A SIMPLE PROOF OF A REMARKABLE CONTINUED FRACTION IDENTITY

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Abstract. We give a simple proof of a generalization of the equality

\[ \sum_{n=1}^{\infty} \frac{1}{2^{[n/2]}} = [0, 2^0, 2^1, 2^1, 2^2, 2^3, 2^5, \ldots], \]

where \( \tau = (1 + \sqrt{5})/2 \) and the exponents of the partial quotients are the Fibonacci numbers, and some closely related results.

Introduction

P. E. Böhmer [3], L. V. Danilov [4], and W. W. Adams and J. L. Davison [1] showed independently that if \( \alpha > 0 \) is irrational, \( b > 1 \) is an integer, and \( S_b(\alpha) = (b - 1) \sum_{k=1}^{\infty} \frac{1}{b^{k^2}} \), then the simple continued fraction for \( S_b(\alpha) \) can be described explicitly in the following way. Let \( \alpha \) have simple continued fraction

\[ \alpha = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \ldots}} = [a_0, a_1, \ldots], \]

with \( \frac{a_n}{a_n} = [a_0, \ldots, a_n], n \geq 0 \). Let \( t_0 = a_0 b, t_n = \frac{b^{a_n-2} - b^{a_n-1}}{b^{a_n-1} - 1}, n \geq 1 \). Then \( S_b(\alpha) = [t_0, t_1, \ldots] \). Thus in the case \( \alpha = \tau = (1 + \sqrt{5})/2 \), the golden ratio, and \( b = 2 \), one gets the remarkable equality \( \sum_{n=1}^{\infty} \frac{1}{2^{[n/2]}} = [0, 2^0, 2^1, 2^1, 2^2, 2^3, 2^5, \ldots] \), where the exponents of the partial quotients are the Fibonacci numbers.

More recently, R. L. Graham, D. E. Knuth, and O. Patashnik [8] indicated how to give a very different proof of the power series version of this result, where the number \( b \) is replaced by an indeterminate (they carried out the proof for the case \( \alpha = (1 + \sqrt{5})/2 \), using the continuant polynomials of Euler [6].

In this note we give a proof, which we feel is simpler than the others, which makes use of a property of the “characteristic sequence” of \( \alpha \) discovered by H. J. S. Smith [13]. The crucial idea of our approach appears in Lemma 2 below, where we regard certain initial segments of the characteristic sequence of \( \alpha \) as base \( b \) representations of integers.

(Böhmer, Danilov, and Adams and Davison also show that \( S_b(\alpha) \) is transcendental for every irrational \( \alpha \). We omit the proof of this fact, which is an

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easy application of a theorem of Roth [11], using Lemma 3 and Theorem B below.)

**Preliminaries.** Let \( \alpha \) be an irrational number with \( 0 < \alpha < 1 \). (At the end, we will remove the restriction \( \alpha < 1 \).) Let \( \alpha = [0, a_1, a_2, \ldots] \) and \( b_n = [0, a_1, \ldots, a_n], \ n \geq 0, \) where \( p_n, q_n \) are relatively prime non-negative integers. (As usual, we put \( p_{-2} = 0, \ p_{-1} = 1, \ q_{-2} = 1, \ q_{-1} = 0, \) so that \( p_n = a_n p_{n-1} + p_{n-2}, \ q_n = a_n q_{n-1} + q_{n-2} \) for all \( n \geq 0 \).) For \( n \geq 1 \), define \( f_\alpha(n) = [(n + 1)\alpha] - [n\alpha], \) and consider the infinite binary sequence \( f_\alpha = (f_\alpha(n))_{n \geq 1}, \) which is sometimes called the characteristic sequence of \( \alpha \). Define binary words \( X_n, \ n \geq 0, \) by \( X_0 = 0, \ X_1 = 0a^{-1}1, \ X_k = X_{k-1}\alpha X_{k-2}, \ k \geq 2, \) where \( X^a \) denotes the word \( X \) repeated \( a \) times, and \( X_1 = 1 \) if \( a_1 = 1 \).

The following result was first proved by Smith [13]. Other proofs can be found in [2], [7], [12], and [14], and further references to the characteristic sequence can be found in [2]. Nishioka, Shiokawa, and Tamura [9] treat the more general case \( [(n + 1)\alpha + \beta] - [n\alpha + \beta] \).

**Lemma 1.** For each \( n \geq 1, \) \( X_n \) is a prefix of \( f_\alpha \). That is, \( X_n = f_\alpha(1)f_\alpha(2) \cdots f_\alpha(s), \) where \( s \) is the length of \( X_n \).

**The main proof.** We are now ready to prove the result stated in the Introduction. (However, we will keep the restriction \( \alpha < 1 \) until the following section.) Let \( b > 1 \) be an integer, let \( 0 < \alpha < 1 \) be irrational, \( \alpha = [0, a_1, a_2, \ldots] \), let \( b_n = [0, a_1, \ldots, a_n], \ n \geq 0, \) and let the binary words \( X_n, \ n \geq 0, \) be defined as above.

According to Lemma 1, the binary word \( X_n \) (which has length \( q_n \)) by a trivial induction using \( q_n = a_n q_{n-1} + q_{n-2} \) is identical with the binary word \( f_\alpha(1)f_\alpha(2) \cdots f_\alpha(q_n) \). If we let \( x_n \) denote the integer whose base \( b \) representation is \( X_n \), i.e. \( x_n = f_\alpha(1)b^{q_n-1} + f_\alpha(2)b^{q_n-2} + \cdots + f_\alpha(q_n)b^0 \), then we can write

\[
x_n = b^{q_n} \cdot \sum_{k=1}^{q_n} \frac{f_\alpha(k)}{b^k}.
\]

Now we come to the crucial step.

**Lemma 2.** For \( n \geq 0, \) let \( t_{n+1} = \frac{b^{q_{n+1}} - b^{q_n}}{b^{q_n} - 1} \). Then for \( n \geq 1, \)

\[
x_{n+1} = t_{n+1} x_n + x_{n-1}.
\]

**Proof.** Using the facts that \( X_n \) has length \( q_n \), \( X_{n-1} \) has length \( q_{n-1} \), \( x_{n+1} \) is the integer whose base \( b \) representation is \( X_{n+1} \), and \( X_{n+1} = X_{q_{n+1}}X_{n-1} \), it follows that

\[
x_{n+1} = b^{q_{n+1}}(1 + b^{q_n} + b^{2q_n} + \cdots + b^{(a_{n+1} - 1)q_n})x_n + x_{n-1}
\]

\[
= b^{q_n}(b^{q_n+q_{n+1}} - 1)(b^{q_n - 1} - 1)x_n + x_{n-1} = t_{n+1} x_n + x_{n-1}.
\]

**Lemma 3.** For \( n \geq 1, \)

\[
[0, t_1, \ldots, t_n] = \frac{b-1}{b^{q_n} - 1} \cdot x_n.
\]

**Proof.** Let \( y_n = \frac{b-1}{b^{q_n} - 1}, \ n \geq 0. \) We show by induction on \( n \) that \( [0, t_1, \ldots, t_n] = \frac{x_n}{y_n} \). We start the induction at \( n = 0 \) by setting \( t_0 = 0 \). Note that \( x_0 = 0, \)
$x_1 = 1, \ y_0 = 1, \ y_1 = \frac{b^n - 1}{b - 1} = t_1$. For the induction step, we simply note that $x_{n+1} = t_{n+1}x_n + x_{n-1}$ and $y_{n+1} = t_{n+1}y_n + y_{n-1}$.

**Theorem A.** Let $b > 1$ be an integer, and let $0 < \alpha < 1$ be irrational, with $f_\alpha(n) = [(n + 1)\alpha] - [n\alpha], \ n \geq 1$. Let $\alpha = [0, a_1, a_2, \ldots]$, let $b_n = [0, a_1, \ldots, a_n], \ n \geq 0$ (where $p_n, q_n$ are relatively prime non-negative integers), and let $t_n = \frac{b_{n+1} - b_n}{b_n - 1}, \ n \geq 1$. Then

$$(b - 1) \sum_{k=1}^{\infty} \frac{f_\alpha(k)}{b^k} = [0, t_1, t_2, \ldots].$$

**Proof.** We have seen that $x_n = b^{a_n} \sum_{k=1}^{a_n} \frac{f_\alpha(k)}{b^k}$. Hence by Lemma 3,

$$(b - 1) \left( \frac{b^{a_n}}{b_{a_n} - 1} \right) \sum_{k=1}^{a_n} \frac{f_\alpha(k)}{b^k} = [0, t_1, \ldots, t_n],$$

and we can take the limit as $n \to \infty$.

**Theorem B.** With the same hypotheses as in Theorem A, we have

$$(b - 1) \sum_{n=1}^{\infty} \frac{1}{b^{[n/\alpha]}} = [0, t_1, t_2, \ldots].$$

**Proof.** This is a restatement of Theorem A, using the easily verified fact (when $0 < \alpha < 1$) that $f_\alpha(k) = 1$ if and only if $k = [n/\alpha]$ for some $n$.

**Theorem C.** With the same hypotheses as in Theorem A, we have

$$(b - 1)^2 \sum_{k=1}^{\infty} \frac{[k\alpha]}{b^k} = [0, t_1, t_2, \ldots].$$

**Proof.** Using $f_\alpha(k) = [(k + 1)\alpha] - [k\alpha]$ and $[\alpha] = 0$, the series in Theorem C is obtained from the series in Theorem A by a slight rearrangement.

**Theorem D.** With the same hypotheses as in Theorem A, we have

$$\sum_{k=1}^{\infty} \frac{f_\alpha(k)}{b^k} = (b - 1) \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(b^{a_k} - 1)(b^{a_k - 1} - 1)}.$$

**Proof.** We saw in the proof of Lemma 3 that $[0, t_1, \ldots, t_n] = \frac{s_n}{y_n}, \ n \geq 1$, where $y_n = \frac{b^{a_n} - 1}{b - 1}, \ n \geq 0$. By a well-known theorem (J. B. Roberts [10, p. 101]), $\frac{s_n}{y_n} = \sum_{k=1}^{n} (-1)^{k-1} \frac{(-1)^{k-1}}{y_k y_{k-1}}, \ n \geq 1$, and Theorem D now follows from Theorem A.

**Removing the restriction** $\alpha < 1$. Now let $\alpha' = a_0 + \alpha$, where $a_0 \geq 0$ is an integer, $\alpha$ is irrational, and $0 < \alpha < 1$.

By Theorem A we get

$$(b - 1) \sum_{k=1}^{\infty} \frac{f_{\alpha'}(k)}{b^k} = (b - 1) \sum_{k=1}^{\infty} \frac{a_0 + f_\alpha(k)}{b^k}$$

$$= (b - 1) a_0 \sum_{k=1}^{\infty} \frac{1}{b^k} + (b - 1) \sum_{k=1}^{\infty} \frac{f_\alpha(k)}{b^k}$$

$$= a_0 + [0, t_1, t_2, \ldots] = [a_0, t_1, t_2, \ldots].$$
To handle Theorem B we need to use the fact, whose simple proof we omit, that if \( \alpha' = a_0 + \alpha \), where \( 0 < \alpha < 1 \), then for each \( k = 0, 1, 2, \ldots \), the value \( k \) is assumed by the expression \( \lfloor n/\alpha' \rfloor \) exactly \( a_0 + 1 \) times if \( \lfloor n/\alpha \rfloor = k \) for some \( n \geq 1 \), and exactly \( a_0 \) times if \( \lfloor n/\alpha \rfloor \) never equals \( k \). It then follows from Theorem B that \( (b - 1) \sum_{n=1}^{\infty} \frac{[\alpha' n]}{k^k} = [a_0 b, t_1, t_2, \ldots] \).

By Theorem C and some careful rearrangement we get \( (b - 1)^2 \sum_{k=1}^{\infty} \frac{[\alpha b]}{k^k} = [a_0 b, t_1, t_2, \ldots] \).

Finally, the modified Theorem D (using the modified Theorem A) is

\[
(b - 1) \sum_{k=1}^{\infty} \frac{f_{\alpha'}(k)}{b^k} = a_0 + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(b - 1)^2}{(b a_k - 1)(b a_{k-1} - 1)}.
\]

**Remark.** This paper grew out of the first author’s consideration of the number \( \sum_{k=1}^{\infty} \frac{f_{\alpha}(k)}{2^k} \), where \( \alpha = \frac{1 + \sqrt{5}}{2} \), as the fixed point of the sequence \( \{g_n(0)\} \), \( n \geq 1 \), where \( g_1(x) = x/2 \), \( g_2(x) = (x + 1)/2 \), \( g_n(x) = g_{n-1}(g_{n-2}(x)) \), \( n \geq 3 \). This quickly leads (upon setting \( g_n(x) = (x + a_n)/b_n \) and solving for \( a_n \) and \( b_n \)) to

\[
\sum_{k=1}^{\infty} \frac{f_{\alpha}(k)}{2^k} = [0, 2^0, 2^1, 2^1, 2^2, 2^3, 2^5, \ldots].
\]

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**References**


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