EMBEDDING THEOREMS
FOR RESIDUALLY ČERNIKOV CC-GROUPS

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Abstract. Embedding theorems for residually Černikov CC-groups are obtained, extending the corresponding results on FC-groups and improving some previous results on CC-groups.

1. Introduction

Groups with Černikov conjugacy classes, or CC-groups, were introduced by Polovickii [10, 11] as an extension of the concept of FC-groups. A group $G$ is said to be a CC-group if $G/C_G(x^G)$ is a Černikov group for each $x \in G$. In the theory of FC-groups, a classical problem introduced by P. Hall [8] was embedding periodic FC-groups, a classical problem introduced by P. Hall [8] was embedding periodic FC-groups, a classical problem introduced by P. Hall [8] was embedding periodic FC-groups, a classical problem introduced by P. Hall [8] was embedding periodic FC-groups. In the theory of FC-groups, a classical problem introduced by P. Hall [8] was embedding periodic FC-groups. In the theory of FC-groups, a classical problem introduced by P. Hall [8] was embedding periodic FC-groups. In the theory of FC-groups, a classical problem introduced by P. Hall [8] was embedding periodic FC-groups. Since then, his work on periodic FC-groups has been continued in a sequence of papers, such as, for example, those of Gorčakov and Tomkinson (see [15] for a complete account of this subject). The main result in this line is the characterization of the periodic residually finite FC-groups as subgroups of centrally restricted products of finite groups.

The aim of this paper is to study the natural extension of embedding theory from FC-groups to CC-groups. There have been a few papers previously written on this subject. In [10], the first of these papers, the following result is presented: a countable periodic residually Černikov CC-group is a subgroup of a direct product of Černikov groups. In [1] Franciosi, de Giovanni and Tomkinson showed that a CC-group with trivial center (and so residually Černikov) is a subgroup of a direct product of Černikov groups. We improve this result in Theorem 5.7, where we obtain the same conclusion if the residually Černikov CC-group has countable center. Finally, in [4] it is proved that a countable periodic CC-group is a section of a direct product of CC-groups. Here (Theorem 4.2) we obtain an analogous result for periodic residually Černikov CC-groups of arbitrary cardinal. In Section 5, we obtain, mainly, embedding results for $G/Z(G)$ and $G'$ and we prove (Theorem 5.7) that a periodic residually Černikov CC-group $G$ with $G'$, $G/G'$ or $Z(G)$ countable is a subgroup of a direct product of Černikov groups.
In the following, we shall use Polovickiǐ's theorem characterization of CC-groups (Theorem 4.36 of [12]), which assures that, if $G$ is a CC-group, then the normal closure $x^G$ is Černikov-by-cyclic and $[G, x]$ is Černikov for every $x \in G$. Our group-theoretic notation is standard and is taken from [12] and [15]. We will refer by $\mathcal{H}(C \cup A_0)$ ($\mathcal{F}(C \cup A_0)$, resp.) to the class of (quotients of, resp.) subgroups of direct products of Černikov and torsion-free abelian groups. We extend Tomkinson's definition of centrally restricted product of finite groups (see [15], p. 29) to the centrally complete product of Černikov groups, denoted by $Z^*_C$, which is the subgroup of the cartesian product where every element has a finite number of noncentral components. Its torsion subgroup is, precisely, the centrally restricted product, denoted by $Z_r C$. A residual system of Černikov groups is a set of normal subgroups $N_i$ of $G$ with trivial intersection and such that $G/N_i$ is a Černikov group for all $i \in I$. We denote it by $\{N_i : i \in I\}$.

2. Auxiliary results

In this section, we shall state some auxiliary results necessary for the following sections.

**Lemma 2.1.** (i) The classes of FC-groups and CC-groups are closed under the formations of centrally restricted or complete products.
(ii) Every abelian group is a $Z^*_C$-group.
(iii) $\mathcal{H}(C \cup A_0) \leq Z^*_C \leq \mathcal{F}(C \cup A_0)$.
(iv) $\mathcal{H} \subseteq Z_r C \subseteq \mathcal{F}(C \cup A_0)$.

**Proof.** The proof is an immediate consequence of the definitions. For (ii) and (iii), note that the additive group of the rational numbers is in the class $Z^*_C$, being a direct summand of the cartesian product of countably many copies of $C_p\infty$. □

Now, we are embedding a particularly simple class of groups, which contains the abelian groups.

**Proposition 2.2.** If $G$ is a central-by-Černikov group, $G \in \mathcal{F}(C \cup A_0)$. Furthermore, if $G$ is periodic, $G \in \mathcal{F}(C \cup A_0)$.

**Proof.** Let $Z = Z(G)$ so that $G/Z$ is Černikov. It is easy to check that any abelian group is residually Černikov, and so $Z$ is residually Černikov. Let $\{Z_i : i \in I\}$ be a Černikov residual system of $Z$. Each $Z_i$ is a normal subgroup of $G$, and since $G/Z$ and $Z/Z_i$ are Černikov groups, so is $G/Z_i$. Therefore $G$ is residually Černikov. Now, by Theorem 4.11 of [12], $G'$ is Černikov. It is easy to see that there exists a normal subgroup $N$ of $G$ such that $G/N$ is a Černikov group and $N \cap G' = 1$. Thus $G \leq (G/N) \times (G/G')$. Since $G/G'$ is an abelian group, $G/G' \in \mathcal{H}(C \cup A_0)$, and so $G \in \mathcal{F}(C \cup A_0)$. □

The following result relates embeddings of certain subgroups with embeddings of the whole group.

**Proposition 2.3.** If $H$ is a subgroup of the CC-group $G$ such that $G = HZ$, with $Z = Z(G)$ one has

(i) $H$ is residually Černikov if and only if $G$ is residually Černikov;
(ii) $H \in Z^*_C$ if and only if $G \in Z^*_C$;
(iii) if $G$ is periodic, $H \in Z_r C$ if and only if $G \in Z_r C$. 

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Proof. Let us observe that the converses are evident, and that we can deduce (iii) from (ii). In order to prove (i), let \( \{H_i; i \in I\} \) be a Černikov residual system of \( H \). Since \( G = HZ \), \( H_i \) is a normal subgroup of \( G \). Thus, \( G \leq \prod \{G/Z_i; i \in I\} \). Since \( H/H_i \) is Černikov and \( ZH/H_i \) is central in \( G/H_i \), we can deduce that \( G/H_i \) is central by Černikov. By Proposition 2.2, \( G/H_i \) is residually Černikov, and therefore so is \( G \). In order to prove (ii), let us assume that \( H \in \text{Zr}^*C \). We deduce that there exists a Černikov residual system \( \{H_i; i \in I\} \) of \( H \) such that \( H \leq \text{Zr}^*(H/H_i) \). Since \( G = HZ \), each \( H_i \) is a normal subgroup of \( G \), and we can embed \( G \) into \( \prod \{G/Z_i; i \in I\} \). It is easy to see that \( G \leq \text{Zr}^*(G/H_i) \). As in (i) \( G/H_i \) is a central-by-Černikov group, and by Lemma 2.1 and Theorem 2.2, \( G/H_i \in \text{Zr}^*C \), and so \( G \in \text{Zr}^*C \). □

The next result shows that the periodicity is not an important hypothesis when considering centrally restricted products of Černikov groups.

Proposition 2.4. If \( G \in \text{Zr}^*C \), then \( G \) is isomorphic to a subgroup of the direct product of a \( \text{Zr}C \)-group and a torsion-free abelian group.

Proof. By hypothesis, \( G \leq \text{Zr}^*\{G_i; i \in I\} \), where \( G_i \) are Černikov groups. \( \text{Zr}^*G_i = ZD \), with \( Z = \prod \{Z(G_i); i \in I\} \) and \( D = \text{Dr}\{G_i; i \in I\} \). It is clear that we can assume \( G = ZD \). Thus \( Z = Z(G) \) and \( G/Z \) is a periodic group. Let \( V \) be a maximal torsion-free subgroup of \( Z \). Then \( Z/V \) is a periodic group, \( V \cap D = 1 \) and \( G \leq (G/V) \times (G/D) \). The abelian group \( G/D = ZD/D \) can be embedded into a torsion-free abelian group and a periodic abelian group, and the latter belongs to \( \mathcal{Z}C \) by Lemma 2.1. Since \( G/V = (Z/V)(DV/V) \), it is a central extension of the \( \text{ZrC} \)-group \( DV/V \cong D \). By Proposition 2.3, \( G/V \in \text{Zr}C \), and the result follows. □

The next result shows that there exist some aspects in the theory of embeddings of \( CC \)-groups that have a better behaviour than in the \( FC \)-case. It is known that the torsion subgroup of the abelian group \( \prod \{C_{p^n}; n \in N\} \) is not a subgroup of a direct product of finite groups (Example 2.6 of [15]). This is an example of a centrally restricted product of a countable number of finite groups which does not belong to the class \( \mathcal{F} \). The next theorem shows, however, that an analogous statement is true for \( CC \)-groups, though the problem is still open for an uncountable set of indices.

Theorem 2.5. If \( G \leq \text{Zr}^*\{G_n; n \in N\} \), with \( G_n \) Černikov, then

\[ G \in \mathcal{F}(C \cup A_0) \]

Proof. Let us suppose first that \( G \) is periodic. Let \( T = T(\prod Z(G_n)) \) and \( D = \text{Dr}G_n \) such that \( G \leq \text{Zr}G_n = TD \). We can assume \( G = TD \). If \( T \) is countable, so is \( G \), and by Theorem 6 of [10] \( G \in \mathcal{F}C \). So let us assume that \( T \) is uncountable. \( T \) is abelian and periodic, so we can suppose \( T \leq \text{Dr}\{E_i; i \in I\} \), where \( E_i \) are Černikov groups and \( I \) is uncountable. Since \( D \) is countable, so is \( D \cap T \), and there exists a countable subset \( J \) of \( I \) such that \( D \cap T \leq \text{Dr}\{E_j; j \in J\} \). Let \( L = T \cap (\text{Dr}\{E_i; i \in I \setminus J\}) \). Thus, \( T/L \) is a countable group. Furthermore, \( L \leq G \) because \( L \leq T \leq Z(G) \) and \( L \cap D = L \cap T \cap D = 1 \). Then \( G \leq (G/L) \times (G/D) \). But \( G/D \) is a periodic abelian group, and so \( G/D \in \mathcal{F}C \). On the other hand, \( G/L = (T/L)(DL/L) \). Since \( DL/L \cong D \),
DL/L is a residually Černikov group, and by Proposition 2.3, so is G/L. But this group is countable and periodic, and so G/L ∈ ICC, and the theorem is proved if G is periodic. In the general case, we can suppose G = ZD, where Z = \[\prod Z(G_n)\]. Proceeding as in Proposition 2.4 and keeping the same notation, we obtain G ≤ (G/V) × (G/D), where G/D is abelian and G/V is a periodic group. Besides, G/V = (Z/V)(DV/V). Proceeding as in the proof of Proposition 2.3, G/V ≤ Zr*[F_n : n ∈ N], where F_n are periodic central-by-Černikov groups. The proof of Proposition 2.2 shows that F_n ≤ C_n × A_n, where C_n is Černikov and A_n is an abelian group. So G ≤ ZrC_n × \[\prod A_n × G/D\]. By the first part of the proof ZrC_n ∈ ICC, and so G ∈ ICC(C ∪ A_0).

The next result represents a crucial point in establishing the general embedding results. It extends from FC-groups, but the proof becomes more complicated and tedious by changing finite to Černikov.

**Theorem 2.6.** Let \( \rho \) be an ordinal limit. Let us assume that \( \{ N_\alpha : \alpha < \rho \} \) is a family of normal subgroups of the CC-group G such that \( \bigcap \{ N_\alpha : \alpha < \rho \} = 1 \), and let us call \( C_\alpha \) to \( C_G(G/N_\alpha) \). Let \( \{ H_\alpha : \alpha < \rho \} \) be an ascending chain of normal subgroups of G satisfying the following properties:

1. \( [G, H_\alpha] ≤ N_\beta \), for all \( \beta ≥ \alpha \).
2. \( G = C_\alpha H_{\alpha+2} \), for all \( \alpha < \rho \).

Then \( G ≤ Zr*[G/N_\alpha : \alpha < \rho] \).

**Proof.** Since \( \{ N_\alpha : \alpha < \rho \} = 1 \), we can suppose that \( g ≤ \prod \{ G/N_\alpha : \alpha < \rho \} \). Let us assume that the theorem is false. Then there must exist an element \( x \in g \) with an infinite number of noncentral components. So, we can take an infinite number of ordinals \( \alpha_1 < \alpha_2 < \cdots < \alpha_n < \cdots \) such that \( x \) does not belong to \( C_{\alpha_n} \) for every \( n \). Since, by condition (ii), \( [G, x]N_\alpha = [H_{\alpha+2}, x]N_\alpha \), it follows that \( [H_{\alpha+2}, x] \) is not contained in \( N_{\alpha_n} \), for each \( n \). Now, by condition (i), \( [H_{\alpha+2}, x] ≤ N_{\alpha_m} \) for all \( m ≥ n + 2 \), because it is clear that \( \alpha_{n+2} ≥ \alpha_n + 2 \) for all \( n \). Let us define \( M_n = \bigcap \{ N_\alpha : i ≥ n \} \). Thus we have an ascending chain \( M_1 ≤ M_2 ≤ \cdots ≤ M_n ≤ \cdots \) such that \( [H_{\alpha+2}, x] ≤ [G, x] \cap M_{n+2} \) but \( [H_{\alpha+2}, x] \) is not contained in \( M_n \). So, \( [G, x] \cap M_2 < [G, x] \cap M_4 < \cdots < [G, x] \cap M_{2n} < \cdots \) is a strict ascending chain in \( [G, x] \).

Since \( [G, x] \) is a Černikov group, there must exist an \( m \) such that for \( n ≥ m \) the quotient \( ([G, x] \cap M_{2n+2})/([G, x] \cap M_{2n}) \) is finite. Since \( [H_{\alpha+2}, x] ≤ [G, x] \cap M_{2n+2} \), \( [H_{\alpha+2}, x]/[H_{\alpha+2}, x] \cap M_{2n} \) is also finite. Now, \( M_{2n} ≤ N_{\alpha_{2n}} \), and thus, \( [H_{\alpha+2}, x]/([H_{\alpha+2}, x] \cap N_{\alpha_{2n}}) \) is finite. This group is isomorphic to \( [G, x]N_{\alpha_{2n}}/N_{\alpha_{2n}} \), and so the factors \( [G, x]N_{\alpha_k}/N_{\alpha_k} \) are finite for \( k = 2m, 2m+2, \ldots \). Let \( \overline{G} \) be \( G/([N_{\alpha_k} : k = 2m, 2m+2, \ldots]) \), and let us denote by \( \overline{C} \) the image of any subset \( C \) of \( G \) under the canonical map. Thus \( \overline{G}, x] ≤ \prod \{ ([G, x]N_{\alpha_k}/N_{\alpha_k} : k ∈ N] \}. \) Since the factors of this cartesian product are finite, it follows that \( \overline{G}, x] \) is residually finite. Since \( \overline{G}, x] \) is Černikov, \( \overline{G}, x] \) must be finite. But for each natural number \( k \), there exists \( g_k ∈ H_{\alpha_k+2} \) such that \( [g_k, x] \) does not belong to \( N_{\alpha_k} \). Since \( [g_k, x] ∈ \{ N_{\alpha_j} : j > k \} \), we conclude that the elements \( [g_k, x] \) are all different. This implies that \( \overline{G}, x] \) is infinite, which is a contradiction, and the result follows.

If \( G \) is an FC-group and \( N \) is a normal subgroup of \( G \) such that \( G/N \) is a finite group, there exists a finite subset \( X \) of \( G \) such that \( G = XG/N \). This
is not true if \( G/N \) is a Černikov factor of a \( CC \)-group (see, for example, a Prufer \( p \)-group). The next lemma is the solution of this problem that we need for our purposes.

**Lemma 2.7.** Let \( N \) be a normal subgroup of a \( CC \)-group \( G \) such that \( G/N \) is a Černikov group. If \( C/N = Z(G/N) \), then there exists a finite subset \( X \) of \( G \) such that \( G = CX^G \). (Observe that \( C = C_G(G/N) \)).

**Proof.** Let \( D/N \) be the radicable part of \( G/N \). Thus, \( G/N \) contains a finite subgroup \( S/N \) such that \( G = SD \). There exists a finite subset \( X \) of \( G \) such that \( S = \langle X \rangle N \), and therefore \( G = \langle X \rangle D \). Since \( D/N \) is radicable and \( S/N \) is finite, by Lemma 3.29.1 of [12], \( D/N = ([D, S]N/N)(C_{D/N}(S/N)) \leq [D, S]C/N \). Thus \( G = \langle X \rangle D = SC[D, S] \). But \([D, S] \leq NX^G\), and so \( G = CX^G \), and the proof is complete. \( \square \)

### 3. Residually Černikov \( CC \)-groups with \( G/Z(G) \) countable

Analogous to the classification of periodic residually finite \( FC \)-groups (cf. [5], [14], [15]) as subgroups of centrally restricted products of finite groups, we try to classify the residually Černikov \( CC \)-groups. The first step was done by Polovickii [10], who showed that a periodic residually Černikov group which is countable is a subgroup of a direct product of Černikov groups. In this section, we generalize this result to residually Černikov groups with \( G/Z(G) \) countable.

**Theorem 3.1.** If \( G \) is a residually Černikov \( CC \)-group with a countable residual system, then \( G \in Zr^*C \). Furthermore, if \( G \) is periodic, then \( G \in ZrC \).

**Proof.** Clearly, the second statement follows from the first. We know that \( G \leq \prod\{F_n : n \in N\} \), where \( F_n \) is a Černikov group for each \( n \). For each \( k \geq 1 \) let \( G_k := G \cap (\prod\{F_n : n > k\}) \). We construct by induction two chains of normal subgroups of \( G \), \( \{H_n : n \in N\} \) and \( \{M_n : n \in N\} \), satisfying the following conditions: (a) \( \{H_n : n \in N\} \) is an ascending chain, and \( H_n \) is the normal closure in \( G \) of a finite subset of \( G \). (b) For each \( n \geq 1 \), \( M_n = G_{s_n} \), where \( s_n \geq n \) and \( s_1 < s_2 < \cdots < s_n \). In particular, \( \{M_n : n \in N\} \) is a descending chain. (c) For every \( n > 1 \), \( G = H_nC_G(G/M_{n-1}) \). (d) For every \( n > 1 \), \( T(H_n) \cap M_n = 1 \). Let us define \( H_1 = 1 \), \( M_1 = G_1 \), and let us suppose that we have constructed \( n - 1 \) elements of both chains: \( H_1 \leq H_2 \leq \cdots \leq H_{n-1} \) and \( M_1 \geq M_2 \geq \cdots \geq M_{n-1} \). \( G/M_{n-1} \) is clearly a Černikov group, and so by Lemma 2.7, there exists a finite subset \( Y \) of \( G \) such that \( G = Y^GCG(G/M_{n-1}) \). By hypothesis, \( H_{n-1} = X^G \), for a finite subset \( X \) of \( G \). If we define \( H_n := \langle X \cup Y \rangle^G \), it is clear that (a) and (c) are satisfied. \( T(H_n) \) is a Černikov group, and since \( \{G_n : n \in N\} \) is a descending chain, there exists \( m \geq 1 \) such that \( T(H_n) \cap G_m \) is minimal. Thus \( T(H_n) \cap G_m = 1 \). Let us define \( s_n = \max\{m, n, s_{n-1} + 1\} \) and \( m_n = G_{s_n} \). Then, it is clear that conditions (b) and (d) hold, and our construction is complete. From (b) we have \( \bigcap\{M_n : n \geq 1\} = \bigcap\{G_n : n \geq 1\} = 1 \). Let us define \( N_0 = N_1 \) and \( N_i = M_{i+1}T(H_i) \), for \( i \geq 1 \). By Lemma 2.20 of [15], \( \bigcap\{N_i : i \geq 0\} = 1 \). For each \( k \in N \), \( [G, H_k] \leq T(H_k) \) and \( T(H_k) \leq T(H_k) \) if \( r \geq k \). Thus \([G, H_k] \leq N_r, r \geq k \). Besides, by (c) \( H_{k+2}G(G/N_k) \geq H_{k+2}CG(G/M_{k+1}) = G \). By Theorem 2.6 \( G \) is isomorphic to a subgroup of the centrally complete product of the \( G/N_i \).
Since $G/N_i$ is a quotient of $G/M_{i+1} = G/G_{i+1}$, $G/N_i$ is Černikov, and the result follows. □

The following result extends Polovickii's theorem given in [10].

**Corollary 3.2.** Let $G$ be a residually Černikov CC-group with $G/Z(G)$ countable. Then, $G \in \mathcal{B}(C \cup A_0)$. Furthermore, if $G$ is periodic, $G \in \mathcal{B}C$.

**Proof.** The second sentence follows from the first. By hypothesis $G = HZ$, where $H$ is normal in $G$ and countable, and $Z = Z(G)$. So, $G' = H'$ is a countable subgroup of $G$. For each nonunit element $x \in G'$, there exists a normal subgroup $N_x$ of $G$ with $G/N_x$ a Černikov group and $x \notin N_x$. So, if $N = \bigcap\{N_x : 1 \neq x \in G'\}$, $N \cap G' = 1$. Then, $G \leq (G/G') \times (G/N)$. But $(G/G') \in \mathcal{B}(C \cup A_0)$ and $\{N_x/N : 1 \neq x \in G'\}$ is a countable residual system for the CC-group $G/N$. By Theorem 3.1, $G/N \in ZrC$, with a countable number of components and, by Theorem 2.5, $G/N \in \mathcal{B}(C \cup A_0)$, and the proof is complete. □

## 4. Residually Černikov CC-groups as sections

Gorčakov [5] showed that periodic residually finite FC-groups are sections of direct products of finite groups. Later, this result was a consequence of the complete characterization of the periodic residually finite FC-groups as the subgroups of centrally restricted products of finite groups due to Tomkinson [14]. To date, an analogous characterization has not been obtained for CC-groups. In this section we extend Gorčakov's result, showing that residually Černikov CC-groups are sections of direct products of Černikov and torsion-free abelian groups. In Example 2.4 of [4] there is an example of a CC-group with $G/Z(G)$ non-periodic, and so it is not a section of this type. So, there are CC-groups that are not in the class $\mathcal{B}(C \cup A_0)$, and this shows that the hypothesis of residually Černikov cannot be omitted. On the other hand, an infinite countable extra special $p$-group (see p. 49 of [15]) is $\mathcal{B}DF$ but it is not residually Černikov. Thus, the classification that we shall obtain in this section is not a characterization because a $\mathcal{B}(C \cup A_0)$-group is not always residually Černikov. The next result represents the induction step, and its proof is very close to the corresponding theorem of [5].

**Theorem 4.1.** Let $G$ be a CC-group subgroup of the cartesian product

$$\prod\{F_i : i \in I\}$$

of an uncountable number of Černikov groups $F_i$. Then $G$ can be embedded as a subgroup of a centrally complete product of CC-groups with cardinal strictly less than $|I|$. Furthermore, if $G$ is periodic, the embedding can be performed in a centrally restricted product.

Now we are able to establish our main result of this section.

**Theorem 4.2.** A residually Černikov CC-group is in the class $\mathcal{B}(C \cup A_0)$. Furthermore, if $G$ is periodic, $G$ is a $\mathcal{B}C$-group.

**Proof.** The second statement follows immediately from the first. Let us suppose that $G \leq \prod\{F_i : i \in I\}$, with $|I| \leq |G|$. If $I$ is countable, the result follows.
from Corollary 3.2. Let us suppose that $|I|$ is uncountable. By Theorem 4.1, there exists a family $\{G_j: j \in J\}$ of CC-groups with $|G_j| < |I|$ for all $j$, and such that $G \leq Zr^*G_j$. We can assume that $G_j = G/K_j$ for all $j \in J$, and thus, $G = H_jK_j$ with $H_j$ normal in $G$ and $|H_j| < |I|$. $H_j$ is residually Černikov, and so, by induction $H_j \in \mathcal{C}(C \cup A_0)$. Then, Lemma 2.1 implies that $G_j \in \mathcal{C}(C \cup A_0)$. Let $Z = \prod Z(G_j)$ and $D = DrG_j$ such that $G \leq Zr^*G_j = ZD$. Then $D \in \mathcal{C}(C \cup A_0)$ and so $G \in \mathcal{C}(C \cup A_0)$.

**Corollary 4.3.** If $G$ is a CC-group, $G/Z(G) \in \mathcal{C}(C \cup A_0)$. If $G$ is periodic, then $G/Z(G) \in \mathcal{C}C$.

**Proof.** It is a consequence of Theorem 4.2. $\square$

Another traditional step in FC-group theory has been embedding the derived group $G'$ of an FC-group $G$. Tomkinson [13] has shown that $G' \in \mathcal{C}F$, for any FC-group $G$. Using Theorem 3.1 of [4], we can prove an analogous theorem for CC-groups. The proof is very close to that of Theorem 3.6 of [15].

**Theorem 4.4.** If $G$ is a CC-group, then $g' \in \mathcal{C}C$.

We finish this section with some arithmetical properties.

**Theorem 4.5.** If $G$ is an infinite residually Černikov CC-group, then $|G| = |G'||Z(G)|$.

**Proof.** Proceeding as in Corollary 3.2, $G/N \leq \prod\{G/N_x: 1 \neq x \in G'\}$ with $N = \bigcap\{N_x: 1 \neq x \in G'\}$. If $x \in N$, then $[G, x] \leq N \cap G' = 1$, and so $x \in Z = Z(G)$. Then it is easy to see that $Z(G/N) = Z/N$, and by Lemma 6.3 of [9] $|G/Z| \leq \max\{n_0, |G'|\}$. So, $|G| = |G/Z||Z| \leq n_0|G'||Z|$, and we have equality because $G$ is infinite. Thus, if $G'$ or $Z(G)$ is infinite, the result follows. Otherwise, there exists a normal subgroup $K$ of $G$ with $G/K$ a Černikov group and $K \cap Z \cap G' = 1$. So $K = 1$ and $G$ is Černikov. By Theorem 4.35 of [12], $G$ is an FC-group and so is central by finite. Consequently, $G$ is finite, which is a contradiction, and the proof is complete. $\square$

The result above is false if the group is not residually as shows any infinite extra special $p$-group.

**Corollary 4.6.** If $G$ is a CC-group with $G/Z(G)$ infinite, then $|G| = |G'||Z_2(G)|$.

**Proof.** It follows from Theorem 4.5. $\square$

5. Some particular cases of embedding

We show in this section some special cases in which a residually Černikov CC-group can be embedded into a direct product of Černikov and abelian groups. Most of these results are, in the FC-case, consequences of the characterization of residually finite periodic FC-groups given by Tomkinson [14]. First, we present some results which are adaptations of some Gorčakov theorems (see Gorčakov [7] and Theorems 2.2, 2.3, 2.4 of [15]). Since our proofs are very similar to those given there, we just state them. We shall denote by $\pi_J(D_r\{H_i: i \in I\})$ the projection from $D_r\{H_i: i \in I\}$ to $D_r\{H_i: i \in J\}$ for a set of groups $\{H_i: i \in I\}$ and where $J$ is contained in $I$. 

Lemma 5.1. Let $G$ be such that $G \leq \text{Dr}\{H_i : i \in I\}$, where the $H_i$ are groups such that $|H_i| < \kappa$ for all $i \in I$, and where $\kappa$ is a fixed uncountable cardinal. Then, the index set $I$ can be seen as a union of an ascending chain of subsets $I(\alpha)$, $\alpha < \rho$, $\rho$ being the least ordinal of cardinality $|I|$, such that the following conditions are satisfied. (a) For each $\alpha$, $J(\alpha) = I(\alpha) - I(\alpha)$ has cardinality strictly less than $\kappa$. (b) If $J \subseteq I(\alpha)$ is finite, then $\text{Dr}_J(G) = \text{Dr}_J(G \cap \text{Dr}\{H_i : i \in I(\alpha)\})$.

Lemma 5.2. Let $G$ be such that $G \leq \text{Dr}\{H_i : i \in I\}$, where the $H_i$ are groups such that $|H_i| < \kappa$ for all $i \in I$ and where $\kappa$ is an uncountable given cardinal. Then $G'$ is a direct product of normal subgroups of $G$ which have cardinality strictly less than $\kappa$.

As consequences of Lemma 5.1 we can state (see Theorem 2.4 of [15]).

Theorem 5.3. If $G \in \mathfrak{I}(C \cup A_0)$, then $G/Z(G) \in \mathfrak{I}C$.

Corollary 5.4. If $G$ is a residually Černikov CC-group, then $G/Z(G) \in \mathfrak{I}C$. So, if $G$ is a CC-group, $G/Z_n(G) \in \mathfrak{I}C$ for $n \geq 2$.

Now we use these previous results to state the more important cases of embedding.

Theorem 5.5. If $G$ is a residually Černikov CC-group, $G' \in \mathfrak{I}C$.

Proof. We will proceed by induction on $|G|$. If $G$ is countable, by Corollary 3.2, $G \in \mathfrak{I}(C \cup A_0)$, and since $G'$ is periodic, $G' \in \mathfrak{I}C$. If $G$ is uncountable, by Theorem 4.1 we have $G \leq \text{Zr}\{G_i : i \in I\}$ with $|I| = |G|$ and $G_i$ is a CC-group with $|G_i| < |G|$. Let $Z = \prod_i Z(G_i)$ and $D = \text{Dr}\{G_i : i \in I\}$. Then $G \leq ZD$. Thus, $G' = (GZ)'$, and since $GZ = (GZ \cap D)Z$, we conclude $G' = (GZ \cap D)'$. By Lemma 5.2 $(GZ \cap D)' = \text{Dr}\{H_j : j \in J\}$, where $H_j$ are normal subgroups of $GZ \cap D$ and $|H_j| < |G|$ for all $j \in J$. For each $j \in J$, since $H_j \leq G'$, there exists $K_j \leq G$ such that $H_j \leq K_j' \leq K_j$ and $|K_j'| < |G|$. By induction, $K_j' \in \mathfrak{I}C$, and so $H_j \in \mathfrak{I}C$. Thus, $G' \in \mathfrak{I}C$, and the proof is complete. □

Corollary 5.6. If $G$ is a CC-group, $G'Z(G)/Z(G) \in \mathfrak{I}C$.

Proof. It is a consequence of Theorem 5.5. □

Theorem 5.7. Let $G$ be a residually Černikov CC-group. If $G'$, $G/G'$ or $Z(G)$ are countable, then $G \in \mathfrak{I}(C \cup A_0)$. Furthermore, if $G$ is periodic, $G \in \mathfrak{I}C$.

Proof. If $G'$ or $Z(G)$ are countable, proceeding as in Corollary 3.2 and applying Corollary 5.4, we conclude that $G \in \mathfrak{I}(C \cup A_0)$. If $G/G'$ is countable, we prove the result by induction, as in Theorem 5.5. Using the same notation, $G' = \text{Dr}\{H_j : j \in J\}$, where $H_j$ are normal subgroups of $GZ \cap D$ and $|H_j| < |G|$ for all $j \in J$. In fact, $H_j \leq G$, $H_j$ are normal subgroups of $(GZ \cap D)Z = GZ$, and so the subgroups $H_j$ are also normal in $G$. Since $G/G'$ is a countable group, there exists a countable normal subgroup $C$ of $G$ such that $G = CG'$. But $C \cap G'$ is also countable, and so there exists a countable subset $J_0 \subseteq J$ with $C \cap G' = H \leq \text{Dr}\{H_j : j \in J_0\}$. Clearly, $H$ is normal in $G$ and, since we are supposing that $G$ is uncountable, $|H| < |G|$. Let $K =: \text{Dr}\{H_j : j \in J - J_0\}$ and $E = CH$. Thus, $|E| < |G|$. Then, $E \cap K = CH \cap G' \cap K = (C \cap G') \cap K = H \cap K = 1$ and, since $G = EK$, we obtain $G = E \times K$. Therefore, $(G/G') = (E/E') \times (K/K')$, and so $E/E'$ and
$K/K'$ are countable. By induction, $E$ and $K$ are $\mathcal{H}(C \cup A_0)$-groups, and so is $G$. The second part of the theorem is a consequence of the first, and our result follows. □

In this previous theorem, we showed that a residually Černikov $CC$-group with $Z(G)$ countable belongs to the class $\mathcal{H}(C \cup A_0)$. The analogous problem is still unsolved in the $FC$-case, that is to say, it is unknown if every periodic residually finite $FC$-group with countable center is in the class $\mathcal{H}F$. The next result gives a partial answer to this problem.

**Theorem 5.8.** Let $G$ be a periodic and residually finite $FC$-group with countable center $Z(G) = Z$. Then $G$ is isomorphic to a subgroup of a direct product of groups which are all finite except for countably many of them, which are Černikov central by finite groups.

**Proof.** We can suppose that $G \leq (G/Z) \times (G/N)$, where $G/N$ is a periodic $FC$-group with a countable residual system of finite groups. By Corollary 2.26 of [15], $G/Z \in \mathcal{H}F$. Tomkinson (Theorem 2.24 of [15]) also showed that $G/N$ is a subgroup of a centrally restricted product of a countable number of finite groups. Proceeding as in Theorem 2.5, we can prove that $G/N \leq Dr\{A_i : i \in I\} \times Dr\{C_n : n \geq 1\}$, with $A_i$ Černikov abelian groups and $C_n$ Černikov $FC$-groups. So, we have $G \leq Dr\{F_j : j \in J\} \times Dr\{A_i : i \in I\} \times Dr\{C_n : n \geq 1\}$ with $F_j$ finite groups. Let $A := Dr\{A_i : i \in I\}$. Clearly, $G \cap A \leq Z$. Since $Z$ is countable, $I_0 = \text{supp}(G \cap A)$ is also countable, but $G \cap (Dr\{A_i : i \in I - I_0\}) = 1$ which implies that $G \leq Dr\{F_j : j \in J\} \times Dr\{A_i : i \in I_0\} \times Dr\{C_n : n \geq 1\}$, and the result follows. □

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