REAL RANK OF TENSOR PRODUCTS OF C*-ALGEBRAS

KAZUNORI KODAKA AND HIROYUKI OSAKA

(Communicated by Palle E. T. Jorgensen)

Abstract. We study the real rank of tensor products of C*-algebras. From the dimension theory: \( \dim(X \times Y) \leq \dim X + \dim Y \), it is naturally hoped that \( RR(A \otimes B) \leq RR(A) + RR(B) \). We then prove that it is false generally. Moreover, we point out that (FS)-property for C*-algebras is not stable under taking tensor products.

1. Introduction

The concept of the non-commutative real rank for a C*-algebra \( A (= RR(A)) \) was defined recently by Brown and Pedersen [4]. An important part of the motivation for introducing it is to have an analogue for C*-algebras of the dimension for topological spaces: if \( X \) is a locally compact Hausdorff space, the dimension \( \dim X \) can be defined as a property of the algebras \( C(X) \) of continuous functions on \( X \) [10]. Thus \( \dim X \leq n \) if for any real-valued functions \( f_1, f_2, \ldots, f_{n+1} \) and any non-negative real number \( \varepsilon \) there exist other real-valued functions \( g_1, g_2, \ldots, g_{n+1} \) such that \( \|f_i - g_i\| < \varepsilon \) and \( \sum C(X)g_i = C(X) \).

Since Gelfand's representation theory identifies commutative C*-algebras with algebras \( C_b(X) \) of continuous functions, vanishing at infinity, on locally compact Hausdorff spaces, it is natural to define the following concept: let \( A \) be a unital C*-algebra and \( A_{sa} \) be the set of all selfadjoint elements in \( A \). \( RR(A) \) is the least integer \( n \) such that \( \{(a_0, a_1, \ldots, a_n) \in A_{sa}^{n+1} : \sum_{k=0}^{n} A_{a_k} = A\} \) is dense in \( A_{sa}^{n+1} \). If \( A \) is non-unital, its real rank is defined by \( RR(A) \), where \( A \) is the C*-algebra obtained by adding a unit to \( A \). From this definition it is obvious that \( \dim X = RR(C(X)) \) for a compact Hausdorff space \( X \).

Brown and Pedersen [4], Zhang [13, 14], and the second author [8, 9], however, studied that the real rank does not always have the parallel properties of the dimension theory: let \( X \) be a locally compact Hausdorff space and \( Y \) be a closed subset of \( X \). Then \( \dim X \leq \max \{\dim Y, \dim X \setminus Y\} \) and \( \dim X = \dim \beta X \), where \( \beta X \) means the Stone-Čech compactification of \( X \). For example, let \( D \) be an irreducible matrix such that \( \det(I - D) = 0 \) and \( O_D \) be the Cuntz-Krieger algebra corresponding to \( D \). Zhang [14] stated that \( RR(O_D) = RR(M(O_D \otimes K) / O_D \otimes K) = 0 \) but \( RR(M(O_D \otimes K)) \neq 0 \), where

Received by the editors November 19, 1993.
1991 Mathematics Subject Classification. Primary 46L05.

©1995 American Mathematical Society
$K$ is the algebra of compact operators on some separable infinite-dimensional Hilbert space and $M(A)$ means the multiplier algebra of $A$.

In this note, we treat the tensor products of $C^*$-algebras. From the dimension theory $\dim(X \times Y) \leq \dim X + \dim Y$ it is natural to conjecture that $RR(A \otimes B) \leq RR(A) + RR(B)$. We prove, however, it is false generally. That is, let $A$ be a unital $C^*$-algebra with non-trivial $K_1$-group of $A(= K_1(A))$, then $RR(A \otimes B(H)) \neq 0$, where $B(H)$ denotes the algebra of all bounded operators on some separable infinite-dimensional Hilbert space $H$. Therefore, if $B$ is one of the Bunce-Dedens algebras, we know $RR(B \otimes B(H)) \neq 0$, and this is a counterexample because it is known that $RR(B) = 0$, $K_1(B) = \mathbb{Z}$ [1][2], and $RR(B(H)) = 0$ [4]. Throughout this note tensor products of $C^*$-algebras mean the minimal tensor products.

We refer the reader to [3][4][6][8][9][11][13][14] for results about the real rank.

2. Result

We recall that $C^*$-algebra $A$ is exact if

$$0 \to A \otimes K \to A \otimes B(H) \to A \otimes B(H)/K \to 0$$

is an exact sequence [5].

**Proposition.** Let $A$ be a unital exact $C^*$-algebra with $K_1(A) \neq 0$. Then $RR(A \otimes B(H)) \neq 0$.

**Proof.** By the six-term exact sequence from $K$-Theory [1],

$$
\begin{array}{cccccc}
K_0(A \otimes K) & \longrightarrow & K_0(A \otimes B(H)) & \longrightarrow & K_0(A \otimes B(H)/K) \\
\uparrow & & \pi_* & & \downarrow \delta \\
K_1(A \otimes B(H)/K) & \longleftarrow & K_1(A \otimes B(H)) & \longleftarrow & K_1(A \otimes K)
\end{array}
$$

Since $K_1(A \otimes B(H)) = 0$ (see [7, Theorem 2.5]), $\pi_*$ is not surjective. For, if $\pi_*$ is surjective, $\text{Ker} \delta = K_0(A \otimes B(H)/K)$, and $\delta = 0$. Since $\text{Ker} \iota_* = \text{Im} \partial$, we know $\iota_*$ is injective, and $K_1(A \otimes B(H)) \neq 0$. This is a contradiction.

Hence, we know there is a projection in $A \otimes B(H)/K \otimes K$ which cannot be lifted to a projection in $A \otimes B(H) \otimes K$.

Consider the following $C^*$-exact sequence:

$$0 \to A \otimes K \otimes K \to A \otimes B(H) \otimes K \to A \otimes B(H)/K \otimes K \to 0.$$ Even if $RR(A \otimes K) = RR(A \otimes B(H)/K) = 0$, by [4, Theorem 3.14] (cf. [13, Proposition 2.3]) and the above argument, $RR(A \otimes B(H) \otimes K) \neq 0$ and $RR(A \otimes B(H)) \neq 0$ (cf. [4, Corollary 3.3]). Otherwise, it is trivially $RR(A \otimes B(H)) \neq 0$, and the proof is completed. \(\Box\)

The next result means that the real rank of tensor products of $C^*$-algebras with real rank zero is not always zero.
Corollary. Let $B$ be one of the Bunce-Deddens algebras. We have, then,

$$RR(B \otimes B(H)) \neq 0.$$ 

Proof. Since the Bunce-Deddens algebras are nuclear, they are exact [5]. By [1][2], we know $RR(B) = 0$ and $K_1(B) = \mathbb{Z}$. □

3. Remarks

(1) Using the idea in Proposition we can produce another example which does not satisfy the conjecture described in the introduction.

Let $B$ be one of the Bunce-Deddens algebras and $O_n$ be the Cuntz algebra. By the Küneth Theorem [12, Theorem 2.14], we know $K_1(B \otimes O_n) = \mathbb{Z}/(n-1)\mathbb{Z}$. As in the same argument $RR(B \otimes M(O_n \otimes K)) \neq 0$. On the other hand, $O_n$ is a purely infinite simple $C^*$-algebra and $K_1(O_n) = 0$. We know $RR(M(O_n \otimes K)) = 0$ by Zhang [14, Examples 2.7(i)].

(2) As Brown and Pedersen pointed out in [4], a $C^*$-algebra has real rank zero if and only if it has the (FS)-property, that is, the set of its all selfadjoint elements has a dense set of elements with finite spectrum. Therefore, Corollary means that (FS)-property is not stable under taking tensor products.

References

9. , Real rank of crossed products by connected compact groups, preprint.

Department of Mathematics, College of Science, Ryukyu University, Okinawa 903-01 Japan

E-mail address, K. Kodaka: b985562@sci.u-ryukyu.ac.jp
E-mail address, H. Osaka: osaka@sci.u-ryukyu.ac.jp