ON THE DETERMINANT AND THE HOLOMONY OF EQUIVARIANT ELLIPTIC OPERATORS

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Abstract. Let $M$ be a closed oriented smooth manifold, $G$ a compact Lie group consisting of diffeomorphisms of $M$, $P \to Z$ a principal $G$-bundle with a connection and $D$ a $G$-equivariant elliptic operator. Then a locally constant family of elliptic operators and its determinant line bundle over $Z$ are naturally defined by $D$. Moreover the holonomy of the determinant line bundle is defined by the connection in $P$. In this note, we give an explicit formula to calculate the holonomy (Theorem 1.4) and give a proof of the Witten holonomy formula (Theorem 1.7) in the special case above.

1. Main results

Let $M$ be a closed oriented smooth manifold, $G$ a compact Lie group consisting of diffeomorphisms of $M$, $P \to Z$ a principal $G$-bundle over a smooth manifold $Z$ with a connection and $D$ a $G$-equivariant elliptic operator. Then, for any $g \in G$, the index of $D$ evaluated at $g$, $\text{Index}(D, g)$, is defined by

$$\text{Index}(D, g) = \text{tr}(g|_{\ker D}) - \text{tr}(g|_{\text{coker} D})$$

and can be calculated by the well-known fixed point formula (cf. [1], [2] or [5]). On the other hand, the determinant of $D$ evaluated at $g$, $\det(D, g)$, is defined by

$$\det(D, g) = \det(g|_{\ker D}) / \det(g|_{\text{coker} D}).$$

Then the next proposition is an immediate consequence of the elementary result of Lemma 1 in Appendix.

Proposition 1.1. Let $g \in G$ be any element of finite order $p$. Then the next equality holds:

$$\det(D, g) = \exp \left( 2\pi i \sum_{k=1}^{p-1} \frac{1}{1 - e^{-2\pi ik/p}} \{ \text{Index}(D) - \text{Index}(D, g^k) \} \right)$$

where $\text{Index}(D) = \text{Index}(D, 1)$ is the numerical index of $D$.

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Remark 1.2. The homomorphism \( \det(D, \cdot): G \to S^1 \) defined by \( \det(D, g) \) is determined by its restriction to the dense subset of \( G \) which consists of all elements of finite order.

Now we assume that \( M \) is a 2\( n \)-dimensional closed Riemannian manifold with a Spin\(^c\)-structure and a connection in the associated \( S^1 \)-bundle of the Spin\(^c\)-structure. We assume that \( G \) acts on \( M \) as isometries and that the action of \( G \) preserves the Spin\(^c\)-structure and the \( S^1 \)-connection. Let \( E \) be a hermitian vector bundle (or virtual vector bundle) over \( M \) with a unitary connection. We assume that the action of \( G \) lifts to a connection-preserving unitary action on \( E \). Then we can define the \( G \)-equivariant Spin\(^c\)-Dirac operator \( D \) on \( M \)

\[
D: \Gamma(S^+ \otimes E) \to \Gamma(S^- \otimes E)
\]

(for the definition of the half spinor bundles \( S^\pm \), see [7, pp. 106-108]) and a line bundle \( \det(D) \) over \( Z \) by

\[
\det(D) = P \times_G ((\wedge' \ker D)^* \otimes (\wedge' \text{coker } D))
\]

where \( r, s \) denote the dimensions of the finite-dimensional (complex) \( G \)-modules \( \ker D, \text{coker } D \). Note that the \( G \)-equivariant elliptic operator \( D \) naturally defines a locally constant elliptic family \( P \times_G D \) parametrized by \( Z \) and the determinant line bundle defined by this elliptic family is isomorphic to \( \det(D) \) above (see [6, pp. 133-134]). The connection in \( P \) naturally defines a connection in \( \det(D) \) and we can regard \( \det(D) \) as a line bundle with a connection. On the other hand, for any \( g \in G \), by considering the mapping torus

\[
M_g = M \times [0, 1]/\sim \quad \text{where} \quad (m, 0) \sim (g(m), 1),
\]

we can also define a locally constant family of Dirac operators parametrized by \( S^1 \). Here the horizontal subspaces of the fibration \( M_g \to S^1 \) is given by the \([0,1] \)-directed vectors. Then we can define as in [7] the determinant line bundle \( L(g) \) over \( S^1 \). Note that, if a horizontal lift \( \tilde{\gamma} \) of an oriented loop \( \gamma \) in \( Z \) connects any fixed base point \( b \) in \( P \) with \( b \cdot g^{-1} \), it is not difficult to see that \( L(g) \) is isomorphic to the restriction of \( \det(D) \) to the loop \( \gamma \) as a line bundle with a connection. Now it can be seen that the holonomy of \( L(g) \) around \( S^1 \), which we denote by \( \text{hol}(D, g) \), is equal to \( \det(D, g) \) and hence the next theorem follows immediately from Proposition 1.1.

Theorem 1.4. Let \( g \in G \) be any element of finite order \( p \). Then we have

\[
\text{hol}(D, g) = \exp \frac{2\pi i}{p} \sum_{k=1}^{p-1} \frac{1}{1 - e^{-2\pi ik/p}} \{\text{Index}(D) - \text{Index}(D, g^k)\},
\]

and hence \( \text{hol}(D, g) \) is calculated explicitly by using the Atiyah-Bott-Singer fixed point formula.

Now the tangent bundle of \( M_g \) splits as the direct sum of the tangent bundle of \( M \) and the trivial real line bundle defined by \([0,1] \)-directed vectors. Hence the Riemannian metric and the Spin\(^c\)-structure on \( M_g \) are naturally defined by those on \( M \) together with the standard metric and the trivial Spin\(^c\)-structure on \([0,1] \). Moreover the associated \( S^1 \)-bundle of the Spin\(^c\)-structure over \( M_g \) and its connection are naturally defined by the \( S^1 \)-bundle over \( M \) and the
standard globally flat connection in the [0,1]-directed trivial real line bundle.
Let $S_g$ be the spinor bundle with respect to the above Spin$^c$-structure on $M_g$
and $E_g$ the hermitian vector bundle (or virtual vector bundle) over $M_g$ with a
unitary connection defined by the mapping torus construction (1.3). Then the
Spin$^c$-Dirac operator

$$A_g : \Gamma(S_g \otimes E_g) \to \Gamma(S_g \otimes E_g)$$

is defined and

$$\xi_g = \frac{1}{2}(\eta_g + \dim \ker A_g)$$

is defined by the eta invariant $\eta_g$ of $A_g$. Note that, using the same argument
as in [4], we can see that $\xi_g$ modulo integer is continuous in $g$.

Now we assume that $g \in G$ has a finite order $p$. Let $X = M \times D^2$, $Y = \partial X = M \times S^1$ be the product Riemannian Spin$^c$-manifolds with the
Spin$^c$-structures induced from the Spin$^c$-structure on $M$ and the trivial Spin$^c$-
structures on $D^2$, $S^1$. We give the metric $p^2 ds^2$ on $S^1$ where $ds^2$ is the
standard metric on $S^1$ and a rotationally symmetric Riemannian metric on $D^2$
which is the product metric near $\partial D^2 = S^1$. The Levi-Civita connection of this
metric defines the connection in the associated $S^1$-bundles over $D^2$, $S^1$. Then
we can define the actions of $\mathbb{Z}_p = \langle g \rangle$ on $X$ and on $\partial X = Y$ as follows:

$$g \cdot (m , \varphi e^{i\theta}) = (g(m) , \varphi e^{i\theta + 2\pi i/p})$$

for $(m , \varphi e^{i\theta}) \in X = M \times D^2 ; \varphi \leq e^{\frac{p}{2\pi}} , 0 \leq \theta \leq 2\pi$. Note that $\mathbb{Z}_p$
acts freely on $Y$ and $Y/\mathbb{Z}_p$ is equal to $M_g$. Let $q_X : X = M \times D^2 \to M$, $q_Y : Y = M \times S^1 \to M$ be projections, and let $E_X$, $E_Y$ denote the hermitian
vector bundles (or virtual vector bundles) $q_X^* E = E \times D^2$, $q_Y^* E = E \times S^1$ on
$X$, $Y$ provided with naturally induced unitary connections. Let

$$B : \Gamma(S_X^+ \otimes E_X) \to \Gamma(S_X^- \otimes E_X) ,$$

$$A : \Gamma(S_Y^+ \otimes E_Y) \to \Gamma(S_Y^- \otimes E_Y)$$

be the $\mathbb{Z}_p$-equivariant Spin$^c$-Dirac operators on $X$, $Y$ where $S_X^\pm$, $S_Y$ are
the spinor bundles over $X$, $Y$ with respect to the Spin$^c$-structures on $X$, $Y$
respectively. Then it is clear that the spinor bundle $S_g$ is equal to the
spinor bundle with respect to the Spin$^c$-structure on $M_g$ induced from the $\mathbb{Z}_p$-
invariant Spin$^c$-structure on $Y$. Moreover it is also clear that $E_g$ is equal to
the quotient $E_Y/\mathbb{Z}_p$ and that the Spin$^c$-Dirac operator $A_g$ on $M_g$ is equal to
the quotient $A/\mathbb{Z}_p$. Here we have the following:

**Proposition 1.6.** Let $g \in G$ be any element of finite order $p$. Then we have
the right-hand side of (1.5) $(-1)^{\text{Index}(D)} e^{-2\pi i \xi_g}$.

**Proof.** For any $h \in \mathbb{Z}_p$, let $\eta_Y(h)$ denote the eta invariant of $A$
evaluated at $h$ (cf. [4]). Then it follows from the same arguments as in [8] that

$$\xi_g = \frac{1}{2}(\eta_g + \dim \ker A_g) = \frac{1}{p} \sum_{k=1}^p \left( \frac{1}{2} \eta_Y(g^k) + \frac{1}{2} \text{tr}(g^k|_{\ker A}) \right).$$

On the other hand, it follows from Theorem 1.2 in [8] that

$$\frac{1}{2} \eta_Y(g^k) + \frac{1}{2} \text{tr}(g^k|_{\ker A}) + \text{Index}(B , g^k)$$

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is equal to the integral
\[ \int_X \text{ch}(E_X) \exp \frac{c_1(S, X)}{2} \tilde{A}(X) \]
if \( k = p \), and is equal to the summation of certain characteristic numbers \( \mathfrak{A}[N] \)
\[ \sum_{N \in \Omega(X)} \mathfrak{A}[N] \]
if \( k \neq p \), where \( \text{Index}(B, g^k) \) is the \( g^k \)-index (i.e., the index evaluated at \( g^k \)) of \( B \) with the global boundary condition considered in Theorem (3.10) in [3], \( \text{ch}(E_X) \) is the Chern character form of \( E_X \), \( c_1(S, X) \) is the first Chern form of the associated \( S^1 \)-bundle of the Spin\(^c\)-structure on \( X \) with respect to the \( S^1 \)-connection, \( \tilde{A}(X) \) is the total \( \tilde{A} \)-form of \( TX \) and \( \Omega(X) \) is the fixed point set of the \( g^k \)-action \( (k \neq p) \) on \( X \) consisting of closed connected submanifolds \( N \). Now it is easy to see that the fixed point set \( \Omega(X) \) coincides with the fixed point set \( \Omega(M) \) of the \( g^k \)-action on \( M = M \times \{0\} \subset M \times D^2 = X \) and the normal bundles \( \nu(N, X) \) of \( N \) in \( X \) is isomorphic to the direct sum of the normal bundles \( \nu(N, M) \) of \( N \) in \( M \) and the trivial bundles \( N \times \mathbb{R}^2 \). Here \( g \) acts on \( N \times \mathbb{R}^2 \) via the \( 2\pi/p \)-rotation of the fiber \( \mathbb{R}^2 \). Hence, considering the fixed point formula (cf. [5]), we can see that the quantity \( \sum_{N \in \Omega(X)} \mathfrak{A}[N] \) is related to the index of the operator \( D \) on \( M \) as follows:
\[ \sum_{N \in \Omega(X)} \mathfrak{A}[N] = \frac{1}{1 - e^{-2\pi ik/p}} \text{Index}(D, g^k). \]
On the other hand, it is clear that
\[ c_1(S, X) = q_0 c_1(S, M) + q_0^2 c_1(D^2) \]
where \( c_1(S, M) \) is the first Chern form of the associated \( S^1 \)-bundle of the Spin\(^c\)-structure on \( M \) with respect to the \( S^1 \)-connection, \( q_0 : X = M \times D^2 \to D^2 \) is the projection and \( c_1(D^2) \) is the first Chern form of \( D^2 \) with respect to the \( S^1 \)-connection which is rotationally symmetric and is product near the boundary. Moreover, since
\[ \text{ch}(E_X) = q_0^* \text{ch}(E), \quad \tilde{A}(X) = q_0^* \tilde{A}(M) \]
and
\[ \int_{D^2} \exp \frac{c_1(D^2)}{2} = \int_{D^2} \frac{c_1(D^2)}{2} = \frac{1}{2}, \]
it follows that
\[ \int_X \text{ch}(E_X) \exp \frac{c_1(S, X)}{2} \tilde{A}(X) = \frac{1}{2} \int_M \text{ch}(E) \exp \frac{c_1(S, M)}{2} \tilde{A}(M) = \frac{1}{2} \text{Index}(D). \]
Hence we can deduce the following equality.
\[ \xi_g = \frac{1}{p} \sum_{k=1}^{p-1} \frac{1}{1 - e^{-2\pi ik/p}} \text{Index}(D, g^k) + \frac{1}{2p} \text{Index}(D) - \frac{1}{p} \sum_{k=1}^{p} \text{Index}(B, g^k). \]
Now it follows from Lemma 2 in Appendix that
\[ \frac{1}{p} \sum_{k=1}^{p} \text{Index}(B, g^k) = 0 \mod{\mathbb{Z}}.\]
and from Lemma 3 in Appendix that
\[
\frac{1}{2p} \text{Index}(D) = \frac{1}{2} \text{Index}(D) - \frac{1}{p} \sum_{k=1}^{p-1} \frac{1}{1 - e^{-2\pi i k/p}} \text{Index}(D).
\]

Thus we can conclude that
\[
e^{-2\pi i \xi} = (-1)^{\text{Index}(D)} \exp \frac{2\pi i}{p} \sum_{k=1}^{p-1} \frac{1}{1 - e^{-2\pi i k/p}} \{\text{Index}(D) - \text{Index}(D, g^k)\}.
\]

This completes the proof.

Since both $\text{hol}(D, g)(= \det(D, g))$ and $e^{-2\pi i \xi}$ are continuous in $g$, it follows from Remark 1.2, Theorem 1.4 and Proposition 1.6 that

**Theorem 1.7 (cf. [7]).** The next equality holds:

\[
\text{hol}(D, g) = (-1)^{\text{Index}(D)} e^{-2\pi i \xi}
\]

for any $g \in G$.

2. An Example

Let $M$ be the non-singular hypersurface of degree $p \geq 2$ in $\mathbb{C}P^{n+1}$ defined by
\[
z_0^p + z_1^p + \cdots + z_{n+1}^p = 0
\]
where $[z_0 : z_1 : \cdots : z_{n+1}]$ is the homogeneous coordinate of $\mathbb{C}P^{n+1}$. Then the action
\[
g \cdot [z_0 : z_1 : \cdots : z_{n+1}] = [e^{2\pi i/p} z_0 : z_1 : \cdots : z_{n+1}]
\]
defines an action of $\mathbb{Z}_p = \langle g \rangle$ on $M$ and the fixed point set of this action is the non-singular hypersurface of degree $p$ in $\mathbb{C}P^n = \{z_0 = 0\} \subset \mathbb{C}P^{n+1}$ defined by
\[
z_1^p + z_2^p + \cdots + z_{n+1}^p = 0.
\]

Let $D$ be the Dolbeault operator on $M$ which is a $\mathbb{Z}_p$-equivariant elliptic operator. Then it follows from the Atiyah-Bott-Singer fixed point formula (see, for example, [9]) that $\text{Index}(D)$ is equal to the $x^n$-coefficient of
\[
\left( \frac{x}{1 - e^{-x}} \right)^{n+2} \left( \frac{1 - e^{-px}}{px} \right) \in \mathbb{C}[x]
\]
multiplied by $p$ and that $\text{Index}(D, g^k)$ is equal to the $x^{n-1}$-coefficient of
\[
\left( \frac{x}{1 - e^{-x}} \right)^{n+1} \left( \frac{1 - e^{-px}}{px} \right) \frac{1}{1 - e^{-x} e^{-2\pi ik/p}} \in \mathbb{C}[x]
\]
multiplied by $p$.

Now, for example, consider the case of $n = 2, 3$. Then we can obtain Tables 1 and 2 only from direct computations using Theorem 1.4 and the fixed point formula above.
Table 1

<table>
<thead>
<tr>
<th>$n = 2$</th>
<th>$p$</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
</tr>
</thead>
<tbody>
<tr>
<td>log(hol)</td>
<td></td>
<td>0</td>
<td>½</td>
<td>0</td>
<td>½</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1/12</td>
<td>0</td>
<td>7/14</td>
<td>0</td>
<td>12/16</td>
<td></td>
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</tbody>
</table>

Table 2

<table>
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<th>$n = 3$</th>
<th>$p$</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
</tr>
</thead>
<tbody>
<tr>
<td>log(hol)</td>
<td></td>
<td>0</td>
<td>4/5</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>8/10</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>12/15</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

where log(hol) denotes $\frac{1}{\Sigma_1} \log \text{hol}(D, g) \mod Z$.

Remark 2.1. If $c_1(M) > 0$ (namely, $p \leq n + 1$), it follows from the Kodaira vanishing theorem that coker$D = \{0\}$ and that ker$D$ is equal to the 1-dimensional space of constant functions on $M$ on which $\mathbb{Z}_p$ acts trivially. Therefore it immediately follows that $\text{hol}(D, g) = \text{det}(D, g) = 1$ and hence that log(hol) = 0. This can also be proved from direct calculations similar as above using the Atiyah-Bott-Singer fixed point formula.

Appendix

Lemma 1. Let $A$ be an $(N \times N)$-matrix which satisfies $A^p = E$ for some positive integer $p$ where $E$ denotes the unit matrix. Then the next equality holds:

$$\det(A) = \exp \frac{2\pi i}{p} \sum_{k=1}^{p-1} \frac{1}{1 - e^{-2\pi ik/p}} \{N - \text{tr}(A^k)\}.$$  

Proof. Let $e^{2\pi i \lambda_j/p}$ ($1 \leq j \leq N$) be the eigenvalues of $A$ where $\lambda_j$'s are integers such that $1 \leq \lambda_j \leq p$. Then the equality of the lemma is equivalent to the next equality:

$$\sum_{k=1}^{p-1} \frac{1}{1 - e^{-2\pi ik/p}} \sum_{j=1}^{N} \left(1 - e^{2\pi i \lambda_j k/p}\right) \mod p.$$  

Therefore it suffices to show that

$$\sum_{k=1}^{p-1} \frac{1 - e^{2\pi i k \lambda/p}}{1 - e^{-2\pi i k/p}} = \lambda \mod p$$

for any integer $\lambda$ such that $1 \leq \lambda \leq p$. Here the left-hand side of (1) is equal to $-\sum_{k=1}^{p-1} \sum_{\nu=1}^{\lambda} e^{2\pi i \nu k/p}$ and hence (1) follows from the equality

$$\sum_{k=1}^{p-1} e^{2\pi i \nu k/p} = -1 \mod p$$

for any integer $\nu$.  □
Lemma 2. Let $V$ be any finite-dimensional $\mathbb{Z}_p$-module and $g \in \mathbb{Z}_p$. Then we have

$$\sum_{k=1}^{p} \text{tr}(g^k|_V) = 0 \mod p.$$ 

Proof. This lemma follows from the equality

$$\sum_{k=1}^{p} \alpha^k = 0 \mod p$$

for any complex number $\alpha$ such that $\alpha^p = 1$. □

Lemma 3. The next equality holds:

$$\frac{1}{2p} = \frac{1}{2} - \frac{1}{p} \sum_{k=1}^{p-1} \frac{1}{1 - e^{-2\pi ik/p}}.$$ 

Proof. This lemma follows from the equality

$$\sum_{k=1}^{p-1} \frac{1}{1 - e^{-2\pi ik/p}} = \frac{p - 1}{2}.$$ □

References


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