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REAL RANK OF TENSOR PRODUCTS OF C^*-ALGEBRAS

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ABSTRACT. We study the real rank of tensor products of C^*-algebras. From the dimension theory: $\dim(X \times Y) \leq \dim X + \dim Y$, it is naturally hoped that $RR(A \otimes B) \leq RR(A) + RR(B)$. We then prove that it is false generally. Moreover, we point out that (FS)-property for C^*-algebras is not stable under taking tensor products.

1. INTRODUCTION

The concept of the non-commutative real rank for a C^*-algebra A ($= RR(A)$) was defined recently by Brown and Pedersen [4]. An important part of the motivation for introducing it is to have an analogue for C^*-algebras of the dimension for topological spaces: if X is a locally compact Hausdorff space, the dimension X ($= \dim X$) can be defined as a property of the algebras $C(X)$ of continuous functions on X [10]. Thus $\dim X \leq n$ if for any real-valued functions $f_1, f_2, \ldots, f_{n+1}$ and any non-negative real number ϵ there exist other real-valued functions $g_1, g_2, \ldots, g_{n+1}$ such that $\|f_i - g_i\| < \epsilon$ and $\sum C(X)g_i = C(X)$.

Since Gelfand's representation theory identifies commutative C^*-algebras with algebras $C_0(X)$ of continuous functions, vanishing at infinity, on locally compact Hausdorff spaces, it is natural to define the following concept: let A be a unital C^*-algebra and A_{sa} be the set of all selfadjoint elements in A. $RR(A)$ is the least integer n such that $\{(a_0, a_1, \ldots, a_n) \in A_{sa}^{n+1} : \sum_{k=0}^{n} Aa_k = A\}$ is dense in A_{sa}^{n+1}. If A is non-unital, its real rank is defined by $RR(\tilde{A})$, where $\tilde{A}$ is the C^*-algebra obtained by adding a unit to A. From this definition it is obvious that $\dim X = RR(C(X))$ for a compact Hausdorff space X.

Brown and Pedersen [4], Zhang [13, 14], and the second author [8, 9], however, studied that the real rank does not always have the parallel properties of the dimension theory: let X be a locally compact Hausdorff space and Y be a closed subset of X. Then $\dim X \leq \max\{\dim Y, \dim X \backslash Y\}$ and $\dim X = \dim \beta X$, where βX means the Stone-Čech compactification of X. For example, let D be an irreducible matrix such that $\det(I - D) = 0$ and O_D be the Cuntz - Krieger algebra corresponding to D. Zhang [14] stated that $RR(O_D) = RR(M(O_D \otimes \mathbf{K})/O_D \otimes \mathbf{K}) = 0$ but $RR(M(O_D \otimes \mathbf{K})) \neq 0$, where

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$\mathbf{K}$ is the algebra of compact operators on some separable infinite-dimensional Hilbert space and $M(A)$ means the multiplier algebra of A.

In this note, we treat the tensor products of C^*-algebras. From the dimension theory $\dim(X \times Y) \leq \dim X + \dim Y$ it is natural to conjecture that $RR(A \otimes B) \leq RR(A) + RR(B)$. We prove, however, it is false generally. That is, let A be a unital C^*-algebra with non-trivial K_1-group of $A(= K_1(A))$, then $RR(A \otimes B(H)) \neq 0$, where $B(H)$ denotes the algebra of all bounded operators on some separable infinite-dimensional Hilbert space H. Therefore, if B is one of the Bunce-Dedens algebras, we know $RR(B \otimes B(H)) \neq 0$, and this is a counterexample because it is known that $RR(B) = 0$, $K_1(B) = \mathbf{Z}$ [1][2], and $RR(B(H)) = 0$ [4]. Throughout this note tensor products of C^*-algebras mean the minimal tensor products.

We refer the reader to [3][4][6][8][9][11][13][14] for results about the real rank.

2. Result

We recall that C^*-algebra A is exact if

$$0 \to A \otimes \mathbf{K} \to A \otimes B(H) \to A \otimes B(H)/\mathbf{K} \to 0$$

is an exact sequence [5].

Proposition. *Let A be a unital exact C^*-algebra with $K_1(A) \neq 0$. Then*

$$RR(A \otimes B(H)) \neq 0.$$

Proof. By the six-term exact sequence from K-Theory [1],

$$\begin{tikzcd}
K_0(A \otimes \mathbf{K}) \arrow[r] & K_0(A \otimes B(H)) \arrow[r, "\pi_*"] & K_0(A \otimes B(H)/\mathbf{K}) \arrow[d, "\partial"] \\
K_1(A \otimes B(H)/\mathbf{K}) \arrow[u] & K_1(A \otimes B(H)) \arrow[l] & K_1(A \otimes \mathbf{K}) \arrow[l, "\iota_*"']
\end{tikzcd}$$

Since $K_1(A \otimes B(H)) = 0$ (see [7, Theorem 2.5]), π_* is not surjective. For, if π_* is surjective, $\operatorname{Ker}\partial = K_0(A \otimes B(H)/\mathbf{K})$, and $\partial = 0$. Since $\operatorname{Ker}\iota_* = \operatorname{Im}\partial$, we know ι_* is injective, and $K_1(A \otimes B(H)) \neq 0$. This ia a contradiction.

Hence, we know there is a projection in $A \otimes B(H)/\mathbf{K} \otimes \mathbf{K}$ which cannot be lifted to a projection in $A \otimes B(H) \otimes \mathbf{K}$.

Consider the following C^*-exact sequence:

$$0 \to A \otimes \mathbf{K} \otimes \mathbf{K} \to A \otimes B(H) \otimes \mathbf{K} \to A \otimes B(H)/\mathbf{K} \otimes \mathbf{K} \to 0.$$

Even if $RR(A \otimes \mathbf{K}) = RR(A \otimes B(H)/\mathbf{K}) = 0$, by [4, Theorem 3.14] (cf. [13, Proposition 2.3]) and the above argument, $RR(A \otimes B(H) \otimes \mathbf{K}) \neq 0$ and $RR(A \otimes B(H)) \neq 0$ (cf. [4, Corollary 3.3]). Otherwise, it is trivially $RR(A \otimes B(H)) \neq 0$, and the proof is completed. □

The next result means that the real rank of tensor products of C^*-algebras with real rank zero is not always zero.

Corollary. *Let B be one of the Bunce-Deddens algebras. We have, then,*

$$RR(B \otimes B(H)) \neq 0.$$

Proof. Since the Bunce-Deddens algebras are nuclear, they are exact [5]. By [1][2], we know $RR(B) = 0$ and $K_1(B) = \mathbf{Z}$. □

3. Remarks

(1) Using the idea in Proposition we can produce another example which does not satisfy the conjecture described in the introduction.

Let B be one of the Bunce-Deddens algebras and O_n be the Cuntz algebra. By the Künneth Theorem [12, Theorem 2.14], we know $K_1(B \otimes O_n) = \mathbf{Z}/(n-1)\mathbf{Z}$. As in the same argument $RR(B \otimes M(O_n \otimes \mathbf{K})) \neq 0$. On the other hand, O_n is a purely infinite simple C^*-algebra and $K_1(O_n) = 0$. We know $RR(M(O_n \otimes \mathbf{K})) = 0$ by Zhang [14, Examples 2.7(i)].

(2) As Brown and Pedersen pointed out in [4], a C^*-algebra has real rank zero if and only if it has the (FS)-property, that is, the set of its all selfadjoint elements has a dense set of elements with finite spectrum. Therefore, Corollary means that (FS)-property is not stable under taking tensor products.

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