < \varepsilon \cdot |I|$$

and

$$|\{x \in I : f_1(x) = f_2(x) = 1\}| \geq |I \setminus G_\alpha| + \sum_n \sum_z \left(1 - \frac{\varepsilon}{2^{n+|z|}} \right) \cdot |S \cap I_{n,z}|$$

$$\geq |I \setminus G_\alpha| + (1 - \varepsilon) \cdot |S \cap I \cap G_\alpha|$$

$$\geq (1 - \varepsilon) \cdot |S \cap I|,$$

i.e. (f) and (g) are also fulfilled, which proves that $I \in \mathscr{J}$ in this case.

II. (b) Assume that the assertion holds for each compact interval contained in G_α and α is odd. First we will prove the following statement:

(∗)   *if I is a bounded interval contained in $G_{\alpha+1}$ and $a = \inf I$, $b = \sup I \in T$, then there exist derivatives f_1, f_2 defined on $\mathrm{cl}\, I$ which satisfy (a)–(e) and such that $f_1 = f_2 = 0$ on $\mathrm{cl}\, I \setminus G_\alpha$.*

Let $\{I_n : n\}$ be a family (maybe empty) of all components of $I \cap G_\alpha$, and let a_n, b_n be endpoints of I_n. For each n, use Lemma 6 to find a strictly increasing sequence $(y_{n,z})_{z \in \mathbb{Z}}$ of elements of $T \cap I_n$ with limit points a_n and b_n, and such that for each $z \in \mathbb{Z}$, $|I_{n,z}| \leq \varphi(I_{n,z}) + [\varrho(I_{n,z}, \{a_n, b_n\})]^2$, where $I_{n,z} = [y_{n,z-1}, y_{n,z}] \subset G_\alpha$. For each n and each $z \in \mathbb{Z}$, let $(f_{n,z,1}, f_{n,z,2}) \in \mathscr{P}(I_{n,z}, 1)$ and define $r(I_{n,z}) = \int_{I_{n,z}} f_{n,z,1}$. Use Lemma 9 for the family $\mathscr{K} = \{I_{n,z} : n, z \in \mathbb{Z}\}$ and the function r, and find a function $t : \mathscr{K} \to \{-1, 1\}$ such that for every interval $J \subset \mathbb{R}$,

$$\left| \sum_{I_{n,z} \subset J} t(I_{n,z}) \cdot r(I_{n,z}) \right| \leq 2 \sup\{r(I_{n,z}) : I_{n,z} \subset J\}.$$

For $j \in \{1, 2\}$, define the function $f_j : \mathrm{cl}\, I \to \mathbb{R}$ as follows:

$$f_j(x) = \begin{cases} t(I_{n,z}) \cdot f_{n,z,j}(x) & \text{if } x \in I_{n,z}, \\ 0 & \text{if } x \in \mathrm{cl}\, I \setminus G_\alpha. \end{cases}$$

The pair (f_1, f_2) obviously satisfies (a)–(e) and f_1, f_2 are derivatives at points of $G_\alpha \cap I \setminus \{a, b\}$. If $x \in I \setminus G_\alpha$ or $x \in \{a, b\}$, then $x \in T$. Let $y > 0$. We may assume that $x + y \in \operatorname{int} I_{s,m}$ for some s and some $m \in \mathbb{Z}$ (otherwise we would drop the last term in the first line of the estimation below). Then for $j \in \{1, 2\}$,

$$\left| \frac{1}{y} \int_x^{x+y} f_j \right| \leq \frac{1}{y} \left(\left| \sum_{I_{n,z} \subset (x, x+y]} \int_{I_{n,z}} f_j \right| + \int_{I_{s,m}} |f_j| \right)$$

$$\leq \frac{1}{y} \left(\left| \sum_{I_{n,z} \subset (x, x+y]} t(I_{n,z}) \cdot r(I_{n,z}) \right| + 2|I_{s,m}| \right)$$

$$\leq \frac{2 \sup\{r(I_{n,z}): I_{n,z} \subset (x, x+y]\}}{y} + \frac{2 \cdot \varphi(I_{s,m})(1+y)}{y + y^2} + 2y$$

$$\leq \frac{4 \sup\{|I_{n,z}|: I_{n,z} \subset (x, x+y]\}}{y}$$

$$+ \frac{2 \cdot \varphi(x, x+y+|I_{s,m}|) \cdot (1+y)}{y + |I_{s,m}| - \varphi(I_{s,m})} + 2y$$

$$\leq \frac{4 \cdot \varphi(x, x+y)}{y} + \frac{2 \cdot \varphi(x, x+y+|I_{s,m}|)}{y + |I_{s,m}|}$$

$$\cdot \frac{1+y}{1 - \frac{\varphi(x, x+y+|I_{s,m}|)}{y+|I_{s,m}|}} + 6y \xrightarrow{y \to 0} 0.$$

(We used the fact that T is nonporous at x.) Similar argument holds for $y < 0$, so f_1, $f_2 \in \mathscr{D}(\operatorname{cl} I)$, which completes the proof of $(*)$.

Now let I be any compact interval contained in $G_{\alpha+1}$ and $\varepsilon \in (0, 1]$.

- Use Lemma 3 to find nonoverlapping compact intervals $J_1, \ldots, J_n$ contained in $I \cap G_\alpha$ such that $\operatorname{fr}(I \cap G_\alpha \setminus \bigcup_{i=1}^n J_i) \subset T$, $|I \cap G_\alpha \setminus \bigcup_{i=1}^n J_i| \leq \frac{\varepsilon \cdot |I \cap G_\alpha|}{6}$ and $|S \cap I \setminus \bigcup_{i=1}^n J_i| \leq \frac{\varepsilon \cdot |S \cap I|}{2}$. For $i \in \{1, \ldots, n\}$, let $(g_{i,1}, g_{i,2}) \in \mathscr{P}(J_i, \varepsilon/2)$.
- Let $I_1, \ldots, I_m$ be components of $I \setminus \bigcup_{i=1}^n J_i$. For $k \in \{1, \ldots, m\}$, use $(*)$ with $I = I_k$ and find derivatives $h_{k,1}$, $h_{k,2}$ defined on $\operatorname{cl} I_k$ which satisfy (a)–(e).
- Use Lemma 7 with $D = I \setminus G_\alpha$ and $\varepsilon_1 = \varepsilon_2 = 1 - \varepsilon$, and find approximately continuous functions $f_{0,1}$, $f_{0,2}$ satisfying conditions (1)–(4) of that lemma.
- For $j \in \{1, 2\}$, define the function $f_j : I \to \mathbb{R}$ as follows:

$$f_j(x) = f_{0,j}(x) + \begin{cases} g_{i,j}(x) & \text{if } x \in J_i, \ i \in \{1, \ldots, n\}, \\ h_{k,j}(x) & \text{if } x \in I_k, \ k \in \{1, \ldots, m\}. \end{cases}$$

Then obviously f_1 and f_2 are derivatives which satisfy (a)–(e). (Note that

$h_{k,j} = 0$ at the endpoints of I_k and on $I_k \backslash G_\alpha$, $k \in \{1, \ldots, m\}$.) Moreover,

$$\left| \int_I f_1 - |I| \right| \leq \left| \int_{\bigcup_{i=1}^n J_i} f_1 - \left| \bigcup_{i=1}^n J_i \right| \right| + \left| \int_{I \cap G_\alpha \backslash \bigcup_{i=1}^n J_i} f_1 - \left| I \cap G_\alpha \backslash \bigcup_{i=1}^n J_i \right| \right|$$

$$+ \left| \int_{I \backslash G_\alpha} f_1 - |I \backslash G_\alpha| \right|$$

$$\leq \frac{\varepsilon}{2} \cdot \left| \bigcup_{i=1}^n J_i \right| + 3 \cdot \left| I \cap G_\alpha \backslash \bigcup_{i=1}^n J_i \right| + \varepsilon \cdot |I \backslash G_\alpha|$$

$$\leq \frac{\varepsilon}{2} \cdot \left| \bigcup_{i=1}^n J_i \right| + \frac{\varepsilon}{2} \cdot |I \cap G_\alpha| + \varepsilon \cdot |I \backslash G_\alpha| \leq \varepsilon \cdot |I|,$$

and

$$|\{x \in I : f_1(x) = f_2(x) = 1\}| \geq \sum_{i=1}^n |\{x \in J_i : f_1(x) = f_2(x) = 1\}|$$

$$\geq \sum_{i=1}^n \left(1 - \frac{\varepsilon}{2}\right) \cdot |S \cap J_i|$$

$$\geq \left(1 - \frac{\varepsilon}{2}\right)\left(1 - \frac{\varepsilon}{2}\right) \cdot |S \cap I| \geq (1 - \varepsilon) \cdot |S \cap I|,$$

so (f) and (g) are also satisfied, which proves that $I \in \mathscr{J}$ in this case.

III. Assume that α is a limit ordinal and that every compact interval in G_β with some $\beta < \alpha$ belongs to $\mathscr{J}$. Let I be an arbitrary compact interval contained in G_α. Then by the compactness of I we get $I \subset G_{\beta_1} \cup \cdots \cup G_{\beta_n} = G_\gamma$, where $\gamma = \max\{\beta_i : i \in \{1, \ldots, n\}\} < \alpha$. Hence by assumption $I \in \mathscr{J}$.

This completes the proof. □

Theorem 11. *Assume that $S \subset \mathbb{R}$ is ambiguous and $T = \mathbb{R} \backslash S$ is nonporous. Then there exist derivatives $f, g \in \mathscr{D}$ such that*

    (i) $f \cdot g = 0$ *on* T,

    (ii) $f = g$ *and* $|f| = 1$ *on* S,

    (iii) $|f| < 2$, $|g| < 2$.

Proof. Let $I_z = [z - 1, z]$ for $z \in \mathbb{Z}$. By Proposition 10, $I_z \in \mathscr{J}$. Let, e.g., $(f_z, g_z) \in \mathscr{P}(I_z, 1)$ $(z \in \mathbb{Z})$. Define

$$f(x) = f_z(x) \quad \text{if } x \in I_z, z \in \mathbb{Z},$$
$$g(x) = g_z(z) \quad \text{if } x \in I_z, z \in \mathbb{Z}.$$

It is easy to verify that f and g satisfy our requirements. □

In 1990 J. Mařík proved the following theorem.

Theorem 12. *Let $S \subset \mathbb{R}$, $T = \mathbb{R} \backslash S$. Then the following three conditions are equivalent:*

    (1) *There is a natural number m and derivatives $f_1, \ldots, f_m$ such that*

$$f_1 \cdot \cdots \cdot f_m = \chi_S.$$

    (2) *S is ambiguous and T is nonporous.*

(3) *There are* f, $g \in \mathscr{D}$ *such that* $f = g = 1$ *on* S *and* $fg = 0$ *on* T [4, *Theorem* 18].

Hence and by Theorem 11 we get the following corollary.

Corollary 13. *Let* $S \subset \mathbb{R}$, $T = \mathbb{R} \backslash S$. *Then the following four conditions are equivalent*:

(1) *There is a natural number* m *and derivatives* $f_1, \ldots, f_m$ *such that*

$$f_1 \cdots f_m = \chi_S.$$

(2) S *is ambiguous and* T *is nonporous*.

(3) *There are* f, $g \in \mathscr{D}$ *such that* $f = g$ *and* $|f| = 1$ *on* S, $fg = 0$ *on* T *and* $|f| < 2$, $|g| < 2$ *on* $\mathbb{R}$.

(4) *There is a natural number* m *and bounded derivatives* $f_1, \ldots, f_m$ *such that*

$$f_1 \cdots f_m = \chi_S.$$

REFERENCES

1. A. M. Bruckner, *Differentiation of real functions*, Lecture Notes in Math., vol. 659, Springer-Verlag, Berlin and New York, 1978.

2. C. Goffman, C. J. Neugebauer, and T. Nishiura, *Density topology and approximate continuity*, Duke Math. J. **28** (1961), 497–506.

3. C. Kuratowski, *Topologie*, Vol. I, PWN, Warszawa, 1958.

4. J. Mařík, *Characteristic functions and products of derivatives*, Real Anal. Exchange **16** (1990–1991), 245–254.

5. G. Petruska and M. Laczkovich, *Baire one functions, approximately continuous functions and derivatives*, Acta Math. Acad. Sci. Hungar. **25** (1974), 189–212.

6. Z. Zahorski, *Sur la première dérivée*, Trans. Amer. Math. Soc. **69** (1950), 1–54.

DEPARTMENT OF MATHEMATICS, PEDAGOGICAL UNIVERSITY, UL. CHODKIEWICZA 30, 85-064 BYDGOSZCZ, POLAND

E-mail address: `wspb05@pltumk11.bitnet`