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ON THE COMPOSITION OF TRANSCENDENTAL ENTIRE AND MEROMORPHIC FUNCTIONS

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ABSTRACT. It is proved that $f(g) - R$ has infinitely many zeros if f is a transcendental meromorphic, g a transcendental entire, and R a non-constant rational function. The exponent of convergence of the sequence of zeros of $f(g) - R$ is also estimated.

1. INTRODUCTION AND MAIN RESULT

The main result of this paper is the following theorem.

Theorem. *Suppose that f is transcendental meromorphic in the plane, that g is transcendental entire, and that R is a non-constant rational function. Then the equation $f(g(z)) = R(z)$ has infinitely many solutions.*

K. Katajamäki, L. Kinnunen, and I. Laine [7] proved this result under the hypothesis that f has finite order and g has finite lower order. In fact, they gave a lower bound for the exponent of convergence of the solutions and also dealt with the case that R is transcendental but of smaller growth than g. While our method does not seem to be suitable to handle transcendental functions R, it does give a bound for the exponent of convergence (compare §3). The results of K. Katajamäki, L. Kinnunen, and I. Laine [7] generalized their results of [6], where they assumed that f is entire, as well as the result of [3], where the above theorem was proved under the additional hypothesis that $f(g)$ has finite order.

The above theorem is known in the case that $R(z) = z$ [2], as well as in the case that f is entire and R is a polynomial [1]. These results confirmed a conjecture of F. Gross [5]. The method used in [2] is based on the observation that $f(g)$ has infinitely many fixpoints if and only if $g(f)$ does. The underlying idea in the present paper is to consider the solutions of $f(g(z)) = R(z)$ as fixpoints of $R^{-1}(f(g(z)))$ and to proceed similarly as in [2]. This requires some modifications of the argument, however, because R^{-1} is, in general, not a single-valued function.

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We note that the method of [1] extends to the case that R is rational. It does not seem to extend, however, to the case that f is meromorphic. On the other hand, the case that f has only one pole can be handled by this method with only minor modifications.

2. Proof of the theorem

We may assume that $R(\infty) = \infty$ and in fact that $R(z) \sim z^n$ for some $n \in \mathbb{N}$ as $z \to \infty$ because otherwise we consider $L(f)$ and $L(R)$ instead of f and R for a suitable linear transformation L. In view of the remarks made at the end of the introduction, we may also assume that f has at least two poles z_1 and z_2. By p_j we denote the order of z_j.

By [2, Lemma 1] there exist functions h_j analytic in a neighborhood of 0 such that $h_j(0) \doteq 0$ and $f(h_j(z) + z_j) = z^{-p_j}$ for $j = 1, 2$. Similarly as in [2] we define $k_1(z) = h_1(z^{p_2 n}) + z_1$ and $k_2(z) = h_2(z^{p_1 n}) + z_2$ so that $f(k_1(z)) = f(k_2(z)) = z^{-p_1 p_2 n}$.

Next we note that $R(z) = S(z)^n$ for some function S which is univalent in a neighborhood of ∞ and satisfies $S(z) \sim z$ as $z \to \infty$. We denote the inverse function of S by T. Then $T(z) \sim z$ as $z \to \infty$ and $R(T(z)) = z^n$. Finally, following [2], we define $u(z) = g(T(z^{-p_1 p_2}))$ and

$$v(z) = \frac{u(z) - k_1(z)}{u(z) - k_2(z)}.$$

Then 0 is an essential singularity of u and hence v. Because $k_1(0) = z_1 \neq z_2 = k_2(0)$, we have $v(z) \neq 1$ for sufficiently small z. Hence Picard's theorem implies that v takes one of the values 0 and ∞ in any punctured neighborhood of 0. Without loss of generality we may assume that this holds for the value 0, that is, there exists a sequence ζ_j tending to 0 such that $v(\zeta_j) = 0$. It follows that $u(\zeta_j) = k_1(\zeta_j)$ and hence that

$$f(g(T(\zeta_j^{-p_1 p_2}))) = f(u(\zeta_j)) = f(k_1(\zeta_j)) = \zeta_j^{-p_1 p_2 n} = R(T(\zeta_j^{-p_1 p_2})),$$

that is, $f(g(\omega_j)) = R(\omega_j)$ for $\omega_j = T(\zeta_j^{-p_1 p_2})$. The conclusion follows since $\zeta_j \to 0$ implies that $\omega_j \to \infty$.

3. A quantitative version of the main result

Denote by $\rho(f)$ and $\lambda(f)$ the order and the lower order of f and by σ the exponent of convergence of the zeros of $f(g(z)) - R(z)$. K. Katajamäki, L. Kinnunen, and I. Laine [7] have shown that $\sigma \geq \lambda(g)$, provided $\lambda(g) < \infty$ and $\rho(f) < \infty$.

We shall show that the slightly stronger inequality $\sigma \geq \rho(g)$ can be obtained without growth restrictions on f or g if we make the assumptions of §2:

If f has at least two poles and $R(\infty) = \infty$, then $\sigma \geq \rho(g)$.

To prove this result, we proceed as in §2. Instead of Picard's theorem, however, we use Nevanlinna theory for functions meromorphic in the neighborhood of an essential singularity (see, e.g., [4, §78ff.]). By $\rho(u)$ and $\rho(v)$ we denote the orders of u and v at 0. It is easily seen from the definitions of u and v that $\rho(v) = \rho(u) = p_1 p_2 \rho(g)$. By Borel's theorem [4, p. 354], we may assume that the exponent of convergence of ζ_j at 0 is equal to $\rho(v)$ so that $\sum_{j=1}^{\infty} |\zeta_j|^{\mu}$

diverges for all $\mu < \rho(v)$. Because $\omega_j = T(\zeta_j^{-p_1p_2}) \sim \zeta_j^{-p_1p_2}$ as $j \to \infty$, this implies that $\sum_{j=1}^{\infty} |\omega_j|^{-\mu}$ diverges for all $\mu < \rho(g)$, that is, the exponent of convergence of the zeros of $f(g) - R$ is at least $\rho(g)$.

4. Remarks

The estimate $\sigma \geq \rho(g)$ is probably far from being best possible. It seems likely to me that the counting function of the zeros of $f(g) - R$ and the Nevanlinna characteristic of $f(g)$ are always of the same order of magnitude. Possibly the Nevanlinna deficiency $\delta(0, f(g) - R))$ is always equal to 0 if f, g, and R are as in the statement of our main theorem. We remark that J. K. Langley [8] proved that if f and g are entire transcendental and if $f(g)$ is of finite order, then $\delta(0, f(g) - R)) < 1$ for any non-constant rational function R. For further results concerning the number of zeros of $f(g) - R$ for entire f we refer to C.-C. Yang and J.-H. Zheng [10].

Finally we mention that the conclusion of our main theorem remains valid if f is a rational function of degree at least 2 (see K. Katajamäki, L. Kinnunen, and I. Laine [7] or G. S. Prokopovich [9]). This follows also from our proof if f has at least two poles.

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