OPTIMAL INTERVALS OF STABILITY
OF A FORCED OSCILLATOR

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Abstract. Consider the differential equation of a nonlinear oscillator with linear friction and a $T$-periodic external force. We find optimal bounds on the derivative of the restoring force and on the period $T$ in order to obtain a unique $T$-periodic solution that is asymptotically stable.

1. Introduction

The purpose of this paper is to complete the results obtained in [7] and [1]. Consider the differential equation

\[ x'' + cx' + g(x) = p(t) \]

where $c > 0$ is a fixed constant, $p \in C(\mathbb{R}/T\mathbb{Z})$ and $g \in C^1(\mathbb{R})$ satisfies

\[ a \leq g'(x) \leq b \quad \text{for each } x \in \mathbb{R} \]

with $a \geq 0$. If $a = 0$ we also need the additional assumption

\[ g(-\infty) < \frac{1}{T} \int_0^T p(t) \, dt < g(+\infty). \]

Recently, in [7], Ortega has studied the case $a = 0$ and obtained sharp conditions on $b$ for the existence, uniqueness and stability of a $T$-periodic solution of (1.1). In fact he has proved that there exists $b_0$, that can be computed, such that if $b \leq b_0$, then (1.1) has a unique $T$-periodic solution that is (locally) asymptotically stable when (1.2) holds. Moreover there exists $\tau(b) > 0$ such that if $b > b_0$ but $T \leq \tau(b)$ the above assertion is still true, while if $T > \tau(b)$, then instability will appear for some $p$ satisfying (1.2).

The case $a > 0$ was considered in [3] and [4] and sufficient conditions on $a$ and $b$ were obtained for the same problem. More recently, in [1], Ortega and the author have also studied this case and found sharp conditions that guarantee global asymptotic stability (g.a.s.) (and independent from the period $T$). In fact we have defined two functions, $A$ and $B$, and we have proved that if $A[a]B[b] < 1$, then there exists a unique $T$-periodic solution of (1.1) that is g.a.s., while if $A[a]B[b] > 1$, then one can find a periodic function $p$ (for suitable period) such that (1.1) has an unstable periodic solution.

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In this paper we shall obtain sharp conditions for a fixed period $T$ when $A[a]B[b] \geq 1$. In fact we shall find a sequence of intervals that can be computed such that if $T$ is in one of these intervals, then (1.1) has a unique $T$-periodic solution that is asymptotically stable. In general this sequence is finite and has several disjoint intervals (see Figure 1). This is different from the case $a = 0$ where only one interval appears.

Thanks to the principle of linearized stability the nonlinear problem can be reduced to the following problem in linear theory: to determine under what conditions on $a$ and $b$ the linear equation
\[
y'' + cy' + \alpha(t)y = 0 \quad (a \leq \alpha(t) \leq b, \text{ a.e. } t \in \mathbb{R})
\]
does not have nontrivial $2T$-periodic solutions. To deal with this question we shall use a technique based in control theory as used by Brockett in [2] and by Ortega in [7].

The main results on (1.1) are stated in Section 2. The linear equation is studied in Sections 3 and 4. The proofs of the main results appear in Section 5.

I would like to thank Professor Ortega for encouraging me to study this problem and for his suggestion to elaborate this paper.

2. The main theorems

Consider the equation
\[
(2.1) \quad x'' + cx' + g(x) = p(t)
\]
where $c > 0$ is a fixed constant, $g \in C^1(\mathbb{R})$ and $p \in C(\mathbb{R}/T\mathbb{Z})$. It will be assumed that there exist positive constants $a$ and $b$ such that
\[
(2.2) \quad a \leq g'(x) \leq b \quad \forall x \in \mathbb{R}.
\]

Consider the functions
\[
A[k] = \begin{cases} 
  k^{-1/2} \exp\{-c\omega_k^{-1}\tanh^{-1}(\omega_k/c)\} & \text{if } 0 < k < c^2/4, \\
  2c^{-1/2}e & \text{if } k = c^2/4, \\
  k^{-1/2} \exp\{-c\omega_k^{-1}\arctan(\omega_k/c)\} & \text{if } k > c^2/4,
\end{cases}
\]
and
\[
B[k] = \begin{cases} 
  0 & \text{if } 0 < k \leq c^2/4, \\
  k^{1/2} \exp\{c\omega_k^{-1}[\arctan(\omega_k/c) - \pi]\} & \text{if } k > c^2/4,
\end{cases}
\]
where $\omega_k = \sqrt{4k - c^2}$. Here $\arctan: (-\infty, +\infty) \rightarrow (-\pi/2, \pi/2)$. As a consequence of Theorem 1.2 in [1] we have that if $A[a]B[b] < 1$, then (2.1) has a unique $T$-periodic solution that is globally asymptotically stable. Therefore we are interested in what happens when $A[a]B[b] \geq 1$.

**Theorem 2.1.** Suppose that $A[a]B[b] \geq 1$ and (2.2) holds. There exist $\tau_1 = \tau_1(a, b)$ and $\tau_2 = \tau_2(a, b)$ such that if
\[
(2.3) \quad T \notin [\tau_1, \tau_2] \quad \forall n \in \mathbb{N}
\]
holds, then (2.1) has a unique $T$-periodic solution that is locally asymptotically stable.

**Remark.** We shall see below that the constants $\tau_1$ and $\tau_2$ can be computed. In particular if $A[a]B[b] = 1$, then $\tau_1 = \tau_2$ and therefore the intervals of stability
are the infinite components of \( \{ t > 0 \mid t \neq n \tau_1, n \in \mathbb{N} \} \). On the other hand, if \( A[a]B[b] > 1 \) we have that \( \tau_1 < \tau_2 \) and hence there must exist some \( n \in \mathbb{N} \) such that

\[
\frac{n + 1}{n} \leq \frac{\tau_2}{\tau_1}.
\]

If \( n_0 \in \mathbb{N} \) is the first number satisfying (2.4), then there exist exactly \( n_0 \) intervals of stability for the equation (2.1) (see Figure 1).

In the next result we shall show that (2.3) is sharp. We shall need the additional assumptions

\[
\inf \{ g'(x) : x \in \mathbb{R} \} = a, \quad \sup \{ g'(x) : x \in \mathbb{R} \} = b.
\]

**Theorem 2.2.** Suppose that (2.5) holds. If there exists \( n \in \mathbb{N} \) such that

\[
T \in n(\tau_1, \tau_2),
\]

then there exists some \( p \in C(\mathbb{R}/\mathbb{Z}) \) such that (2.1) has an unstable \( T \)-periodic solution.

Now we shall say how one can calculate the constants, but first we introduce some notation. If \( k > 0 \) define

\[
\xi_k = \begin{cases} \frac{2\pi}{\omega_k}, & \text{if } k > \frac{c^2}{4}, \\ +\infty, & \text{if } k \leq \frac{c^2}{4}, \end{cases}
\]

where \( \omega_k = \sqrt{4k - c^2} \). Notice that \( \xi_k \) is the first positive zero of any nontrivial function \( \phi \) satisfying \( \phi'' + c\phi' + k\phi = 0 \), \( \phi(0) = 0 \). Suppose that \( b > \frac{c^2}{4} \) and define the switching function : \( \gamma \in L^\infty(\mathbb{R}) \) as follows:

\[
\gamma(t) = \begin{cases} a, & \text{if } t \leq -\xi_b, \\ b, & \text{if } -\xi_b < t < 0, \\ a, & \text{if } 0 \leq t. \end{cases}
\]

For each \( s \in [0, \xi_b + \xi_a) \) consider the initial value problem

\[
(P_s) \begin{cases} y''(t) + cy'(t) + \gamma(t-s)y(t) = 0, \\ y(0) = 0, \ y'(0) = 1. \end{cases}
\]

When the solution \( y_s(t) \) of \((P_s)\) vanishes for some \( t > 0 \), we define \( T(s) \) as the first positive zero of \( y_s(t) \) and define \( R(s) = y_s'(T(s)) \) where \( y_s'(t) \) is the derivative of \( y_s(t) \) with respect to \( t \). In the equation the function \( \alpha(t) = \gamma(t-s) \) is a piece-wise constant function that at most has one switch in the interval \((0, T(s))\). This equation is similar to the Meissner equation studied for example in [6, p. 115] with \( c = 0 \).

The problem \((P_s)\) can be integrated and one can obtain explicitly the functions \( T \) and \( R \). In particular these functions are defined in an interval \( I \subset \)
Figure 2. The functions $T$ and $R$ when $a > c^2/4$

$[0, \xi_b + \xi_a)$ such that $(\xi_b, \xi_b + \xi_a) \subset I$. Figure 2 shows the graph of these functions.

The explicit expression of $T(s)$ and $R(s)$ and its properties are given in an appendix at the end of the paper. Then one can compute $\tau_1 > 0$ solving

$$(2.7) \quad \tau_1 = T(\bar{s}_1), \quad R(\bar{s}_1) = -1, \quad \bar{s}_1 \in (\xi_b, \sigma],$$

and $\tau_2 > 0$ solving

$$(2.8) \quad \tau_2 = T(\bar{s}_2), \quad R(\bar{s}_2) = -1, \quad \bar{s}_2 \in [\sigma, \xi_b + \xi_a),$$

where $\sigma$ is the point where $R$ has its minimum. The solvability and uniqueness of these equations follows from Proposition 1 in the appendix.

3. A PROBLEM IN CONTROL THEORY

Let $0 < a \leq b$ be such that $b > c^2/4$. For $\lambda > 0$ consider the problem

$$(P_\lambda) \begin{cases} y'' + cy' + \alpha(t)y = 0, \\ y(0) = y(L) = 0, \quad y'(0) = 1, \quad y'(L) = -\lambda, \\ y(t) \neq 0 \quad \forall t \in (0, L), \end{cases}$$

where $L > 0$ and $\alpha \in L^\infty(0, L)$ is such that

$$(3.1) \quad a \leq \alpha(t) \leq b \quad \text{a.e. } t \in (0, L).$$

Consider the functions $T(s)$ and $R(s)$ defined on the interval $I$ as in Section 2. We have the following result:

**Lemma 3.1.** If the problem $(P_\lambda)$ has a solution, then

(i) The equation

$$(3.2) \quad R(s) = -\lambda, \quad s \in I,$$

is solvable.
(ii) If \( s_1, s_2 \) are the solutions of (3.2) and \( T(s_1) \leq T(s_2) \), then \( L \in [T(s_1), T(s_2)] \).

**Proof.** We shall use the language and methods of control theory (see for example [5]). Consider the control process

\[
x' = C[u]x, \quad x = \text{col}(x_1, x_2), \quad C[u] = \begin{pmatrix} 0 & 1 \\ -u & -c \end{pmatrix}.
\]

The class of admissible controllers is \( U = \{ u \in L^\infty(0, r) | r > 0, a \leq u(t) \leq b \} \) a.e. \( t \in (0, r) \), the initial point is \( X_0 = \text{col}(0, 1) \) and the target set is \( K = \{(0, d) | d < 0\} \). For each \( u \in U \) attaining the target set denote by \( \tau(u) \) the first positive zero of \( x_1(t) \), where \( x = \text{col}(x_1, x_2) \) is the corresponding response. Consider the set \( V = \{ u \in U | x_2(\tau(u)) = -\lambda \} \). Note that the lemma will be proved if there exist \( s_1, s_2 \in I \) such that \( T(V) \subset [T(s_1), T(s_2)] \) and \( R(s_1) = R(s_2) = -\lambda \).

To prove this consider for each \( n \in \mathbb{N} \) the cost functional

\[
F_n[u] = \tau(u) + n(x_2(\tau(u)) + \lambda)^2.
\]

Note that \( \alpha \in V \) and that if \( v \in V \), then \( F_n[v] = \tau(v) \). Let \( u^*_n \) be an optimal control minimizing \( F_n \) with optimal response \( x^*_n = \text{col}(x^*_n, x^*_n) \). Its existence follows for example from Theorem 4 in [5, p. 259]. Consider the Hamiltonian function \( H(\eta, x, u) = \eta \cdot C[u]x = (\eta_1 - c\eta_2)x_2 - u\eta_2x_1 \), and the function \( M(\eta, x) = \max\{H(\eta, x, u) | a \leq u \leq b\} = (\eta_1 - c\eta_2)x_2 + b(\eta_2x_1)^- - a(\eta_2x_1)^+ \). The maximal principle of Pontryagin says that there exists \( \bar{\eta} = \text{col}(\bar{\eta}_1, \bar{\eta}_2) \) a nontrivial solution of \( \eta' = -C[u^*_n]^\top \eta \) such that \( H(\bar{\eta}, x^*_n, u^*_n) = M(\bar{\eta}, x^*_n) \) a.e. \( t \in (0, \tau(u^*_n)) \). In consequence

\[
u^*_n(t) = \begin{cases} a & \text{if } \bar{\eta}_2(t) > 0, \\
b & \text{if } \bar{\eta}_2(t) < 0.
\end{cases}
\]

Since \( \bar{\eta}_2(t) \) and \( x_1^*(t) \) are solutions of adjoint equations and \( x_1^*(t) \neq 0 \) \( \forall t \in (0, \tau(u^*_n)) \) it follows that \( \bar{\eta}_2(t) \) has at most one zero in \( (0, \tau(u^*_n)) \). Therefore \( u^*_n \) has at most one jump in \( (0, \tau(u^*_n)) \) and only takes the values \( a \) and \( b \). Hence \( u^*_n \) is a switching function as considered in Section 2 and there must exist \( s^*_n \) such that \( u^*_n(t) = \gamma(t - s^*_n) \) a.e. \( t \in (0, \tau(u^*_n)) \). Moreover \( \tau(u^*_n) = T(s^*_n) \) and \( x_2^*(\tau(u^*_n)) = R(s^*_n) \).

The sequence \( \{s^*_n\} \) is bounded. This follows from

\[
T(s^*_n) = \tau(u^*_n) \leq F_n[u^*_n] \leq F_n[\alpha] = \tau(\alpha) = L
\]

and the properties of \( T(s) \) (see Proposition 1(iii) in the appendix). Hence we can suppose that \( \{s^*_n\} \) converge (in the other case one can take a convergent subsequence). If \( a \leq c^2/4 \), then (3.3) implies that the limit of \( \{s^*_n\} \) is in \( I \). If \( a > c^2/4 \), then \( I = [0, \xi_b + \xi_a] \) but \( T(0) = T(\xi_b + \xi_a) \) and \( R(0) = R(\xi_b + \xi_a) \). Therefore in both cases there exists \( s_1 \in I \) such that \( T(s^*_n) \to T(s_1) \) and \( R(s^*_n) \to R(s_1) \). Let \( x^* = \text{col}(x^*_1, x^*_2) \) be the solution of \( x' = C[\gamma(t - s_1)]x \), \( x(0) = \text{col}(0, 1) \). Note that \( T(s_1) = \tau(\gamma(-s_1)) \) and \( R(s_1) = x_2^*(T(s_1)) \).

We assert that \( R(s_1) = -\lambda \). If the claim is not true there must exist a subsequence such that \( (R(s^*_n) + \lambda) \to r \neq 0 \). But then \( F_n[u^*_n] = T(s^*_n) + n_k(R(s^*_n) + \lambda)^2 \to +\infty \) contradicting that \( u^*_n \) is the minimal optimal control for \( F_n \) since \( F_n[\alpha] = L < +\infty \).
Finally, if \( v \in V \), then \( \tau(v) \geq T(s_1) \). In the other case, since \( T(s_n^*) \to T(s_1) \), for \( n \) sufficiently large we should have \( F_n[v] = \tau(v) \leq T(s_n^*) \leq F_n[u_n^*] \) contradicting again the optimality of \( u_n^* \).

To calculate \( s_2 \) we can use the same reasoning minimizing the cost functional
\[
G_n[u] = e^{-\tau(u)} + n(x_2(\tau(u)) + \lambda)^2
\]
for \( n \in \mathbb{N} \) such that \( 1 + n\lambda^2 > e^{-L} \).

For the next result suppose that \( A[a]B[b] \geq 1 \). Then as consequence of Proposition 1 in the appendix there exist unique \( \tau_1 > 0 \) and \( \tau_2 > 0 \) satisfying (2.7) and (2.8).

**Lemma 3.2.** Suppose \( A[a]B[b] \geq 1 \) and let \( \tau_1, \tau_2 \) be as in (2.7) and (2.8).

(i) If the problem \((P_1)\) has a solution, then \( L \in [\tau_1, \tau_2] \).

(ii) If \( L \in [\tau_1, \tau_2] \), then there exists \( \hat{\alpha} \in L^\infty(0, L) \) satisfying (3.1) such that the corresponding problem \((\tilde{P}_1)\) has a solution.

**Proof.** (i) This is a particular case of Lemma 3.1.

(ii) If \( L \in [\tau_1, \tau_2] \), there exists \( \tilde{s} \in [\tilde{s}_1, \tilde{s}_2] \) such that \( T(\tilde{s}) = \tilde{L} \). Then consider \( \tilde{\alpha}(t) = \gamma(t - \tilde{s}) \) and the problem
\[
\begin{cases}
  y'' + cy' + \tilde{\alpha}(t)y = 0, \\
  y(0) = y(\tilde{L}) = 0, \quad y'(0) = 1, \quad y'(\tilde{L}) = -1, \\
  y(t) \not\equiv 0 \quad \text{in } (0, \tilde{L})
\end{cases}
\]
has a solution.

**Corollary 3.3.** With the same conditions as in the previous lemma, if \( T \in (\tau_1, \tau_2) \), then there exists \( \beta \in L^\infty(\mathbb{R}/T\mathbb{Z}) \) satisfying (3.1) such that the equation
\[
y'' + cy' + \beta(t)y = 0
\]
is unstable.

**Proof.** If \( T \in (\tau_1, \tau_2) \), then \( A[a]B[b] > 1 \) and there exists \( s \in (\tilde{s}_1, \tilde{s}_2) \) such that \( T(s) = T \). Consider \( \beta \in L^\infty(\mathbb{R}/T\mathbb{Z}) \) such that \( \beta(t) = \gamma(t - s) \) \( \forall t \in (0, T) \). The solution \( y(t) \) of (3.4) satisfies \( y(t + T) = R(s)y(t) \) \( \forall t \in \mathbb{R} \). Then \( R(s) < -1 \) is a Floquet multiplier of (3.4) and thus is unstable.

4. **The linear equation**

Let \( 0 < a \leq b \) be such that \( A[a]B[b] \geq 1 \). Consider the linear equation
\[
y'' + cy' + \alpha(t)y = 0
\]
where \( \alpha \in L^\infty(\mathbb{R}/T\mathbb{Z}) \) satisfies
\[
a \leq \alpha(t) \leq b \quad \text{a.e. } t \in \mathbb{R}.
\]

**Proposition 4.1.** Let \( \tau_1 \leq \tau_2 \) be as in (2.7) and (2.8) respectively. If
\[
T \not\in n[\tau_1, \tau_2] \quad \forall n \in \mathbb{N},
\]
then there does not exist any nontrivial 2T-periodic solution of (4.1).

**Proof.** Suppose, contrary to the assertion of the proposition, that there exists a nontrivial 2T-periodic solution of (4.1). If \( \phi(t) \) is such a solution, then it must
vanish (to see this integrate (4.1) over a period). Let $t_0$ be such that $\phi(t_0) = 0$. Since the zeros of $\phi$ are simple and $\phi$ is periodic, the number of zeros in the interval $[t_0, t_0 + 2T]$ is even. Let $2n$ ($n \in \mathbb{N}$) be the number of zeros of $\phi$ in $[t_0, t_0 + 2T]$.

We shall again use the language of control theory, but now we use polar coordinates: $y_1(t) = \rho(t) \sin \theta(t)$, $y_2(t) = \rho(t) \cos \theta(t)$. Consider the control process

$$\begin{cases} \theta' = \cos^2 \theta + u \sin^2 \theta + c \sin \theta \cos \theta, \\ \rho' = \rho((1 - u) \sin \theta \cos \theta - c \cos^2 \theta). \end{cases} \tag{4.4}$$

The class of admissible controllers is $U = \{u \in L^\infty(t_0, r) | r > t_0, a \leq u(t) \leq b \text{ a.e. } t \in (t_0, r)\}$, the initial point is $X_0 = \text{col}(0, 1)$ and the target state is $X_1 = \text{col}(2n\pi, 1)$. Consider the control problems of minimal and maximal time to attain the target state. Note that if $\text{col}(\theta(t), \rho(t))$ is a solution of (4.4), then $\gamma(t) = \rho(t) \sin \theta(t)$ is a solution of (4.1). Also note that $\theta(t)$ is increasing when $\theta(t) = j\pi$, $j \in \mathbb{Z}$. Let $u^*$ and $\text{col}(\rho^*, \lambda^*)$ be a minimal optimal control and the corresponding response. Let

$$t_0 < t_1 < \cdots < t_{2n} = \tau^*$$

be such that $\theta^*(t_k) = k\pi$ ($k = 1, \ldots, 2n$). Observe that $\rho^*(t_0) = \rho^*(t_{2n}) = 1$. As a consequence of the principle of optimality and Lemma 3.1 there exist $s_1^* < \cdots < s_{2n}^*$ such that $R(s_k^*) = -\rho^*(t_k)/\rho^*(t_{k-1})$ and $t_k - t_{k-1} = T(s_k^*)$ for each $k = 1, \ldots, 2n$. Hence we have $\tau^* = T(s_1^*) + \cdots + T(s_{2n}^*)$ and

$$R(s_1^*) \cdots R(s_{2n}^*) = 1. \tag{4.5}$$

Thus $(s_1^*, \ldots, s_{2n}^*)$ is a solution of the problem of minimizing the function $f(s_1, \ldots, s_{2n}) = T(s_1) + \cdots + T(s_{2n})$ in the set $C = \{(s_1, \ldots, s_{2n}) | g(s_1, \ldots, s_{2n}) = 0, s_k \in I \forall k\}$ where $g(s_1, \ldots, s_{2n}) = R(s_1) \cdots R(s_{2n}) - 1$. Let $\sigma$ be as in Section 2. From (4.5) and Proposition 1(iii) in the appendix we have that $R(\sigma) \geq 1$. We distinguish two cases:

Case 1: $|R(\sigma)| = 1$. This happens when $A[a]B[b] = 1$. Furthermore $\tau_1 = T(\sigma)$ and $C = \{\sigma, \ldots, \sigma\}$. Hence $s_1^* = \cdots = s_{2n}^* = \sigma$ and $\tau^* = 2n\tau_1$.

Case 2: $|R(\sigma)| > 1$. Lemma 2 and Proposition 1(iv) in the appendix imply $s_k^* \in (\xi_b, \xi_b + \frac{a}{b})$. Now by using Lagrange’s multipliers there exists $\lambda \in \mathbb{R}$ such that

$$\nabla f(s_1^*, \ldots, s_{2n}^*) = \lambda \nabla g(s_1^*, \ldots, s_{2n}^*),$$

or equivalently

$$T'(s_1^*) = \lambda R'(s_1^*) \cdots R'(s_{2n}^*)$$

$$T'(s_{2n}^*) = \lambda R'(s_1^*) \cdots R'(s_{2n}^*).$$

As in this case $T'(\sigma) \neq 0$ and $R'(\sigma) = 0$ one has that $s_k^* \neq \sigma$ and hence $R'(s_k^*) \neq 0$ for each $k = 1, \ldots, 2n$. Thus one can divide each equation by the corresponding $R'(s_k^*)$ and multiply by $R(s_k^*)$ to obtain

$$\frac{T'(s_1^*) R(s_1^*)}{R'(s_1^*)} = \cdots = \frac{T'(s_{2n}^*) R(s_{2n}^*)}{R'(s_{2n}^*)} = \lambda$$

thanks to (4.5). Now, since the function $h(s) = T'(s)R(s)/R'(s)$ is one-to-one in $(\xi_b, \xi_b + \frac{a}{b})\{\sigma\}$, then $s_1^* = \cdots = s_{2n}^*$. Hence $R(s_1^*)^{2n} = 1$ and consequently we have that $s_1^* = \bar{s}_1$ and $\tau^* = 2n\tau_1$. 

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Therefore in both cases the minimal optimal time is $2n\tau_1$. We use the same reasoning to show that the maximal optimal time is $2n\tau_2$. But since $u = \alpha$ is a control that allows one to reach the target at the time $2T$ we have $2n\tau_1 \leq 2T \leq 2n\tau_2$ leading to $n\tau_1 \leq T \leq n\tau_2$, contradicting (4.3).

5. PROOFS OF THE MAIN THEOREMS

Now the results in Section 2 follow from standard methods:

Proof of Theorem 2.1. Repeating the reasoning of [8, p. 168] one obtains that (2.1) has at least one $T$-periodic solution. If $x_1$ and $x_2$ are $T$-periodic solutions of (2.1), then $y(t) = x_1(t) - x_2(t)$ is a $T$-periodic solution of (4.1)

$$
\alpha(t) = \begin{cases} 
\frac{g(x_1(t)) - g(x_2(t))}{x_1(t) - x_2(t)} & \text{if } x_1(t) \neq x_2(t), \\
\alpha & \text{if } x_1(t) = x_2(t),
\end{cases}
$$

that satisfies (4.2). From Proposition 4.1 one obtains that $y \equiv 0$ and hence (2.1) has a unique $T$-periodic solution. If $z(t)$ is such a solution, consider for each $s \in [0, 1]$ the linear equation

$$
y''(t) + cy'(t) + [sg'(z(t)) + a(1 - s)]y(t) = 0.
$$

Note that for $s = 1$ (5.1) is the linearized equation of (2.1) at $z(t)$. Let $\Delta[s]$ be the discriminant of (5.1), that is, the trace of a monodromy matrix. This is a continuous function of $s$ (see for instance Lemma 2.1 in [7]) and it is well known that the existence of $2T$-periodic solutions of (5.1) is equivalent to

$$
|\Delta[s]| = 1 + e^{-cT}.
$$

Therefore from Proposition 4.1 it follows that $|\Delta[s]| \neq 1 + e^{-cT}$ and since $|\Delta[0]| < 1 + e^{-cT}$ one obtains that $|\Delta[s]| < 1 + e^{-cT}$ for each $s \in [0, 1]$. Therefore the Floquet multipliers lie on the open unit disk. In particular this is true for $s = 1$ and the asymptotic stability of $z(t)$ is a consequence of the principle of linearized stability.

Proof of Theorem 2.2. Since $T/n \in (\tau_1, \tau_2)$ one can use the same argument of [7, Theorem II] together with Corollary 3.3 to state the theorem.

APPENDIX

Here we study the properties of the functions $T(s)$ and $R(s)$ defined in Section 2 related to the linear initial value problem $(P_s)$.

Proposition 1. (i) $T(s)$ and $R(s)$ are $C^1$ functions defined in an interval $I \subset [0, \xi_b + \xi_a]$ such that $(\xi_b, \xi_b + \xi_a) \subset I$.

(ii) $T(s)$ is strictly increasing in $(\xi_b, \xi_b + \xi_a)$. Moreover if $a \leq c^2/4$, then

$$
\lim_{s \to \inf I} T(s) = \lim_{s \to \sup I} T(s) = +\infty.
$$

(iii) $R'(s)$ has only one zero in $(\xi_b, \xi_b + \xi_a)$ where $R(s)$ reaches a minimum. If $\sigma$ is such a zero, then $R(\sigma) = -A[a]B[b]$.

(iv) If $s \not\in (\xi_b, \xi_b + \xi_a)$, then $|R(s)| < 1$.

(v) The equation

$$
R(s) = r, \quad s \in I,
$$

has at most two solutions for each $r \in \mathbb{R}$. 
Proof. After a simple but tedious computation one can obtain explicitly the functions $T$ and $R$, the interval $I$ and the minimum $\sigma$. We distinguish three cases. In all cases the properties follow from simple verification.

Case 1: $a > c^2/4$. Then $I = [0, \xi_b + \xi_a)$ and

$$T(s) = \begin{cases} 
  s + \frac{2}{\omega_b} (\pi - \arctan(\frac{\omega_b}{\omega_a} \tan(\frac{\omega_a}{2} s))) & \text{if } 0 \leq s < \frac{\pi}{\omega_b}, \\
  \frac{\pi}{\omega_a} + \frac{\pi}{\omega_b} & \text{if } s = \frac{\pi}{\omega_b}, \\
  s - \frac{2}{\omega_b} \arctan(\frac{\omega_a}{\omega_b} \tan(\frac{\omega_b}{2} s)) & \text{if } \frac{\pi}{\omega_b} < s \leq \xi_b,
\end{cases}$$

$$R(s) = \begin{cases} 
  \cos^2(\frac{\omega_b}{2} s)(-\sin^2(\frac{\omega_b}{2} s)e^{-\frac{s}{T(s)}}) & \text{if } 0 \leq s < \xi_b, \\
  \frac{1}{2} \sqrt{4 + \omega_b^2(s - \xi_b)^2} & \text{if } \xi_b \leq s.
\end{cases}$$

Here $R(s)$ has its minimum in $\sigma = \xi_b + 2\arctan(\omega_a/c)/\omega_a$.

Case 2: $a = c^2/4$. Then $I = (\pi/\omega_b, +\infty)$ and

$$T(s) = \begin{cases} 
  s - \frac{2}{\omega_b} \tan(\frac{\omega_b}{2} s) & \text{if } \frac{\pi}{\omega_b} < s < \xi_b, \\
  s - \xi_b + \frac{2}{\omega_b} (\pi - \arctan(\frac{\omega_b}{\omega_a} (s - \xi_b))) & \text{if } \xi_b \leq s,
\end{cases}$$

$$R(s) = \begin{cases} 
  \cos(\frac{\omega_b}{2} s)e^{-\frac{s}{T(s)}} & \text{if } \frac{\pi}{\omega_b} < s < \xi_b, \\
  -\frac{1}{2} \sqrt{4 + \omega_b^2(s - \xi_b)^2} & \text{if } \xi_b \leq s.
\end{cases}$$

Case 3: $a < c^2/4$. Then $I = (2(\pi - \arctan(\omega_b/\omega_a))/\omega_b, +\infty)$ and

$$T(s) = \begin{cases} 
  s - \frac{2}{\omega_b} \tanh^{-1}(\frac{\omega_a}{\omega_b} \tan(\frac{\omega_b}{2} s)) & \text{if } \frac{2}{\omega_b} (\pi - \arctan(\frac{\omega_b}{\omega_a})) < s \leq \xi_b, \\
  s - \xi_b + \frac{2}{\omega_b} (\pi - \arctan(\frac{\omega_b}{\omega_a} \tanh(\frac{\omega_a}{2} (s - \xi_b))) & \text{if } \xi_b \leq s,
\end{cases}$$

$$R(s) = \begin{cases} 
  -\sqrt{\cos^2(\frac{\omega_b}{2} s) - \sin^2(\frac{\omega_b}{2} s)e^{-\frac{s}{T(s)}}} & \text{if } \frac{2}{\omega_b} (\pi - \arctan(\frac{\omega_b}{\omega_a})) < s \leq \xi_b, \\
  -\sqrt{\cosh^2(\frac{\omega_b}{2} (s - \xi_b)) + \sinh^2(\frac{\omega_b}{2} (s - \xi_b))} & \text{if } \xi_b \leq s.
\end{cases}$$

In Section 4 we used the following lemma.

Lemma 2. (i) The function $h(s) = R(s)T'(s)/R'(s)$ is one-to-one on $(\xi_b, \xi_b + \xi_a)\setminus\{\sigma\}$. Let $s_1, s_2 \in I$ be such that $s_1 \not\in (\xi_b, \xi_b + \xi_a)$ and $|R(s_2)| > 1$. Then

(ii) There exist $\tilde{s}_1, \tilde{s}_2 \in I$ such that $R(\tilde{s}_1)R(\tilde{s}_2) = R(s_1)R(s_2)$ and $T(\tilde{s}_1) + T(\tilde{s}_2) < T(s_1) + T(s_2)$.

(iii) There exist $\tilde{s}_1, \tilde{s}_2 \in I$ such that $R(\tilde{s}_1)R(\tilde{s}_2) = R(s_1)R(s_2)$ and $T(\tilde{s}_1) + T(\tilde{s}_2) > T(s_1) + T(s_2)$.

Proof. We study the case $a > c^2/4$. The other cases are similar and are left to the reader.
(i) Here \( h(s) = 2 \sin(\omega_a(s-\xi_b)/2)/(\omega_a \cos(\omega_a(s-\xi_b)/2) - c \sin(\omega_a(s-\xi_b)/2)) \) is strictly increasing in \( (\xi_b, \xi_b + \xi_a) \setminus \{\xi\} \).

(ii) Let \( \sigma_0 = 2\arctan(\omega_b/c)/\omega_b \). \( R(s) \) reaches a maximum in \( \sigma_0 \). We distinguish several cases. If \( s_1 \in [0, \sigma_0) \), then there exists \( s_1 \in (\sigma_0, \xi_b) \) such that \( R(s_1) = R(s_1) \) and \( T(s_1) < T(s_1) \) (see Figure 2). Similarly, if \( s_2 \in (\sigma, \xi_b + \xi_a) \), then there exists \( s_2 \in (\xi_b, \sigma) \) such that \( R(s_2) = R(s_2) \) and \( T(s_2) < T(s_2) \), proving the assertion.

Therefore suppose that \( s_1 \in [\sigma_0, \xi_b] \) and \( s_2 \in [\xi_b, \sigma] \). Since \( R(s) \) is strictly decreasing in \( [\sigma_0, \sigma] \), for each \( s \in [s_1, s_2] \) there exists a unique \( \varphi(s) \) such that \( R(s)R(\varphi(s)) = R(s_1)R(s_2) \). The function \( \varphi(s) \) is continuous and decreasing. Moreover \( T(s) \) is strictly decreasing in \( [\sigma_0, \xi_b] \) and increasing in \( [\xi_b, \sigma] \). In consequence we obtain (ii) for \( s_1 \in [\sigma_0, \xi_b] \) taking \( s_1 = s, s_2 = \varphi(s) \) for any \( s \in (\sigma_0, \xi_b) \).

Finally suppose that \( s_1 = \xi_b \). Consider the function \( f(s) = T(s) + T(\varphi(s)) \), \( s \in [s_1, s_2] \), and let \( r \in (\xi_b, s_2) \) be such that \( r = \varphi(r) \). We assert that \( f(s) \) is decreasing in \( (\xi_b, r) \). Note that this implies (ii). From the inverse function theorem, \( \varphi(s) \) is differentiable in \( (\xi_b, r) \) and

\[
\varphi'(s) = -\frac{R'(s)R(\varphi(s))}{R(s)R'(\varphi(s))};
\]

also \( f(s) \) is differentiable and \( f'(s) = T'(s) + T'(\varphi(s))\varphi'(s) \). Multiplying by \( R(s)/R'(s) > 0 \) one obtains

\[
\frac{R(s)}{R'(s)} f'(s) = h(s) - h(\varphi(s)) < 0
\]

and hence \( f'(s) < 0 \) in \( (\xi_b, r) \), proving the assertion.

(iii) Similar.

REFERENCES