ON THE DETERMINANT AND THE HOLOMONY
OF EQUIVARIANT ELLIPTIC OPERATORS

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Abstract. Let \( M \) be a closed oriented smooth manifold, \( G \) a compact Lie group consisting of diffeomorphisms of \( M \), \( P \to Z \) a principal \( G \)-bundle with a connection and \( D \) a \( G \)-equivariant elliptic operator. Then a locally constant family of elliptic operators and its determinant line bundle over \( Z \) are naturally defined by \( D \). Moreover the holonomy of the determinant line bundle is defined by the connection in \( P \). In this note, we give an explicit formula to calculate the holonomy (Theorem 1.4) and give a proof of the Witten holonomy formula (Theorem 1.7) in the special case above.

1. Main Results

Let \( M \) be a closed oriented smooth manifold, \( G \) a compact Lie group consisting of diffeomorphisms of \( M \), \( P \to Z \) a principal \( G \)-bundle over a smooth manifold \( Z \) with a connection and \( D \) a \( G \)-equivariant elliptic operator. Then, for any \( g \in G \), the index of \( D \) evaluated at \( g \), \( \text{Index}(D, g) \), is defined by

\[
\text{Index}(D, g) = \text{tr}(g|_{\ker D}) - \text{tr}(g|_{\text{coker }D})
\]

and can be calculated by the well-known fixed point formula (cf. [1], [2] or [5]). On the other hand, the determinant of \( D \) evaluated at \( g \), \( \det(D, g) \), is defined by

\[
\det(D, g) = \det(g|_{\ker D}) / \det(g|_{\text{coker }D}).
\]

Then the next proposition is an immediate consequence of the elementary result of Lemma 1 in Appendix.

Proposition 1.1. Let \( g \in G \) be any element of finite order \( p \). Then the next equality holds:

\[
\det(D, g) = \exp\left(\frac{2\pi i}{p} \sum_{k=1}^{p-1} \frac{1}{1 - e^{-2\pi ik/p}} \{\text{Index}(D) - \text{Index}(D, g^k)\}\right)
\]

where \( \text{Index}(D) = \text{Index}(D, 1) \) is the numerical index of \( D \).

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Remark 1.2. The homomorphism \( \text{det}(D, \cdot) : G \to S^1 \) defined by \( \text{det}(D, g) \) is determined by its restriction to the dense subset of \( G \) which consists of all elements of finite order.

Now we assume that \( M \) is a \( 2n \)-dimensional closed Riemannian manifold with a Spin\(^c\)-structure and a connection in the associated \( S^1 \)-bundle of the Spin\(^c\)-structure. We assume that \( G \) acts on \( M \) as isometries and that the action of \( G \) preserves the Spin\(^c\)-structure and the \( S^1 \)-connection. Let \( E \) be a hermitian vector bundle (or virtual vector bundle) over \( M \) with a unitary connection. We assume that the action of \( G \) lifts to a connection-preserving unitary action on \( E \). Then we can define the \( G \)-equivariant Spin\(^c\)-Dirac operator \( D \) on \( M \)

\[
D : \Gamma(S^+ \otimes E) \to \Gamma(S^- \otimes E)
\]

(for the definition of the half spinor bundles \( S^\pm \), see [7, pp. 106-108]) and a line bundle \( \text{det}(D) \) over \( Z \) by

\[
\text{det}(D) = P \times_G ((\Lambda^r \ker D)^* \otimes (\Lambda^s \text{coker } D))
\]

where \( r, s \) denote the dimensions of the finite-dimensional (complex) \( G \)-modules \( \ker D, \text{coker } D \). Note that the \( G \)-equivariant elliptic operator \( D \) naturally defines a locally constant elliptic family \( P \times_G D \) parametrized by \( Z \) and the determinant line bundle defined by this elliptic family is isomorphic to \( \text{det}(D) \) above (see [6, pp. 133-134]). The connection in \( P \) naturally defines a connection in \( \text{det}(D) \) and we can regard \( \text{det}(D) \) as a line bundle with a connection. On the other hand, for any \( g \in G \), by considering the mapping torus

\[
(1.3) \quad M_g = M \times [0, 1] / \sim \quad \text{where} \quad (m, 0) \sim (g(m), 1),
\]

we can also define a locally constant family of Dirac operators parametrized by \( S^1 \). Here the horizontal subspaces of the fibration \( M_g \to S^1 \) is given by the \([0,1]\)-directed vectors. Then we can define as in [7] the determinant line bundle \( L(g) \) over \( S^1 \). Note that, if a horizontal lift \( \tilde{y} \) of an oriented loop \( y \) in \( Z \) connects any fixed base point \( b \) in \( P \) with \( b \cdot g^{-1} \), it is not difficult to see that \( L(g) \) is isomorphic to the restriction of \( \text{det}(D) \) to the loop \( \gamma \) as a line bundle with a connection. Now it can be seen that the holonomy of \( L(g) \) around \( S^1 \), which we denote by \( \text{hol}(D, g) \), is equal to \( \text{det}(D, g) \) and hence the next theorem follows immediately from Proposition 1.1.

Theorem 1.4. Let \( g \in G \) be any element of finite order \( p \). Then we have

\[
(1.5) \quad \text{hol}(D, g) = \exp \frac{2\pi i}{p} \sum_{k=1}^{p-1} \frac{1}{1 - e^{-2\pi ik/p}} \left\{ \text{Index}(D) - \text{Index}(D, g^k) \right\},
\]

and hence \( \text{hol}(D, g) \) is calculated explicitly by using the Atiyah-Bott-Singer fixed point formula.

Now the tangent bundle of \( M_g \) splits as the direct sum of the tangent bundle of \( M \) and the trivial real line bundle defined by \([0,1]\)-directed vectors. Hence the Riemannian metric and the Spin\(^c\)-structure on \( M_g \) are naturally defined by those on \( M \) together with the standard metric and the trivial Spin\(^c\)-structure on \([0,1]\). Moreover the associated \( S^1 \)-bundle of the Spin\(^c\)-structure over \( M_g \) and its connection are naturally defined by the \( S^1 \)-bundle over \( M \) and the
standard globally flat connection in the [0,1]-directed trivial real line bundle. Let $S_g$ be the spinor bundle with respect to the above Spin$^c$-structure on $M_g$ and $E_g$ the hermitian vector bundle (or virtual vector bundle) over $M_g$ with a unitary connection defined by the mapping torus construction (1.3). Then the Spin$^c$-Dirac operator

$$A_g : \Gamma(S_g \otimes E_g) \rightarrow \Gamma(S_g \otimes E_g)$$

is defined and

$$\xi_g = \frac{1}{2}(\eta_g + \dim \ker A_g)$$

is defined by the eta invariant $\eta_g$ of $A_g$. Note that, using the same argument as in [4], we can see that $\xi_g$ modulo integer is continuous in $g$.

Now we assume that $g \in G$ has a finite order $p$. Let $X = M \times D^2$, $Y = \partial X = M \times S^1$ be the product Riemannian Spin$^c$-manifolds with the Spin$^c$-structures induced from the Spin$^c$-structure on $M$ and the trivial Spin$^c$-structures on $D^2$, $S^1$. We give the metric $p^2 ds^2$ on $S^1$ where $ds^2$ is the standard metric on $S^1$ and a rotationally symmetric Riemannian metric on $D^2$ which is the product metric near $\partial D^2 = S^1$. The Levi-Civita connection of this metric defines the connection in the associated $S^1$-bundles over $D^2$, $S^1$. Then we can define the actions of $Z_p = \langle g \rangle$ on $X$ and on $\partial X = Y$ as follows:

$$g \cdot (m, re^{i\theta}) = (g(m), rew + 2\pi \theta/p)$$

for $(m, re^{i\theta}) \in X = M \times D^2$; $0 \leq r \leq \frac{p}{2\pi}$, $0 \leq \theta \leq 2\pi$. Note that $Z_p$ acts freely on $Y$ and $Y/Z_p$ is equal to $M_g$. Let $q_X : X = M \times D^2 \rightarrow M$, $q_Y : Y = M \times S^1 \rightarrow M$ be projections, and let $E_X$, $E_Y$ denote the hermitian vector bundles (or virtual vector bundles) $q_X^*E = E \times D^2$, $q_Y^*E = E \times S^1$ on $X$, $Y$ provided with naturally induced unitary connections. Let

$$B : \Gamma(S_X^+ \otimes E_X) \rightarrow \Gamma(S_X^- \otimes E_X),$$

$$A : \Gamma(S_Y \otimes E_Y) \rightarrow \Gamma(S_Y \otimes E_Y)$$

be the $Z_p$-equivariant Spin$^c$-Dirac operators on $X$, $Y$ where $S_X^\pm$, $S_Y$ are the spinor bundles over $X$, $Y$ with respect to the Spin$^c$-structures on $X$, $Y$ respectively. Then it is clear that the spinor bundle $S_g$ is equal to the spinor bundle with respect to the Spin$^c$-structure on $M_g$ induced from the $Z_p$-invariant Spin$^c$-structure on $Y$. Moreover it is also clear that $E_g$ is equal to the quotient $E_Y/Z_p$ and that the Spin$^c$-Dirac operator $A_g$ on $M_g$ is equal to the quotient $A/Z_p$. Here we have the following:

**Proposition 1.6.** Let $g \in G$ be any element of finite order $p$. Then we have

$$\text{the right-hand side of (1.5) } = (-1)^{\text{Index}(D)}e^{-2\pi i \xi_g}.$$

**Proof.** For any $h \in Z_p$, let $\eta_Y(h)$ denote the eta invariant of $A$ evaluated at $h$ (cf. [4]). Then it follows from the same arguments as in [8] that

$$\xi_g = \frac{1}{2}(\eta_g + \dim \ker A_g) = \frac{1}{p} \sum_{k=1}^{p} \left( \frac{1}{2} \eta_Y(g^k) + \frac{1}{2} \text{tr}(g^k |_{\ker A}) \right).$$

On the other hand, it follows from Theorem 1.2 in [8] that

$$\frac{1}{2} \eta_Y(g^k) + \frac{1}{2} \text{tr}(g^k |_{\ker A}) + \text{Index}(B, g^k)$$
is equal to the integral
\[ \int_X \text{ch}(E_X) \exp \frac{c_1(S, X)}{2} \hat{A}(X) \]
if \( k = p \), and is equal to the summation of certain characteristic numbers \( \mathfrak{A}[N] \)
\[ \sum_{N \in \Omega(X)} \mathfrak{A}[N] \]
if \( k \neq p \), where \( \text{Index}(B, g^k) \) is the \( g^k \)-index (i.e., the index evaluated at \( g^k \)) of \( B \) with the global boundary condition considered in Theorem (3.10) in [3], \( \text{ch}(E_X) \) is the Chern character form of \( E_X \), \( c_1(S, X) \) is the first Chern form of the associated \( S^1 \)-bundle of the Spin\(^c\)-structure on \( X \) with respect to the \( S^1 \)-connection, \( \hat{A}(X) \) is the total \( \hat{A} \)-form of \( TX \) and \( \Omega(X) \) is the fixed point set of the \( g^k \)-action \( \langle k \neq p \rangle \) on \( X \) consisting of closed connected submanifolds \( N \). Now it is easy to see that the fixed point set \( \Omega(X) \) coincides with the fixed point set \( \Omega(M) \) of the \( g^k \)-action on \( M = M \times \{0\} \subset M \times D^2 = X \) and the normal bundles \( \nu(N, X) \) of \( N \) in \( X \) is isomorphic to the direct sum of the normal bundles \( \nu(N, M) \) of \( N \) in \( M \) and the trivial bundles \( N \times \mathbb{R}^2 \). Here \( g \) acts on \( N \times \mathbb{R}^2 \) via the \( 2\pi/p \)-rotation of the fiber \( \mathbb{R}^2 \). Hence, considering the fixed point formula (cf. [5]), we can see that the quantity \( \sum_{N \in \Omega(X)} \mathfrak{A}[N] \) is related to the index of the operator \( D \) on \( M \) as follows:
\[ \sum_{N \in \Omega(X)} \mathfrak{A}[N] = \frac{1}{1 - e^{-2\pi i k/p}} \text{Index}(D, g^k). \]
On the other hand, it is clear that
\[ c_1(S, X) = q^*c_1(S, M) + q_D^*c_1(D^2) \]
where \( c_1(S, M) \) is the first Chern form of the associated \( S^1 \)-bundle of the Spin\(^c\)-structure on \( M \) with respect to the \( S^1 \)-connection, \( q_D : X = M \times D^2 \rightarrow D^2 \) is the projection and \( c_1(D^2) \) is the first Chern form of \( D^2 \) with respect to the \( S^1 \)-connection which is rotationally symmetric and is product near the boundary. Moreover, since
\[ \text{ch}(E_X) = q_X^* \text{ch}(E), \quad \hat{A}(X) = q_X^* \hat{A}(M) \]
and
\[ \int_{D^2} \exp \frac{c_1(D^2)}{2} = \int_{D^2} \frac{c_1(D^2)}{2} = \frac{1}{2}, \]
it follows that
\[ \int_X \text{ch}(E_X) \exp \frac{c_1(S, X)}{2} \hat{A}(X) = \frac{1}{2} \int_M \text{ch}(E) \exp \frac{c_1(S, M)}{2} \hat{A}(M) = \frac{1}{2} \text{Index}(D). \]
Hence we can deduce the following equality.
\[ \xi_g = \frac{1}{p} \sum_{k=1}^{p-1} \frac{1}{1 - e^{-2\pi i k/p}} \text{Index}(D, g^k) + \frac{1}{2p} \text{Index}(D) - \frac{1}{p} \sum_{k=1}^{p} \text{Index}(B, g^k). \]
Now it follows from Lemma 2 in Appendix that
\[ \frac{1}{p} \sum_{k=1}^{p} \text{Index}(B, g^k) = 0 \mod \mathbb{Z} \]
and from Lemma 3 in Appendix that
\[
\frac{1}{2p} \text{Index}(D) = \frac{1}{2} \text{Index}(D) - \frac{1}{p} \sum_{k=1}^{p-1} \frac{1}{1 - e^{-2\pi i k/p}} \text{Index}(D).
\]
Thus we can conclude that
\[
e^{-2\pi i \xi g} = (-1)^{\text{Index}(D)} \exp \frac{2\pi i}{p} \sum_{k=1}^{p-1} \frac{1}{1 - e^{-2\pi i k/p}} \{\text{Index}(D) - \text{Index}(D, g^k)\}.
\]
This completes the proof.

Since both hol(D, g)(= det(D, g)) and \(e^{-2\pi i \xi g}\) are continuous in \(g\), it follows from Remark 1.2, Theorem 1.4 and Proposition 1.6 that

**Theorem 1.7** (cf. [7]). *The next equality holds:
\[
\text{hol}(D, g) = (-1)^{\text{Index}(D)} e^{-2\pi i \xi g}
\]
for any \(g \in G\).

2. **An Example**

Let \(M\) be the non-singular hypersurface of degree \(p \geq 2\) in \(\mathbb{C}P^{n+1}\) defined by
\[
z_0^p + z_1^p + \cdots + z_{n+1}^p = 0
\]
where \([z_0 : z_1 : \cdots : z_{n+1}]\) is the homogeneous coordinate of \(\mathbb{C}P^{n+1}\). Then the action
\[
g \cdot [z_0 : z_1 : \cdots : z_{n+1}] = [e^{2\pi i/p} z_0 : z_1 : \cdots : z_{n+1}]
\]
defines an action of \(Z_p = \langle g \rangle\) on \(M\) and the fixed point set of this action is the non-singular hypersurface of degree \(p\) in \(\mathbb{C}P^n = \{z_0 = 0\} \subset \mathbb{C}P^{n+1}\) defined by
\[
z_1^p + z_2^p + \cdots + z_{n+1}^p = 0.
\]
Let \(D\) be the Dolbeault operator on \(M\) which is a \(Z_p\)-equivariant elliptic operator. Then it follows from the Atiyah-Bott-Singer fixed point formula (see, for example, [9]) that \(\text{Index}(D)\) is equal to the \(x^n\)-coefficient of
\[
\left( \frac{x}{1 - e^{-x}} \right)^{n+2} \left( \frac{1 - e^{-px}}{px} \right) \in \mathbb{C}[x]
\]
multiplied by \(p\) and that \(\text{Index}(D, g^k)\) is equal to the \(x^{n-1}\)-coefficient of
\[
\left( \frac{x}{1 - e^{-x}} \right)^n \left( \frac{1 - e^{-px}}{px} \right) \frac{1}{1 - e^{-x} e^{-2\pi ik/p}} \in \mathbb{C}[x]
\]
multiplied by \(p\).

Now, for example, consider the case of \(n = 2, 3\). Then we can obtain Tables 1 and 2 only from direct computations using Theorem 1.4 and the fixed point formula above.
Table 1

<table>
<thead>
<tr>
<th>( n = 2 )</th>
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<tbody>
<tr>
<td>( p )</td>
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<tr>
<td>log(hol)</td>
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Table 2

<table>
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<tr>
<th>( n = 3 )</th>
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<tbody>
<tr>
<td>( p )</td>
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<tr>
<td>log(hol)</td>
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</table>

where log(hol) denotes \( \frac{1}{2\pi i} \log \text{hol}(D, g) \mod \mathbb{Z} \).

Remark 2.1. If \( c_1(M) > 0 \) (namely, \( p \leq n + 1 \)), it follows from the Kodaira vanishing theorem that \( \ker D = \{0\} \) and that \( \ker D \) is equal to the 1-dimensional space of constant functions on \( M \) on which \( \mathbb{Z}_p \) acts trivially. Therefore it immediately follows that \( \text{hol}(D, g) = \det(D, g) = 1 \) and hence that \( \log(\text{hol}) = 0 \). This can also be proved from direct calculations similar as above using the Atiyah-Bott-Singer fixed point formula.

Appendix

Lemma 1. Let \( A \) be an \((N \times N)\)-matrix which satisfies \( A^p = E \) for some positive integer \( p \) where \( E \) denotes the unit matrix. Then the next equality holds:

\[
\det(A) = \exp \frac{2\pi i}{p} \sum_{k=1}^{p-1} \frac{1}{1 - e^{-2\pi ik/p}} \{N - \text{tr}(A^k)\}
\]

Proof. Let \( e^{2\pi i\lambda_j/p} \) \((1 \leq j \leq N)\) be the eigenvalues of \( A \) where \( \lambda_j \)'s are integers such that \( 1 \leq \lambda_j \leq p \). Then the equality of the lemma is equivalent to the next equality:

\[
\lambda_1 + \cdots + \lambda_N = \sum_{k=1}^{p-1} \frac{1}{1 - e^{-2\pi ik/p}} \sum_{j=1}^{N} \left(1 - e^{2\pi i\lambda_j/k/p}\right) \mod p.
\]

Therefore it suffices to show that

\[
\sum_{k=1}^{p-1} \frac{1 - e^{2\pi ik\lambda/p}}{1 - e^{-2\pi ik/p}} = \lambda \mod p
\]

for any integer \( \lambda \) such that \( 1 \leq \lambda \leq p \). Here the left-hand side of (1) is equal to \( -\sum_{k=1}^{p-1} \sum_{\nu=1}^{\lambda} e^{2\pi i\nu k/p} \) and hence (1) follows from the equality

\[
\sum_{k=1}^{p-1} e^{2\pi i\nu k/p} = -1 \mod p
\]

for any integer \( \nu \). \( \square \)
Lemma 2. Let $V$ be any finite-dimensional $\mathbb{Z}_p$-module and $g \in \mathbb{Z}_p$. Then we have
\[
\sum_{k=1}^{p} \text{tr}(g^k |_V) = 0 \mod p.
\]

Proof. This lemma follows from the equality
\[
\sum_{k=1}^{p} \alpha^k = 0 \mod p
\]
for any complex number $\alpha$ such that $\alpha^p = 1$. □

Lemma 3. The next equality holds:
\[
\frac{1}{2p} = \frac{1}{2} - \frac{1}{p} \sum_{k=1}^{p-1} \frac{1}{1 - e^{-2\pi i k/p}}.
\]

Proof. This lemma follows from the equality
\[
\sum_{k=1}^{p-1} \frac{1}{1 - e^{-2\pi i k/p}} = \frac{p - 1}{2}. □
\]

References


