COMPOSITION OPERATORS BETWEEN HARDY 
AND WEIGHTED BERGMAN SPACES 
ON CONVEX DOMAINS IN $\mathbb{C}^N$

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(Communicated by Palle E. T. Jorgensen)

Abstract. Suppose $\Omega$ is a bounded, strongly convex domain in $\mathbb{C}^N$ with smooth boundary and $\phi: \Omega \rightarrow \Omega$ is an arbitrary holomorphic map. While in general the composition operator $C_\phi$ need not map the Hardy space $H^p(\Omega)$ into itself when $N > 1$, our main theorem shows that $C_\phi$ does map $H^p(\Omega)$ boundedly into a certain weighted Bergman space on $\Omega$, where the weight function depends on the dimension $N$. We also consider properties of $C_\phi$ on $H^p(\Omega)$ when $\phi(\Omega)$ is contained in an approach region in $\Omega$.

1. Introduction

For $\Omega$ a domain in $\mathbb{C}^N$ and $\phi: \Omega \rightarrow \Omega$ holomorphic, the composition operator $C_\phi$ with symbol $\phi$ is defined by $C_\phi(f) = f \circ \phi$, for $f$ holomorphic on $\Omega$. When $\Omega$ is the unit disc $A$ in $\mathbb{C}$, it is well known that for every holomorphic $\phi: A \rightarrow A$, $C_\phi$ will be a bounded operator on the Hardy spaces $H^p(A)$, for all $p$. However when $\Omega$ is the unit ball $B_N$, $N > 1$, this is no longer the case; various examples have been given to show that $C_\phi$ may fail to be bounded on the Hardy spaces $H^p(B_N)$ ($0 < p < \infty$), and several authors [CW, M, MS, W] have considered the problem of characterizing those $\phi$ for which $C_\phi$ is bounded on $H^p(B_N)$. A completely satisfactory answer to this question is not yet known.

Here we will consider more generally the case that $\Omega$ is a bounded, strongly convex domain in $\mathbb{C}^N$ with smooth boundary. Our main result will show that for every holomorphic $\phi: \Omega \rightarrow \Omega$, $C_\phi$ is a bounded map of $H^p(\Omega)$ into the weighted Bergman space $A^p, \alpha(\Omega) = \{f$ holomorphic: $\int_\Omega |f(z)|^p d\lambda_\alpha^\alpha(z) < \infty\}$, where $\alpha = N - 2$ and $d\lambda_\alpha^\alpha(z) = (\text{dist}(z, \partial \Omega))^{\alpha} d\lambda_\alpha(z)$ with $d\lambda_\alpha(z)$ normalized volume measure on $\Omega$ and $\text{dist}(z, \partial \Omega)$ the Euclidean distance from $z$ to $\partial \Omega$. In particular, when $\Omega = B_N$, this says that every $C_\phi$ maps $H^p(B_N)$ boundedly into $A^p, N-2(B_N) = \{f$ holomorphic: $\int_{B_N} |f(z)|^p (1 - |z|)^{N-2} d\lambda_B(z)\}$.

Received by the editors October 15, 1993.

1991 Mathematics Subject Classification. Primary 47B38; Secondary 32H02.

The first author's research was supported in part by a grant from the National Science Foundation.

The second author's research was supported in part by N.S.E.R.C. Canada.

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We also consider compactness properties of \( C\phi : H^p(\Omega) \to H^p(\Omega) \) when \( \phi(\Omega) \) is contained in an approach region at some \( \zeta \in \partial \Omega \). Our result here generalizes a previously known result in the ball ([M, Theorem 2.2]) which showed that if \( \phi(B_N) \) is contained in a Koranyi approach region of appropriately small aperture (depending on the dimension \( N \)), then \( C\phi \) is compact as an operator from \( H^p(B_N) \) to \( H^p(B_N) \).

2. Notation and background

In this section we fix our notation and collect some relevant background material. In all of what follows, \( \Omega \subseteq \mathbb{C}^N \) will be a bounded strongly convex domain with \( C^\infty \) boundary, and \( B_N \) will be the unit ball in \( \mathbb{C}^N \), except that when \( N = 1 \) we write instead \( A \) for the unit disc. Normalized surface area measure on \( \partial \Omega \) and \( \partial B_N \) will be denoted by \( \sigma_\Omega \) and \( \sigma_{B_N} \) (or just \( \sigma_B \) if the dimension need not be explicitly shown) respectively. Similarly \( \lambda_\Omega \), \( \lambda_B \) denote normalized volume measure on \( \Omega \) and \( B_N \). Recall that a holomorphic map from \( \Delta \) to \( \Omega \) is called an extremal map ([L1]), or complex geodesic ([V]) if it is an isometry with respect to the Kobayashi distances on \( \Delta \) and \( \Omega \). For \( z_0 \in \Omega \) and \( x \in \partial \Omega \) there is a unique extremal map \( \phi_x : \Delta \to \Omega \) satisfying \( \phi_x(0) = z_0 \) and \( \phi_x(1) = x \); \( \phi_x \) is \( C^\infty \) on \( \overline{\Delta} \) with \( \phi_x(\partial \Delta) \subseteq \partial \Omega \) ([L1, A]).

Associated with each such extremal map \( \phi_x \) is a retraction \( p_x \), which is a holomorphic map of \( \Omega \) onto \( \phi_x(\Delta) \subseteq \Omega \) satisfying

\[
p_x \circ \phi_x(\lambda) = \phi_x(\lambda) \quad \forall \lambda \in \Delta,
\]

and

\[
p_x \circ p_x = p_x
\]

[L1, RW].

Note that in the ball \( B_N \) with \( z_0 = 0 \) we have \( \phi_x(\lambda) = \lambda x \quad (x \in \partial B_N, \lambda \in \Delta) \) and \( p_x(z) = \langle z, x \rangle x \quad (z \in B_N) \), where \( \langle , \rangle \) denotes the usual inner product in \( \mathbb{C}^N \).

There is also, for each fixed direction \( v \neq 0 \) in \( \mathbb{C}^N \) and \( z_0 \in \Omega \), a unique extremal map \( \phi_v : \Delta \to \Omega, \quad C^\infty \) on \( \partial \Delta \), satisfying

\[
\phi_v(0) = z_0, \\
\phi_v'(0) = rv
\]

with \( r > 0 \) maximal [L1, RW]. From these extremal maps Lempert has constructed a canonical mapping \( \Psi : \overline{B}_N \to \overline{\Omega} \) called the spherical representation of \( \Omega \), defined (for \( z_0 \) fixed in \( \Omega \)) by

\[
\Psi(0) = z_0, \\
\Psi(v) = \phi_v(\|v\|), \quad v \neq 0 \text{ in } \overline{B}_N.
\]

It is easy to see that \( \Psi \) is holomorphic on slices through the origin. Moreover, \( \Psi \) is a homeomorphism between \( \overline{B}_N \) and \( \overline{\Omega} \) which is a smooth diffeomorphism off any neighborhood of \( 0 \) (but including \( \partial B_N \)) [L1, Théorème 3; L3, Theorem 5.1]. Its inverse \( \Psi^{-1} : \overline{\Omega} \to \overline{B}_N \) is given by

\[
\Psi^{-1}(z) = \tanh k_\Omega(z_0, z) \frac{\phi_v'(0)}{\|\phi_v(0)\|},
\]
where \( k_\Omega \) is the Kobayashi distance on \( \Omega \) and \( \tanh k_\Omega(z_0, z) = 1 \) if \( z \in \partial \Omega \). Moreover, \( \varphi_x(re^{i\theta}) = \Psi(re^{i\theta}\Psi^{-1}(x)) \) for \( r \in [0, 1] \), \( \theta \) real, and \( x \in \partial \Omega \) ([A]). The reader may check that with \( \Omega = B_N \) and \( z_0 = 0 \), \( \Psi \) is the identity.

The map \( \Psi \) allows us to prove a version of slice integration on \( \partial \Omega \), generalizing Proposition 1.4.7(i) of [R] for the ball. This lemma will be used in §4.

**Lemma 1.** For integrable \( f \) on \( \partial \Omega \)

\[
\int_{\partial \Omega} f \, d\sigma_\Omega \approx \int_{\partial \Omega} \int_0^{2\pi} f \circ \varphi_x(e^{i\theta}) \, d\theta \, d\sigma_\Omega(x),
\]

where \( \approx \) indicates that the ratio of the two quantities is bounded above and below by finite positive constants independent of \( f \).

**Proof.** Fix \( z_0 \in \Omega \) and let \( \Psi \) be the spherical representation of \( \Omega \). Use the fact that \( \Psi \) is a diffeomorphism between \( \partial B_N \) and \( \partial \Omega \) together with slice integration in \( B_N \) to see that

\[
\begin{align*}
\int_{\partial \Omega} f \, d\sigma_\Omega &= \int_{\partial B_N} f \circ \Psi \, d\sigma_\Omega \circ \Psi \\
&\approx \int_{\partial B_N} f \circ \Psi \, d\sigma_B \\
&= \frac{1}{2\pi} \int_{\partial B_N} \int_0^{2\pi} f \circ \Psi(e^{i\theta}\eta) \, d\theta \, d\sigma_B(\eta) \\
&= \frac{1}{2\pi} \int_{\partial \Omega} \int_0^{2\pi} f \circ \Psi(e^{i\theta}\Psi^{-1}(x)) \, d\theta \, d\sigma_B \circ \Psi^{-1}(x) \\
&\approx \int_{\partial \Omega} \int_0^{2\pi} f \circ \Psi(e^{i\theta}\Psi^{-1}(x)) \, d\theta \, d\sigma_\Omega(x) \\
&= \int_{\partial \Omega} \int_0^{2\pi} f \circ \varphi_x(e^{i\theta}) \, d\theta \, d\sigma_\Omega(x).
\end{align*}
\]

The following lemma due to Lempert will play an essential role in our main theorems in §§3 and 4. Let \( d_\Omega(z) \) be the Euclidean distance from \( z \) to \( \partial \Omega \).

**Lemma 2** [L1, Proposition 12]. Let \( \varphi: \Delta \to \Omega \) be an extremal map. There exists a finite constant \( c \) depending only on \( \varphi(0) \) so that

\[
d_\Omega(\varphi(\lambda)) \leq c(1 - |\lambda|)
\]

for all \( \lambda \in \Delta \).

### 3. Main theorem

For \( 0 < p < \infty \) the spaces \( H^p(\Omega) \) are the usual holomorphic Hardy spaces on \( \Omega \) ([S; K, Chapter 8]). For \( \alpha \geq 0 \) define the weighted Bergman space \( A^{p,\alpha}(\Omega) \) to be those holomorphic \( f \) on \( \Omega \) satisfying

\[
\|f\|_{\alpha}^p \equiv \int_{\Omega} |f(z)|^p \, d_\Omega^\alpha(z) \, d\lambda_\Omega(z) < \infty,
\]

where \( d_\Omega^\alpha(z) = (\text{dist}(z, \partial \Omega))^\alpha \). The weighted volume measure \( d_\Omega^\alpha(z) \, d\lambda_\Omega(z) \) will be abbreviated as \( d\lambda_\Omega^\alpha \). In the case \( \Omega = B_N \) we have the standard weighted Bergman spaces.

Our first lemma is a quantitative formulation of the fact that in the disc every composition operator is bounded on the standard weighted Bergman spaces.
This result is well known (it is a consequence of Proposition 3.4 and Theorem 4.3 of [MS]); we include its proof for completeness. For $e^{i\theta} \in \partial \Delta$ and $t > 0$, $S(e^{i\theta}, t) = \{ z \in \Delta : |1 - ze^{-i\theta}| < t \}$, and let $\alpha$ be non-negative.

**Lemma 3.** Let $\gamma : \Delta \to \Delta$ be holomorphic with $\gamma(0) = 0$. There exists an absolute constant $C$ (independent of the particular choice of $\gamma$) so that

$$\lambda^\alpha_{\Delta}(\gamma^{-1}(S(e^{i\theta}, t))) \leq Ct^{\alpha+2}$$

for all real $\theta$ and $t > 0$.

**Proof.** Littlewood's subordination principle shows that the operator $C_\gamma$ is bounded on the weighted Bergman space $A^{p, \alpha}(\Delta)$, with the norm of $C_\gamma = 1$. In particular, for all $f \in A^{2, \alpha}(\Delta)$ we have

$$\int_{\Delta} |f \circ \gamma|^2 \, d\lambda^\alpha_{\Delta}(z) \leq \int_{\Delta} |f|^2 \, d\lambda^\alpha_{\Delta}(z)$$

or

$$\int_{\Delta} |f|^2 (\lambda^\alpha_{\Delta} \gamma^{-1}) \leq \int_{\Delta} |f|^2 \, d\lambda^\alpha_{\Delta}(z).$$

Apply this to the test functions

$$f_w = (1 - z\overline{w})^{-(\alpha+2)} \in A^{2, \alpha}(\Delta)$$

(which have norms in $A^{2, \alpha}(\Delta)$ comparable to $(1 - |w|^2)^{-(\alpha+2)/2}$ for the choice $w = (1 - t)e^{i\theta}$. Using the estimate

$$|f_w(z)|^2 \geq c_1 \frac{1}{(t^{\alpha+2})^2} \quad \text{on } S(e^{i\theta}, t)$$

for some absolute constant $c_1$ we obtain

$$c_1 \frac{1}{t^{2(\alpha+2)}} \lambda^\alpha_{\Delta} \gamma^{-1}(S(e^{i\theta}, t)) \leq c_2 \frac{1}{t^{\alpha+2}},$$

which gives the desired result. $\square$

Now we return to our map $\phi : \Omega \to \Omega$. Pick an arbitrary base point $z_0 \in \Omega$ and let $\phi(z_0) = w_0$. Fix $\zeta \in \partial \Omega$, and let $\varphi_{\zeta} : \Delta \to \Omega$ be the extremal map with $\varphi_{\zeta}(0) = w_0$ and $\varphi_{\zeta}(1) = \zeta$. Denote its associated retraction by $p_{\zeta}$. For each $x \in \partial \Omega$ we consider the holomorphic self-map $\tau_x$ of the disc $\Delta$ defined by

$$\tau_x = \varphi_{\zeta}^{-1} \circ p_{\zeta} \circ \phi \circ \varphi_x,$$

where the extremal maps $\varphi_x : \Delta \to \Omega$ have $\varphi_x(0) = z_0$ and $\varphi_x(1) = x$. Notice that for each $x \in \partial \Omega$, $\tau_x(0) = 0$. Also define sets $S(\zeta, t)$ for $t > 0$ by

$$S(\zeta, t) = \{ z \in \Omega : |1 - \varphi_{\zeta}^{-1} \circ p_{\zeta}(z)| < t \}.$$

In the special case $\Omega = B_N$, $z_0 = w_0 = 0$ the maps $\tau_x$ are just

$$\tau_x(\lambda) = (\phi(\lambda x), \zeta)$$

and $S(\zeta, t) = \{ z \in B_N : |1 - (z, \zeta)| < t \}$, which is the usual definition of a Carleson set (based at $\zeta$) in $B_N$ and consistent with our previous use of the notation $S(e^{i\theta}, t)$ in $\Delta$. We will see later that in general these sets $S(\zeta, t)$ are equivalent to the usual Carleson sets in $\Omega$ as defined by Hormander.

The next proposition uses the maps $\tau_x$ to estimate

$$\lambda^\alpha_\Omega \phi^{-1} S(\zeta, t).$$

This estimate will be the key ingredient in the proof of our main theorem.
Proposition 4. There exists a finite constant $C$, independent of $\zeta \in \partial \Omega$ and $t > 0$, so that
\[
\lambda_\Omega^{\alpha} \phi^{-1}S(\zeta, t) \leq Ct^{\alpha+2}.
\]
In particular, when $\alpha = N - 2$, we have
\[
\lambda_\Omega^{N-2} \phi^{-1}S(\zeta, t) = O(t^N)
\]
for $\zeta \in \partial \Omega$, $t > 0$.

Proof. Clearly it is enough to show this for all $t < t_0$, where $t_0$ is an arbitrary positive number. Fix a neighborhood $V$ of $z_0$, $V \subset \Omega$, and find $t_0 > 0$ so that $\phi^{-1}S(\zeta, t) \cap V$ is empty for all $\zeta \in \partial \Omega$ and $t < t_0$. Then find $\epsilon > 0$ so that if $\Psi: \overline{B}_N \to \overline{\Omega}$ is the spherical representation of $\Omega$ described in §2 (with $\Psi(0) = z_0$) we have $\Psi(\overline{B}_\epsilon) \subset V$, where $\overline{B}_\epsilon$ is the ball of radius $\epsilon$ centered at 0.

Writing $\chi_{\phi^{-1}S}$ for the characteristic function of a set $\phi^{-1}S(\zeta, t)$ we have the following estimate for all $0 < t < t_0$:
\[
\lambda_\Omega^{\alpha} \phi^{-1}S(\zeta, t) = \int_{\Omega} \chi_{\phi^{-1}S}(w)d\lambda_\Omega^\alpha(w) d\lambda_\Omega(w)
\]
\[
= \int_{\partial \Omega \setminus V} \chi_{\phi^{-1}S}(w) d\lambda_\Omega^\alpha(w) d\lambda_\Omega(w)
\]
\[
\leq c_1 \int_{\overline{B}_\epsilon \setminus \overline{B}_\epsilon} \chi_{\phi^{-1}S} \circ \Psi(z) d\lambda_\Omega^\alpha(\Psi(z)) d\lambda_B(z),
\]
where $c_1$ is a constant independent of $\zeta$ and $t$, since $\Psi$ is a diffeomorphism on $\overline{B}_N \setminus \overline{B}_\epsilon$. Changing to polar coordinates and using slice integration in the ball gives that this last integral is
\[
c_2 \int_{\epsilon}^1 r^{2N-1} \int_{\partial B} \int_{0}^{2\pi} \chi_{\phi^{-1}S} \circ \Psi(re^{i\theta} \eta) d\lambda_\Omega^\alpha(\Psi(re^{i\theta} \eta)) d\theta d\sigma_B(\eta) dr
\]
\[
= c_2 \int_{\epsilon}^1 r^{2N-1} \int_{\partial \Omega} \int_{0}^{2\pi} \chi_{\phi^{-1}S} \circ \Psi(re^{i\theta} \Psi^{-1}(x)) d\theta d\sigma_B \circ \Psi^{-1}(x) dr
\]
\[
\leq c_3 \int_{\epsilon}^1 r^{2N-1} \int_{\partial \Omega} \int_{0}^{2\pi} \chi_{\phi^{-1}S} \circ \Psi(re^{i\theta} \Psi^{-1}(x)) d\theta d\sigma_B(x) d\sigma_\Omega(x) dr,
\]
again because $\Psi$ is a diffeomorphism on $\partial B_N$. Since
\[
\Psi(re^{i\theta} \Psi^{-1}(x)) = \phi_x(re^{i\theta}),
\]
the integrand in the inner integral of the last line is $\chi_{\phi^{-1}S}(\phi_x(re^{i\theta})) d\lambda_\Omega^\alpha(\phi_x(re^{i\theta}))$, which, by Lemma 2, is bounded above by a constant multiple of $\chi_{\phi^{-1}S}(\phi_x(re^{i\theta})) \cdot (1 - r)^\alpha$ (where we use the fact that $\phi_x(0) = z_0$ for all $x \in \partial \Omega$). At this point we have
\[
\lambda_\Omega^{\alpha} \phi^{-1}S(\zeta, t) \leq c_4 \int_{\epsilon}^1 r^{2N-1} \int_{\partial \Omega} \int_{0}^{2\pi} \chi_{\phi^{-1}S}(\phi_x(re^{i\theta})) d\theta d\sigma_B(x)(1 - r)^\alpha dr
\]
\[
\leq c_4 \int_{\partial \Omega} \int_{0}^{\Delta} \chi_{\phi^{-1}S}(\phi_x(u)) d\lambda_\Omega^\alpha(u) d\sigma_B(u).
\]
Now
\[ \chi_{\phi^{-1}S}(u) = 1 \iff \phi \circ \varphi_x(u) \in S(\zeta, t) \]
\[ \iff |1 - \varphi_x^{-1} \circ \varphi \circ \varphi_x(u)| < t \]
\[ \iff \tau_x(u) \in S(1, t). \]

Applying Lemma 3 gives
\[ \lambda_\Omega^\alpha \varphi^{-1}S(\zeta, t) \leq c_5 t^{\alpha+2}, \]
where \( c_5 \) depends on neither \( \zeta \) nor \( t \), and we are done. □

To use this proposition to prove the main theorem we need to relate the sets \( S(\zeta, t) \) to the Carleson sets defined by Hormander [H] (in the general setting of bounded strictly pseudoconvex domains with \( C^2 \) boundary). These sets, denoted \( A(\zeta, t) \) for \( \zeta \in \partial \Omega \) and \( t > 0 \), are defined as follows: Let \( \pi_\zeta \) be the complex tangent space at \( \zeta \), and let \( A(\zeta, t) \) be all points in \( \Omega \) whose distance to the ball in \( \pi_\zeta \) with center \( \zeta \) and radius \( \sqrt{t} \) is at most \( t \). We claim that the sets \( S(\zeta, t) \) and \( A(\zeta, t) \) are comparable in the sense that there exist finite positive constants \( k_1, k_2 \) so that
\[ A(\zeta, k_2 t) \subset S(\zeta, t) \subset A(\zeta, k_1 t). \]

To see this we make use of a special biholomorphic map of \( \Omega \) to a domain \( \Omega' \) which flattens out the image of the extremal disc \( \varphi_\zeta(\Delta) \) in \( \Omega \). The existence of this biholomorphism and the relevant properties of it and the domain \( \Omega' \) are contained in the following theorem.

**Theorem 5** [L1, L2, L4]. Given a strongly convex domain \( \Omega \subset \subset \mathbb{C}^N \) with \( C^\infty \) boundary and an extremal map \( \varphi: \Delta \rightarrow \Omega \) with associated retraction \( p \), there exists a domain \( \Omega' \subset \subset \mathbb{C}^N \) and a biholomorphism \( \Lambda: \Omega \rightarrow \Omega' \) which is \( C^\infty \) on \( \overline{\Omega} \) such that:

(i) \( \Lambda \circ \varphi(\lambda) = (\lambda, 0') \) \( \forall \lambda \in \Delta. \)
(ii) \( \Lambda \circ p = \pi \circ \Lambda \), where \( \pi(z_1, z') = (z_1, 0'). \)
(iii) For each \( \xi \in \partial \Delta \) we have \( (\xi, 0') \in \partial \Omega' \) with the unit outward normal there \( (\xi, 0'). \)
(iv) \( \Omega' \) is strongly convex in a neighborhood of \( \{ (\xi, 0') : |\xi| = 1 \} \).

**Lemma 6.** The sets \( S(\zeta, t) \) and \( A(\zeta, t) \) are comparable.

**Proof.** The sets \( A(\zeta, t) \) are invariant under biholomorphic maps ([H, p. 73]), so it suffices to show that the sets \( \Lambda(S(\zeta, t)) \) and \( A(\Lambda(\zeta), t) \) are comparable, where \( \zeta \in \partial \Omega \). We use Theorem 5, with \( \varphi_\zeta = \varphi \) and \( p_\zeta = p \). By (i) and (ii) we have \( \Lambda(S(\zeta, t)) = \{ z \in \Omega' : |1 - z_1| < t \} \). By (iii) and (iv) and the definition of \( A(\zeta, t) \) there are constants \( k_1, k_2 > 0 \) such that
\[ A(\Lambda(\zeta), k_2 t) \subset \Lambda(S(\zeta, t)) \subset A(\Lambda(\zeta), k_1 t) \] (\( \forall t > 0 \)).

Now the sets \( \varphi_\zeta(\Delta) \) vary continuously with \( \zeta \in \partial \Omega \) ([L1, Proposition 11; A, Lemma 1.9]), so we may assume that \( k_1, k_2 \) are independent of \( \zeta \), and we are done. □

Our main theorem is as follows.
Theorem 7. If $\Omega$ is a bounded, strongly convex domain in $\mathbb{C}^N$ ($N \geq 2$) with $C^\infty$ boundary, and if $\phi: \Omega \to \Omega$ is homomorphic, then $C_\phi$ maps $H^p(\Omega)$ boundedly into $A^{p,N-2}(\Omega)$, for each $0 < p < \infty$.

Proof. We wish to show that there exists $C < \infty$ so that
\[ \int_{\Omega} |f \circ \phi(z)|^p \, d\lambda^{N-2}_\Omega(z) \leq C \int_{\Omega} |f|^p \, d\sigma_\Omega, \]
whenever $f \in H^p(\Omega)$. By Hormander's Carleson measure theorem ([H, Theorem 4.3]) it suffices to show that there exists $C' < \infty$ satisfying
\[ \lambda^{N-2}_\Omega \phi^{-1} A(\zeta, t) \leq C't^N \]
for all $\zeta \in \partial \Omega$, $t > 0$. Lemma 6 shows that we may replace $A(\zeta, t)$ in (*) by $S(\zeta, t)$, and then the result follows from Proposition 4. \(\Box\)

4. Maps into admissible regions

In Theorem 2.2 of [M] it was shown that if the holomorphic map $\phi: B_N \to B_N$ has $\phi(B_N)$ contained in a Koranyi approach region of sufficiently small aperture (depending on the dimension $N$), then $C_\phi$ will be compact on $H^p(B_N)$; moreover, this result is sharp in a natural sense. In this section we give a theorem in the same spirit for holomorphic maps of a bounded strongly convex domain $\Omega$ with $C^\infty$ boundary. As the ideas are similar to those in the last section we will omit some details of the arguments.

Analogous to Lemma 3 in §3 we begin with a one variable result, which considers maps which take $A$ into a non-tangential approach region in $A$.

Lemma 8 [M, Lemma 2.3]. If $\gamma: \Delta \to A$ is holomorphic with $\gamma(0) = 0$ and $\gamma(\Delta) \subseteq \{z \in A: |1 - z| < a(1 - |z|)\}$, then there exists $C < \infty$ depending only on $a$ so that
\[ \sigma_\Delta(\gamma^{-1}S(1, t)) \leq C t^b, \]
where $b = \frac{\pi}{2 \cos^{-1}(1/\alpha)}$. In particular $C_\gamma$ will be compact (for all $\alpha > 1$).

Recall that if we fix a base point $z_0$ in our strongly convex domain $\Omega$ and consider any extremal map $\phi: \Delta \to \Omega$ with $\phi(0) = z_0$, then Lemma 2 guarantees that $d_\Omega(\phi(\lambda)) \leq c_1(1 - |\lambda|)$ for all $\lambda \in \Delta$, where $c_1$ is a finite constant not depending on a particular map $\phi$. By Theorem 5 there is a positive constant $c_2$ such that $d_\Omega(z) \leq c_2 d_\Omega(p(z))$ for every $z \in \Omega$, where $p$ is the retraction associated with $\phi$. Just as in the proof of Lemma 6, $c_2$ is independent of $\phi(1)$. Thus
\[ d_\Omega(z) \leq c_1 c_2 (1 - |\phi^{-1} \circ p(z)|) \]
for all $z$ in $\Omega$. For $\zeta \in \partial \Omega$, define approach regions $D_\alpha(\zeta)$ by
\[ D_\alpha(\zeta) = \{z \in \Omega: |1 - \phi^{-1} \circ p_\zeta(z)| < \alpha d_\Omega(z)\}. \]
Notice that by (4.1) $D_\alpha(\zeta)$ is empty for $\alpha < \frac{1}{c_1 c_2}$.

Theorem 9. Let $\Omega$ be a smooth, strongly convex bounded domain. There exist positive constants $\alpha_0$, $\alpha_1$ (which depend on $\Omega$, a chosen base point $z_0$ in $\Omega$, and for $\alpha_1$ explicitly on the dimension $N$) so that for each $\zeta \in \partial \Omega$ we have:

1. $D_\alpha(\zeta)$ is empty for $\alpha < \alpha_0$. 
2. $D_\alpha(\zeta)$ is compact for $\alpha > \alpha_1$. 
3. $D_\alpha(\zeta)$ is not empty for $\alpha_1 < \alpha < \alpha_0$.
(2) If \( \phi(\Omega) \subset D_\alpha(\zeta) \) for \( \alpha_0 < \alpha < \alpha_1 \), then \( C_\phi \) is compact from \( H^p(\Omega) \) to \( H^p(\Omega) \), \( 0 < p < \infty \).

(3) If \( \phi(\Omega) \subset D_\alpha(\zeta) \), then \( C_\phi \) is bounded from \( H^p(\Omega) \) to \( H^p(\Omega) \), \( 0 < p < \infty \).

Sketch of proof. We have already observed (1) holds when \( \alpha_0 = \frac{1}{c_1c_2} \). By a recent theorem of Li and Russo [LR] and Lemma 6, \( C_\phi \) is compact from \( H^p(\Omega) \) to \( H^p(\Omega) \) if

\[
\sigma_\Omega \phi^{-1}S(\eta, t) = o(t^N)
\]

as \( t \to 0 \), uniformly in \( \eta \in \partial \Omega \); and \( C_\phi \) is bounded from \( H^p(\Omega) \) to \( H^p(\Omega) \) when

\[
\sigma_\Omega \phi^{-1}S(\eta, t) = O(t^N)
\]

Thus (2) follows if \( \alpha \in (\alpha_0, \alpha_1) \) and \( \phi(\Omega) \subset D_\alpha(\zeta) \Rightarrow \sigma_\Omega \phi^{-1}S(\eta, t) = o(t^N) \), and (3) follows if \( \phi(\Omega) \subset D_\alpha(\zeta) \Rightarrow \sigma_\Omega \phi^{-1}S(\eta, t) = O(t^N) \). Here we are identifying the map \( \phi \) with its extension a.e. to \( \partial \Omega \) along inner normal vectors ([K, Proposition 8.5.1]). This was also implicit in Lemma 8.

We first consider the process of estimating \( \sigma_\Omega \phi^{-1}S(\eta, t) \) when \( \eta = \zeta \) and \( \phi(\Omega) \subset D_\alpha(\zeta) \). Exactly as in the proof of Proposition 4 define maps \( \tau_x : \Delta \to \Delta (x \in \partial \Omega) \) by \( \tau_x = \varphi^{-1}_x \circ p_\zeta \circ \phi \circ \varphi_x \). We have \( \tau_x(0) = 0 \), and by hypothesis and (4.1),

\[
|1 - \tau_x(\lambda)| \leq c_1c_2\alpha(1 - |\varphi^{-1}_x \circ p_\zeta \circ \phi \circ \varphi_x(\lambda)|)
\]

which says \( \tau_x(\Delta) \subseteq \{ \lambda \in \Delta : |1 - \lambda| \leq c_1c_2\alpha(1 - |\lambda|) \} \). By Lemma 8

\[
\sigma_\Delta \tau_x^{-1}S(1, t) \leq Ct^b,
\]

where \( C \) depends on \( c_1c_2\alpha \) and \( b = \pi/(2\cos^{-1}(\frac{1}{c_1c_2\alpha})) \).

By Lemma 1,

\[
\sigma_\Omega(\phi^{-1}S(\zeta, t) \cap \partial \Omega) \equiv \sigma_\Omega(A)
\]

\[
\leq C \int_{\partial \Omega} \int_0^{2\pi} \chi_A \circ \varphi_x(e^{i\theta}) \, d\theta \, d\sigma_\Omega(x)
\]

\[
\leq Ct^b,
\]

since \( \chi_A \circ \varphi_x(e^{i\theta}) = 1 \iff |1 - \tau_x(e^{i\theta})| \leq t \).

Now set \( \alpha_1 = \frac{1}{c_1c_2} \sec^2\frac{\pi}{2N} \). If \( \phi(\Omega) \subset D_\alpha(\zeta) \) where \( \alpha < \alpha_1 \), then \( b > N \) and the above calculation shows \( \sigma_\Omega(\phi^{-1}S(\zeta, t) \cap \partial \Omega) = o(t^N) \) while if \( \phi(\Omega) \subset D_\alpha(\zeta) \) we have

\[
\sigma_\Omega(\phi^{-1}S(\zeta, t) \cap \partial \Omega) = O(t^N).
\]

To obtain these same estimates for

\[
\sigma_\Omega(\phi^{-1}S(\eta, t) \cap \partial \Omega)
\]

where \( \eta \in \partial \Omega \) is arbitrary, we observe the following: there is a constant \( k_1 \), independent of \( \eta \) and \( t \), so that if \( S(\eta, t) \cap D_\alpha(\zeta) \) is non-empty, then \( S(\zeta, k_1t) \supset S(\eta, t) \). To see this first notice that there is a constant \( k_2 \) (depending only on \( \Omega \)) so that if \( S(\eta, t) \) and \( S(\zeta, t) \) intersect, then \( S(\zeta, k_2t) \supset S(\eta, t) \) (see, for example, [H, p. 73] where the analogous property is proved for the comparable
sets \( A(\zeta, t) \). Now observe that if \( z \in S(\eta, t) \cap D_\alpha(\zeta) \), then \( z \in S(\zeta, k_3 \alpha t) \) for some constant \( k_3 \) and therefore \( S(\eta, t) \subseteq S(\zeta, k_2 k_3 \alpha t) \equiv S(\zeta, k_1 t) \). Thus

\[
\sigma_\Omega(\phi^{-1}S(\eta, t) \cap \partial \Omega) \leq \sigma_\Omega(\phi^{-1}S(\zeta, k_1 t) \cap \partial \Omega)
\]

\[
\leq c(k_1 t)^b = c't^b,
\]

with \( b \) as before, as desired. \( \square \)

In the case \( \Omega = B_N \) and \( z_0 = 0 \) both \( c_1 \) and \( c_2 \) can be taken to be 1 and the regions \( D_\alpha(\zeta) \) are the usual Koranyi approach regions in \( B \):

\[
D_\alpha(\zeta) = \{ z \in B : |1 - \langle z, \zeta \rangle| < \alpha(1 - |z|) \}.
\]

In this setting Theorem 9 becomes exactly Theorem 2.2 of [M], with \( \alpha_0 = 1 \) and \( \alpha_1 = \sec \frac{\pi}{2N} \).

We finish with some brief remarks on examples. Consider the map \( \phi: B_N \rightarrow B_N \) defined by

\[
\phi(z_1, z_2, \ldots, z_N) = (N^{\frac{1}{2}} z_1 z_2 \cdots z_N, 0').
\]

A direct computation ([MS, p. 904]) shows that for \( t \) small

\[
\lambda_B^\alpha(\phi^{-1}(S(e_1, t)) \simeq t^{\alpha+(N-1)+2},
\]

where \( e_1 = (1, 0') \in \partial B_N \).

Thus in this example \( C_\phi \) is not bounded from \( H^p(B_N) \) to \( A^p,\alpha(B_N) \) for any \( \alpha < \frac{N-3}{2} \) (notice that the spaces \( A^p,\alpha(B_N) \) may be defined for any \( \alpha > -1 \), whereas Theorem 7 shows that \( C_\phi \) must be bounded into \( A^p,\alpha(B_N) \) for all \( \alpha \geq N-2 \). This leads naturally to the question of whether the weight \( \alpha = N-2 \) is optimal in Theorem 7. We believe it is, and in support of this we offer the following observation. For any \( \phi: B_N \rightarrow B_N \) the same sort of slice integration argument used in the proof of Proposition 4 (specialized to the ball) shows that

\[
\sigma_B^\alpha \phi^{-1}S(\zeta, t) = O(t)
\]

for \( \zeta \in \partial B_N \) and \( t > 0 \) (again, we identify \( \phi \) with its a.e. radial extension to \( \partial B_N \)). Moreover, the worst case situation \( \sigma_B^\alpha \phi^{-1}S(\zeta, t) \simeq t \) can occur: if \( \gamma \) is a non-constant inner function on \( B_N \) with \( \gamma(0) = 0 \) define \( \phi \) on \( B_N \) by \( \phi = (\gamma, 0') \). Then since \( \gamma \) is measure-preserving as a map from \( \partial B_N \) to \( \partial \Delta \), \( \sigma_B^\alpha \phi^{-1}S(e_1, t) = t \). Unfortunately this example is not amenable to calculation of \( \lambda_B^\alpha \phi^{-1}S(e_1, t) \) due to the bad oscillatory behavior of \( \gamma \) near points of \( \partial B_N \).

We do not know if \( \sigma_B^\alpha \phi^{-1}S(\zeta, t) \simeq t \) can occur, say, with a Lip 1 mapping \( \phi \); such an example would give \( \lambda_B^\alpha \phi^{-1}S(\zeta, t) \simeq t^{\alpha+2} \), which in turn would show that the exponent \( \alpha = N-2 \) in Theorem 7 cannot be replaced by anything smaller.

The above inner function example can be used to show that Theorem 9 is optimal in the ball, as the relevant computations involve \( \sigma_B^\alpha \phi^{-1} \) rather than \( \lambda_B^\alpha \phi^{-1} \). Specifically there exist \( \phi: B_N \rightarrow B_N \) with \( \phi(B_N) \subset D_\alpha(\zeta) \) (\( \alpha_1 = \sec \frac{\pi}{2N} \)) and \( C_\phi \) bounded but not compact from \( H^p(B_N) \) to \( H^p(B_N) \), and for \( \beta > \alpha_1 \), there exist \( \phi: B_N \rightarrow B_N \) with \( \phi(B_N) \subset D_\beta(\zeta) \) yet \( C_\phi \) not bounded from \( H^p(B_N) \) to \( H^p(B_N) \). See [M] for the details.
References


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