A MEASURE WITH A LARGE SET OF TANGENT MEASURES

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Abstract. There exists a Borel regular, finite, non-zero measure \( \mu \) on \( \mathbb{R}^d \) such that for \( \mu \)-a.e. \( x \) the set of tangent measures of \( \mu \) at \( x \) consists of all non-zero, Borel regular, locally finite measures on \( \mathbb{R}^d \).

Introduction

Tangent measures were introduced in [1] in order to investigate the local behaviour of measures. The main advantage of tangent measures is that they often possess more regularity than the original measure and thus a wider range of analytical techniques may be used upon them. The object of this note is to show that in general this does not necessarily hold.

Let \( \mathcal{M} \) be the set of all Borel regular, locally finite, measures on \( \mathbb{R}^d \). A sequence \( (\mu_k) \) of measures in \( \mathcal{M} \) converges to \( \mu \) in \( \mathcal{M} \) if \( \int f \, d\mu_k \to \int f \, d\mu \) as \( k \to \infty \) for all continuous functions \( f \) with bounded support. This is equivalent to requiring that \( \int g \, d\mu_k \to \int g \, d\mu \) as \( k \to \infty \) for all nonnegative functions \( g \) with Lipschitz constant less than or equal to 1 and bounded support.

\( \mathcal{M} \) together with this notion of convergence is metrisable and the resulting space is both complete and separable. For further information about these results see either [1] or [2].

For \( \mu \in \mathcal{M} \), \( x \in \mathbb{R}^d \) and \( r > 0 \) define for \( E \subset \mathbb{R}^d \)

\[
\mu_{x \cdot r}(E) := \mu(x + rE) := \mu(\{x + re : e \in E\}).
\]

Suppose that \( \mu \in \mathcal{M} \) and \( x \in \mathbb{R}^d \). A measure \( \nu \in \mathcal{M} \) is said to be a tangent measure of \( \mu \) at \( x \) if \( \nu \) is not the zero measure (denoted by \( 0 \)) and there exist sequences \( r_k \downarrow 0 \) and \( c_k > 0 \) such that

\[
c_k \mu_{x \cdot r_k} \to \nu \quad \text{as } k \to \infty.
\]

The set of all tangent measures to \( \mu \) at \( x \) will be denoted by \( \text{Tan}(\mu, x) \).

\( \text{Tan}(\mu, x) \) has the following properties:

1. \( c\nu \in \text{Tan}(\mu, x) \) whenever \( \nu \in \text{Tan}(\mu, x) \) and \( c > 0 \).
2. \( \nu_0, r \in \text{Tan}(\mu, x) \) whenever \( \nu \in \text{Tan}(\mu, x) \) and \( r > 0 \).

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3. \( \text{Tan}(\mu, x) \) is a closed set with respect to the space of all non-zero, Borel regular, locally finite measures.

As a direct consequence we have:

**Lemma 1.** If \( \mathcal{N} \subset \text{Tan}(\mu, x) \), then \( \bigcup_{r,s>0} r\mathcal{N}_{0,s} \subset \text{Tan}(\mu, x) \) where \( r\mathcal{N}_{0,s} := \{rv_0,s : v \in \mathcal{N}\} \).

**Lemma 2.** If \( \mathcal{N} \subset \text{Tan}(\mu, x) \) and \( \mathcal{N} \) is dense in \( \mathcal{M} \), then \( \text{Tan}(\mu, x) = \mathcal{M}\setminus\{0\} \).

**Construction of the measure**

**Theorem 3.** There exists a non-zero measure \( \mu \in \mathcal{M} \) such that for \( \mu \)-a.e. \( x \), \( \text{Tan}(\mu, x) = \mathcal{M}\setminus\{0\} \).

**Proof.** First let us define for \( x \in \mathbb{R}^d \) the Dirac measure at \( x \) as follows

\[
\delta_x(E) := \begin{cases} 
1 & \text{if } x \in E, \\
0 & \text{otherwise.}
\end{cases}
\]

Additionally let \( \mathbb{Q}^+ \) denote the positive rationals and \( \mathbb{Q}^d \) denote the rational \( d \)-tuples, that is, \( d \)-tuples whose coordinates are all rational numbers. We have that

\[
\mathcal{S} = \left\{ \alpha_0\delta_0 + \sum_{i=1}^{n-1} \alpha_i\delta_{x_i} : n \in \{2, 3, \ldots\}, \alpha_i \in \mathbb{Q}^+, x_i \in \mathbb{Q}^d, |x_i| \leq 1 \text{ for } i \in \{0, \ldots, n-1\} \text{ and } \sum_{i=0}^{n-1} \alpha_i = 1 \text{ and } i \neq j \Rightarrow x_i \neq x_j \right\}
\]

is a countable set, and if \( \nu \in \mathcal{S} \), then it is a probability measure with support in \( B(0, 1) \) (the closed ball with centre the origin and radius 1). Moreover

\[
\bigcup_{p,q \in \mathbb{Q}^+} p\mathcal{S}_{0,q}
\]

is a countable set which is dense in \( \mathcal{M} \). Thus by Lemmas 1 and 2 it suffices to construct a measure \( \mu \) such that \( \text{Tan}(\mu, x) \supset \mathcal{S} \) for \( \mu \)-a.e. \( x \).

Let \( (\mu_k)_{k=1}^\infty \) be a sequence of elements of \( \mathcal{S} \) such that every element of \( \mathcal{S} \) occurs infinitely many times in this sequence. Thus each \( \mu_k \) is of the form

\[
\mu_k = \alpha(k,0)\delta_0 + \sum_{i=1}^{n_k-1} \alpha(k,i)\delta_{x(k,i)}
\]

where the \( \alpha(k,i) \), \( x(k,i) \) fulfill the appropriate conditions of \( \mathcal{S} \) (in particular \( x(k,0) = 0 \)). For each \( \mu_k \) define

\[
\sigma_k = \min_{0 \leq i,j \leq n-1} \{|x(k,i) - x(k,j)| : i \neq j\}.
\]

From this define an increasing sequence of real numbers \( (r_k) \) by setting \( r_1 = 8 \) and choosing \( r_{k+1} > 8^{k+2}r_k/\sigma_k \).

Let \( \Sigma := \prod_{k=1}^\infty \{0, \ldots, n_k - 1\} \), and let \( P \) be the probability measure on \( \Sigma \) obtained by setting

\[
P(\eta_j) := \prod_{k=1}^j \alpha_{k,\eta_k}
\]
where \( \eta_j := (\eta_1, \ldots, \eta_j) \times \prod_{k=j+1}^{\infty} \{0, \ldots, n_k - 1\} \). Define \( \pi: \Sigma \to B(0, 1) \) by

\[
\pi(\eta) := \sum_{k=1}^{\infty} (r_k)^{-1} x(k, \eta_k).
\]

Notice that \( \pi \) is a well-defined 1-1 map. Set \( \mu := \pi_* P \), that is, for \( E \subset \mathbb{R}^d \) define

\[
\mu(E) := P[\pi^{-1}(E)].
\]

I claim that \( \mu \) is our required measure. The Borel regularity of \( \mu \) follows from the continuity of the mapping \( \pi \) with respect to the product topology on \( \Sigma \).

**Lemma 4.** For a given \( \nu \in \mathcal{S} \), let \( (v_i)_{i=1}^{\infty} \) be a strictly increasing sequence such that \( \mu_{v_i} = \nu \) for all \( i \). Let

\[
V_{\nu} = \{ \eta \in \Sigma: \eta_{v(i)} = 0 \ i.o. \}.
\]

Then \( P(V_{\nu}) = 1 \) and so \( \mu[\pi(V_{\nu})] = 1 \).

**Proof.** We have that for all \( i \)

\[
P(\eta_{v(i)} = 0) = \alpha(v(i), 0) = \alpha > 0;
\]

therefore \( \sum P(\eta_{v(i)} = 0) = \infty \) and so, by the Borel-Cantelli lemma and independence, the lemma follows. \( \square \)

Let \( V = \bigcap_{\nu \in \mathcal{S}} V_{\nu} \). Then as \( \mathcal{S} \) is countable \( P(V) = 1 \) and so \( \mu[\pi(V)] = 1 \).

For \( x \in \pi(\Sigma) \) define \( x_i := x(i, \lfloor x^{-1}(x)_i \rfloor) \) and so \( x = \sum_{i=1}^{\infty} x_i/r_i \). Let \( \bar{x} \in \pi(V) \), and let \( \bar{\eta} \) be the associated element of \( V \). Fix \( \nu \in \mathcal{S} \), and define \( (v_i)_{i=1}^{\infty} \) as in the lemma (so \( \mu_{v(i)} = \nu \)). Then, as \( \bar{\eta} \in V \), there is an infinite set \( N \subset \bigcup_{i=1}^{\infty} \{v_i\} \) such that for all \( k \in N \), \( \bar{x}_k = 0 \) and \( \mu_k = \nu \).

We wish to show that \( \nu \in \Tan(\mu, \bar{x}) \). So we need to find sequences \( c_j > 0 \) and \( s_j \searrow 0 \) such that \( c_j \mu_{\bar{x}, s_j} \to \nu \) as \( j \to \infty \).

Let \( s_j = 1/r_k(j) \) where \( k(j) \) is the \( j \)th element of \( N \) and so \( s_j \searrow 0 \).

Define

\[
c_j = \left[ \mu \{ x \in \pi(\Sigma): x_i = \bar{x}_i \ for \ i = 1, \ldots, k(j) - 1 \} \right]^{-1}.
\]

By the equivalence from the introduction, \( \phi_k \to \phi \iff \int g \, d\phi_k \to \int g \, d\phi \) where \( \text{Lip}(g) \leq 1 \) and \( \text{spt}(g) \) is bounded and \( g \) is nonnegative. So fix such a \( g \) and suppose \( \text{spt}(g) \subset B(0, R) \) for some \( R \geq 0 \).

Choose \( J \in \mathbb{N} \) such that \( \frac{28}{28}^{k(j)} > R \). For \( j \geq J \) we have (letting \( k := k(j) \))

\[
\int_{\mathbb{R}^d} g \, d(c_j \mu_{\bar{x}, s_j}) = c_j \int_{\mathbb{R}^d} g(r_{k(j)}(x - \bar{x})) \, d\mu(x)
\]

\[
= c_j \int_{\pi(\Sigma)} g \left( \sum_{k=1}^{\infty} \frac{x_i - \bar{x}_i}{r_i} \right) \, d\mu(x).
\]

Let us consider \( r_k \sum_{i=1}^{\infty} \frac{x_i - \bar{x}_i}{r_i} \) in more detail. There are two possible cases:

**Case 1.** \( x_i = \bar{x}_i \) for \( i = 1, \ldots, k - 1 \). Then since

\[
\sum_{i=1}^{\infty} \frac{x_i - \bar{x}_i}{r_i} = \bar{x}_k - \bar{x}_{k+1} + \sum_{i=k+1}^{\infty} \frac{x_i - \bar{x}_i}{r_i}
\]

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and as
\[ r_k \sum_{i=k+1}^\infty \frac{x_i - \bar{x}_i}{r_i} \leq \frac{2}{7} 8^{-k}, \]
we have (as \( \bar{x}_k = 0 \))
\[ x_k - r_k \sum_{i=1}^\infty \frac{x_i - \bar{x}_i}{r_i} \leq \frac{2}{7} 8^{-k}. \]

Case 2. There exists \( u \in \{1, \ldots, k-1\} \) such that \( x_i = \bar{x}_i \) for \( i = 1, \ldots, u-1 \) but \( x_u \neq \bar{x}_u \). Thus
\[ \sum_{i=1}^\infty \frac{x_i - \bar{x}_i}{r_i} = \frac{x_u - \bar{x}_u}{r_u} + \sum_{i=u+1}^\infty \frac{x_i - \bar{x}_i}{r_i} \]
and both
\[ \left| \sum_{i=u+1}^\infty \frac{x_i - \bar{x}_i}{r_i} \right| \leq \frac{\sigma_u}{r_u} 8^{-k} \quad \text{and} \quad \left| \frac{x_u - \bar{x}_u}{r_u} \right| \geq \frac{\sigma_u}{r_u}; \]
therefore
\[ \left| r_k \sum_{i=1}^\infty \frac{x_i - \bar{x}_i}{r_i} \right| \geq \frac{27}{28} \frac{r_k}{\sigma_u} > \frac{27}{28} 8^k > R. \]
Thus in Case 2, \( g[r_k(x - \bar{x})] = 0 \) and so
\[ c_j \int_{\pi(\Sigma)} g[r_k(x - \bar{x})] d\mu(x) = c_j \int_X g[r_k(x - \bar{x})] d\mu(x) \]
where \( X = \{x \in \pi(\Sigma): x_i = \bar{x}_i \text{ for } i = 1, \ldots, k-1 \} \). Notice that \( c_j = [\mu(X)]^{-1} \). As \( \text{Lip}(g) \leq 1 \), we have by Case 1 that for \( x \in X \)
\[ |g[r_k(x - \bar{x})] - g(x_k)| \leq \frac{2}{7} 8^{-k}. \]
Thus integrating over \( X \) and multiplying by \( c_j \) gives
\[ \left| c_j \int_{\pi(\Sigma)} g[r_k(x - \bar{x})] d\mu(x) - \frac{1}{\mu(X)} \int_X g(x_k) d\mu(x) \right| \leq \frac{2}{7} 8^{-k}, \]
but by independence,
\[ \int_X g(x_k) d\mu(x) = \mu(X) \int_{\pi(\Sigma)} g(x_k) d\mu(x) = \mu(X) \int_{R^d} g(x) d\nu(x) \]
and so the theorem follows. \( \square \)

References


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