NON-SINGULAR MODULES AND $R$-HOMOGENEOUS MAPS

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Abstract. A non-singular $R$-module $M$ is a ray for the class of all non-singular modules if every $R$-homogeneous map from $M$ into a non-singular module is additive. Every essential extension of a non-singular locally cyclic module is a ray. We investigate the structure of rays, and determine those semi-prime Goldie-rings for which all non-singular modules are rays and those rings for which the only rays are essential extensions of locally cyclic modules.

1. Introduction

In a recent paper, one of the authors studied endomorphal modules over a principal ideal domain $R$. These are precisely those modules $M$ for which every $R$-homogeneous map $M \rightarrow M$ is an endomorphism of $M$. The discussion of endomorphal modules has its origins in the theory of near rings since the collection of $R$-homogeneous maps $M \rightarrow M$ forms a near ring. Fuchs, Maxson, and Pilz investigated the class of rings for which all $R$-modules are endomorphal in [1]. While they were not able to describe this class of rings completely, they showed in [1, Theorem II.8] that it contains all rings of the form $\prod_{i \in I} \text{Mat}_{n_i}(R)$ such that $n_i \geq 2$ for all $i \in I$. If we try to find rings for which the class of endomorphal modules is as small as possible, we observe that the class of endomorphal modules always contains the locally cyclic modules [3, Corollary 3.2]. Here an $R$-module $M$ is locally cyclic if, for $x, y \in M$, there are $a \in M$ and $r, s \in R$ with $x = ar$ and $y = as$. There are rings for which all endomorphal modules are locally cyclic; for instance, an abelian group $G$ is endomorphal if and only if it is locally cyclic [3, Propositions 2.4 and 2.7 and Theorem 3.4].

It is the goal of this paper to investigate further the relationship between locally cyclic modules and endomorphal modules, especially in the case that $R$ is not necessarily a principal ideal domain. The methods used in [3] for the discussion of endomorphal abelian groups do not carry over to the generality of the rings under consideration in this paper. We show in Section 2 that there exists a ring $R$ without zero-divisors that has a right ideal $I$ which is not endomorphal. On the other hand, every ideal in a commutative integral domain is endomorphal, but not necessarily locally cyclic.

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We focus our attention mostly on non-singular modules. In Theorem 2.1, we show that a right non-singular ring \( R \) is a semi-prime right Goldie-ring if and only if the following conditions are equivalent for every non-singular right \( R \)-module \( M \):

(a) \( M \) contains an essential locally cyclic submodule.
(b) \( M \) contains an essential cyclic submodule.
(c) \( M \) is isomorphic to a submodule of its maximal right ring of quotients.

We consider \( R \)-modules \( M \) and \( N \), and call \( M \) an \( N \)-ray if every \( R \)-homogeneous map \( \alpha: M \rightarrow N \) is an element of \( \text{Hom}_R(M, N) \). Given a class \( \mathcal{C} \) of \( R \)-modules, we say that \( M \) is a ray for \( \mathcal{C} \) if it is an \( N \)-ray for all \( N \in \mathcal{C} \).

Proposition 3.1 establishes that a non-singular module \( M \) which has a non-zero essential locally cyclic submodule is a ray for any class \( \mathcal{C} \) of non-singular \( R \)-modules. We show that there is a semi-prime right Goldie-ring \( R \) such that every non-singular right \( R \)-module is a ray for the class of all non-singular modules (Theorem 3.5 and Corollary 3.6), but has the additional property that not all modules are endomorphical. Moreover, we describe the semi-prime right Goldie-rings \( R \) for which non-singular modules, which are rays for the class of non-singular modules, are submodules of \( Q(R) \) (Theorem 3.4). Theorems 3.4 and 3.6, in particular, show that the class of endomorphical modules may be very large even if not all \( R \)-modules are endomorphical.

2. Non-singular locally cyclic modules and Goldie-rings

Let \( R \) be a ring. The word module denotes a unital right \( R \)-module. The module \( M \) is non-singular if \( Z(M) = \{ x \in M | xI = 0 \text{ for some essential right ideal } I \text{ of } R \} = 0 \). The ring \( R \) itself is (right) non-singular if it is non-singular as a right \( R \)-module. In particular, \( Z(R) \) is a two-sided ideal of \( R \). We define the second singular submodule \( Z_2(M) \) by \( Z_2(M)/Z(M) = Z(M/Z(M)) \). It is well known that \( M/Z_2(M) \) is a non-singular module. Furthermore, we observe \( MZ_2(R) = 0 \) for all non-singular \( R \)-modules \( M \). Therefore, every non-singular \( R \)-module can be viewed as an \( R/Z_2(R) \)-module, and vice-versa. Therefore, we restrict our discussion in this paper to the case that \( R \) is a right non-singular ring. The module \( M \) has Goldie-dimension 1 if it is non-zero and every non-zero submodule of \( M \) is essential in \( M \). In the following, the symbol \( E(M) \) denotes the injective hull of \( M \).

We want to remind the reader that a ring \( R \) is a semi-prime right Goldie-ring if it does not contain any non-zero nilpotent ideals and every direct sum of non-zero right ideals has only finitely many summands. In particular, \( Z(R) = 0 \) for a semi-prime right Goldie-ring. These rings are precisely those which have a semi-simple Artinian maximal right quotient ring, \( Q(R) \). In particular, \( Q(R) \) is the classical right ring of quotients of \( R \) [2]. For further details on Goldie-rings, see [5].

**Theorem 2.1.** The following conditions are equivalent for a right non-singular ring \( R \):

(a) \( R \) is a semi-prime right Goldie-ring.
(b) The following conditions are equivalent for an \( R \)-module \( M \):
   (i) \( M \) contains an essential non-singular cyclic submodule.
   (ii) \( M \) contains an essential non-singular locally cyclic submodule.
   (iii) \( M \) is isomorphic to a submodule of \( Q(R) \).
Proof. (a) ⇒ (b): The implication (i) ⇒ (ii) is trivial.

(ii) ⇒ (iii): We show that the injective hull of $M$ is isomorphic to a submodule of $Q(R)$. For this, it is enough to show that $M$ contains an essential submodule which is isomorphic to a right ideal $I$ of $R$. Then, $E(M) = E(I) \subseteq E(R) = Q(R)$. Suppose that $U$ is an essential locally cyclic submodule of $M$. Since $E(M) = E(U)$, no generality is lost if we assume that $M$ itself is locally cyclic.

Assume that $M$ has infinite Goldie-dimension. If $n$ is the right Goldie-dimension of $R$, then $M$ contains $n+1$ non-zero cyclic submodules $a_0R, \ldots, a_nR$ whose sum is direct. Since $M$ is locally cyclic, there are $a \in M$ and $r_0, \ldots, r_n \in R$ with $a_i = ar_i$ for $i = 0, \ldots, n$. Since $\bigoplus_{i=0}^n a_i R \subseteq aR$, we obtain that $aR$ has Goldie-dimension at least $n + 1$. On the other hand, if $I$ is the annihilator of $a$, then $I$ is not essential in $R$ since $M$ is non-singular. There is a non-zero right ideal $J$ of $R$ such that $I \oplus J$ is essential in $R$. Since $R/I$ is non-singular and $[R/I]/[(I \oplus J)/I] \cong R/(I \oplus J)$ is singular, we obtain that $(J \oplus I)/I$ is an essential submodule of $R/I$. Thus, $aR$ contains an essential submodule isomorphic to $J$, and $n \geq \dim J = \dim aR \geq n + 1$, which is not possible. Therefore, the Goldie-dimension of $M$ has to be finite.

In particular, we can choose non-zero cyclic submodules $b_1R, \ldots, b_kR \subseteq M$ of $M$ whose sum is direct and essential in $M$. Using the arguments of the previous paragraph, there is $b \in M$ such that $\bigoplus_{i=1}^k b_i R \subseteq bR$. Thus, $bR$ is an essential submodule of $M$. By what has been shown, $bR$ contains an essential submodule which is isomorphic to a right ideal of $R$.

(iii) ⇒ (i): Assume that $M$ is a submodule of $Q(R)$. Choose a submodule $V$ of $Q(R)$ such that $M \cap V = 0$ and $M \oplus V$ is essential in $Q(R)$. Suppose we have shown that $M \oplus V$ contains an essential cyclic submodule $U = xR$. Write $x = a + b$ with $a \in M$ and $b \in V$. Then, $xR \subseteq aR \oplus bR$ is an essential submodule of $M \oplus V$. Therefore, $(M \oplus V)/(aR \oplus bR) \cong (M/aR) \oplus (V/bR)$ is singular. Since $M$ is non-singular, $aR$ is essential in $M$. We may, hence, assume that $M$ is essential in $Q(R)$. Observe that $R$ is essential in $Q(R)$. Therefore, $M \cap R$ is an essential right ideal of $R$. Thus, $M \cap R$ contains $dR$ for some regular element $d$ of $R$. Since $dR$ is essential in $R$, we obtain that $dR$ is an essential submodule of $Q(R)$ inside $M$, i.e., $dR$ is essential in $M$. Thus, $M$ satisfies (i).

(b) ⇒ (a): Let $I$ be an essential right ideal of $R$. Since $I$ is an $R$-submodule of $Q(R)$, it contains an essential cyclic submodule $dR$ for some $d \in I$ by the equivalence of (i) and (iii) in (b). Therefore, the family of essential right ideals of $R$ contains a cofinal subfamily of cyclic ideals. Since $R$ is a non-singular ring, it is a semi-prime right Goldie-ring by [2].

However, $Q(R)$ need not be locally cyclic even if $R$ is a semi-prime right Goldie-ring.

Theorem 2.2. The following conditions are equivalent for a ring $R$ such that $R$ is a semi-prime right Goldie-ring:

(a) $Q(R)$ is locally cyclic.

(b) $R$ is a left Goldie-ring.

Proof. (b) ⇒ (a): Suppose that $x$ and $y$ are elements of $Q(R)$. There are $r, s \in R$ and a regular element $d$ of $R$ with $x = d^{-1}r$ and $y = d^{-1}s$ since
R satisfies the left Ore-condition as a semi-prime left Goldie-ring. This proves that \( Q(R) \) is a locally cyclic \( R \)-module.

(a) \( \Rightarrow \) (b): It suffices to show that \( R \) satisfies the left Ore-condition with respect to its regular elements. Since left and right quotient rings are isomorphic if they exist, \( R \) has to be a left Goldie-ring. Let \( r, c \in R \) with \( c \) a regular element of \( R \). Because \( Q(R) \) is a locally cyclic right \( R \)-module, there are \( a \in Q(R) \) and \( s, t \in R \) with \( rc^{-1} = as \) and \( 1 = at \). Write \( a = uv^{-1} \) for \( u, v \in R \) with \( v \) regular in \( R \).

Since \( 1 = uv^{-1}t \), the element \( t \) is a right regular element of \( R \). The fact that \( R \) is a right Goldie-ring yields that \( t \) is regular in \( R \). Therefore, \( u = t^{-1}v \) is a regular element of \( R \), and \( rc^{-1} = uv^{-1}s = t^{-1}vv^{-1}s = t^{-1}s \). This yields \( tr = sc \) for some \( s, t \in R \) with \( t \) regular; and \( R \) satisfies the left Ore-condition.

3. Non-singular modules and rays

In this section, we investigate how locally cyclic non-singular modules and rays are related for the class of rights discussed in Section 2.

**Proposition 3.1.** Let \( R \) be a ring.

(a) A locally cyclic \( R \)-module is a ray for any class \( \mathcal{C} \) of \( R \)-modules.

(b) If an \( R \)-module \( M \) is a ray for a class \( \mathcal{C} \) of non-singular \( R \)-modules, then so is every essential extension of \( M \).

*Proof.* Since (a) is an immediate consequence of the definitions, it remains to show (b). Suppose \( N \) is a module which contains \( M \) as an essential submodule of \( E(M) \). Consider \( x, y \in N \). There is an essential right ideal \( I \) of \( R \) such that \( xI, yI \subseteq M \). If \( \phi: M \to X \) for some \( X \in \mathcal{C} \) is \( R \)-homogeneous, then \( \phi(x + y)i = \phi(xi + yi) = \phi(xi) + \phi(yi) \) for all \( i \in I \). Therefore,

\[
(\phi(x + y) - \phi(x) - \phi(y))I = 0.
\]

Since \( X \) is non-singular, we have \( \phi(x + y) = \phi(x) + \phi(y) \).

**Corollary 3.2.** Let \( R \) be a semi-prime right Goldie-ring. Submodules of \( Q(R) \) are rays for the class of non-singular modules.

*Proof.* Suppose that there are \( r, d \in R \) such that \( d \) is regular in \( R \) and \( rd^{-1} \in Z(Q(R)) \). Then, \( r \in R \cap Z(Q(R)) \subseteq Z(R) = 0 \). Therefore, \( Q(R) \) is a non-singular \( R \)-module, which contains \( R \) as an essential cyclic submodule. Apply Theorem 2.1 and Proposition 3.1.

However, the previous result may fail if \( R \) is not a Goldie-ring.

**Example 3.3.** There exists a ring \( R \) without zero-divisors which has a right ideal which is not endomorphical.

*Proof.* Let \( R \) be a ring without zero-divisors which has infinite Goldie-dimension. Then, \( R \) contains a right ideal isomorphic to \( \bigoplus_\omega R \). Let \( \{x_n | n < \omega\} \) be an \( R \)-basis for \( \bigoplus_\omega R \). It is easy to see that \( \bigcup_{n<\omega} x_n R \) is an \( R \)-closed, strongly \( R \)-pure subset of \( \bigoplus_\omega R \). Then, there is an \( R \)-homogeneous map \( \phi: \bigoplus_\omega R \to \bigoplus_\omega R \) which is the identity on \( \bigcup_{n<\omega} R \) and takes value 0 elsewhere. This shows that \( \bigoplus_\omega R \) is not endomorphical.

The next result determines for which of the rings in Theorem 2.1 the class of non-singular modules which are rays for the class of non-singular modules coincides with the class of submodules of \( Q(R) \).
Theorem 3.4. The following conditions are equivalent for a ring \( R \) such that \( R \) is a semi-prime Goldie-ring:

(a) Every non-singular module \( M \) which is an \( E(M) \)-ray contains an essential locally cyclic submodule.
(b) Every non-singular module \( M \) which is an \( E(M) \)-ray has finite Goldie-dimension.
(c) If \( M \) is a non-singular ray for the class of all non-singular modules, then \( M \) contains an essential locally cyclic submodule.
(d) \( Q(R) \cong \prod_{i=1}^{n} D_i \) where each \( D_i \) is a division algebra.

Proof. (a) \( \Rightarrow \) (b): Condition (a) implies that every non-singular \( R \)-module \( M \) which is an \( E(M) \)-ray is isomorphic to a submodule of \( Q(R) \), and therefore has finite Goldie-dimension.

(b) \( \Rightarrow \) (d): Since \( R \) is a semi-prime Goldie-ring, we have
\[
Q(R) \cong \prod_{i=1}^{n} S_i
\]
where \( S_i = \text{Mat}_{m_i}(D_i) \) for some division algebra \( D_i \). Assume that \( m_i > 1 \) for some \( i \). No generality is lost if we assume \( i = 1 \). We construct a non-singular module \( M \) of infinite Goldie-dimension which is an \( E(M) \)-ray. The construction is based on a series of results which are established in the next paragraph.

We show that a \( Q(R) \)-module \( M \) is an \( E(M) \)-ray as an \( R \)-module if it is endomorphally as a \( Q(R) \)-module. Suppose that \( \phi: M \rightarrow M \) is a \( R \)-homogeneous map. If \( x, y \in M \), then \( \phi(xr) = \phi(xd^{-1})d \) for all \( d \in D_i \) yields that \( \phi \) is \( Q(R) \)-homogeneous. Therefore, \( \phi \) is a homomorphism. Clearly, \( M \) is an \( E(M) \)-ray once we have shown that it is an injective \( R \)-module. Since \( Q(R) \) is semi-simple Artinian, it is enough to observe that \( \bigoplus_{J} Q(R) \) is injective as an \( R \)-module for all index sets \( J \). But this is an immediate consequence of [5, IX, Proposition 2.7].

Since \( m_1 > 1 \), we have that every \( S_i \)-module is endomorphally by [1]. This holds in particular for \( M = \bigoplus_{\omega} S_i \). We view \( M \) as an \( Q(R) \)-module on which \( S_2, \ldots, S_n \) operate trivially. In particular, a map \( \phi: M \rightarrow M \) is \( Q(R) \)-homogeneous if and only if it is \( S_1 \)-homogeneous. Therefore, \( M \) is \( Q(R) \)-endomorphically. By the results of the previous section, \( M \) is a non-singular \( R \)-module of infinite Goldie-dimension which is an \( E(M) \)-ray. The resulting contradiction shows \( m_i = 1 \) for all \( i = 1, \ldots, n \).

(d) \( \Rightarrow \) (a): Let \( M \) be a non-singular \( E(M) \)-ray. Then, \( E(M) \) is a \( Q(R) \)-module and has a unique decomposition \( M = \bigoplus_{i=1}^{n} M_i \) where \( M_i \) is a \( D_i \)-vector space on which \( D_j \) operates trivially for \( j \neq i \). Suppose that \( \phi: E(M) \rightarrow E(M) \) is a \( Q(R) \)-homogeneous map; and choose \( x, y \in E(M) \). There is a regular element \( d \in R \) with \( xd, yd \in M \). Since \( \phi|M \) is an \( R \)-homogeneous map from \( M \) to \( E(M) \), we have \( \phi(x + y)d = \phi(xd + yd) = \phi(xd) + \phi(yd) = (\phi(x) + \phi(y))d \). The fact that \( E(M) \) is non-singular yields \( \phi(x + y) = \phi(x) + \phi(y) \). Therefore, \( E(M) \) is an endomorphically \( Q(R) \)-module. By [3], each \( M_i \) is an endomorphically \( Q(R) \)-module. Since \( D_j \) operates trivially on \( M_i \) if \( j \neq i \), we obtain that \( M_i \) is an endomorphically \( D_i \)-module. Suppose that \( M_i \) has dimension at least 2 as a \( D_i \)-vector-space. Then, \( V = D_i \oplus D_i \) is endomorphically. On the other hand, let \( U \) be the subspace of \( V \) which is generated by \( (1, 0) \). If \( 0 \neq (x, y) \) then \( d \neq 0 \) and \( y = 0 \). This shows that...
$U$ is strongly $R$-pure in $V$. By [3], the map $\phi: V \rightarrow V$ which is defined by

$$
\phi(x) = \begin{cases} x & \text{if } x \in U, \\ 0 & \text{otherwise} \end{cases}
$$

is an $R$-homogeneous map which is not additive. The resulting contradiction shows that each $M_i$ has at most dimension 1 as a $D_i$-vector-space. Hence, $E(M)$ is a submodule of $Q(R)$. Now apply Theorem 2.1.

Since the proof of implication (a) $\Rightarrow$ (c) is trivial, it remains to show the converse in order to complete the proof: Let $M$ be a non-singular $E(M)$-ray. Suppose $M \notin Q(R)$, and consider a decomposition $E(M) = \bigoplus_{i=1}^{n} M_i$ into $S_i$-modules as in the proof of (b) $\Rightarrow$ (d). Assume (a) fails. By the equivalence of (a) and (d) which has been shown, we may assume that $M_1 \not\subseteq S_1$ and that all $S_1$-modules are endomorphal. Let $X$ be a non-singular $R$-module, and decompose $E(X) = \bigoplus_{i=1}^{n} X_i$ where each $X_i$ is an $S_i$-module. If $\alpha: M_1 \rightarrow X$ is $R$-homogeneous, then $\alpha(M_1) \subseteq X_1$. Thus, $\alpha$ is a $S_1$-homogeneous map from $M_1$ to $X_1$. Observe that $\alpha$ induces an $S_1$-homogenous map $\beta: M_1 \oplus X_1 \rightarrow M_1 \oplus X_1$ by $\beta(m, x) = (0, \alpha(m))$ for all $m \in M_1$ and $x \in X_1$. Since all $S_1$-modules are endomorphal, $\beta$, and hence $\alpha$, is additive. This shows that $M_1$ is a ray for the class of all non-singular modules. In particular, $M_1 \subseteq Q(R)$ by (c) and Theorem 2.1, which is not possible.

We want to remind the reader that condition (a) in the last result is equivalent to the requirement that every non-singular module $M$ which is an $E(M)$-ray is isomorphic to a submodule of $Q(R)$ by Theorem 2.1.

We now determine the semi-prime right Goldie-rings $R$ for which all non-singular modules are endomorphal.

**Theorem 3.5.** The following conditions are equivalent for a ring $R$ such that $R$ is a semi-prime right Goldie-ring:

(a) Every non-singular $R$-module is a ray for the class of all non-singular modules.

(b) Every non-singular $R$-module $M$ is an $E(M)$-ray.

(c) Every non-singular $R$-module is endomorphal.

(d) $Q(R) \cong \prod_{i=1}^{n} \text{Mat}_m(D_i)$ where $D_i$ is a division algebra and $m_i > 1$ for $i = 1, \ldots, n$.

**Proof.** Observe that implications (a) $\Rightarrow$ (b) and (b) $\Rightarrow$ (c) are obvious.

(c) $\Rightarrow$ (d): Since $R$ is a semi-prime Goldie-ring, $Q(R) \cong \prod_{i=1}^{n} S_i$ where $S_i = \text{Mat}_m(D_i)$ for some division algebra $D_i$. If $m_i = 1$ for some $i$, then $S_i$ itself is the only endomorphal $S_i$-module by Theorems 2.1 and 3.4. On the other hand, $S_i \times S_i$ is a non-singular $R$-module on which $S_i$ operates trivially for all $j \neq i$. Therefore, $S_i \times S_i$ is endomorphal as an $R$-module. The resulting contradiction verifies (d).

(d) $\Rightarrow$ (a): By [1], we obtain that every $Q(R)$-module is endomorphal. If $M$ is a non-singular, injective $R$-module, then $M$ can be viewed as a $Q(R)$-module.

Let $M$ and $N$ be non-singular modules, and consider an $R$-homogenous map $\sigma: M \rightarrow N$. We show that $\sigma$ extends to an $R$-homogeneous map $\tau: E(M) \rightarrow E(N)$. To define $\tau$, let $x \in E(M)$, and choose a regular element $c \in R$ such that $xc \in M$. Since $E(N)$ is a non-singular $Q(R)$-module, there is a unique $y \in E(N)$ with $\sigma(xc) = yc$. We claim that $y$ is independent of the
chosen regular element \( c \). If \( d \) is a regular element of \( R \) with \( xd \in M \) and \( z \in E(M) \) satisfies \( \sigma(xd) = zd \), then there are \( r, s \in R \) with \( s \) regular such that \( c^{-1}d = rs^{-1} \). Since \( ds = cr \) is regular, we obtain that \( zds = \sigma(xd)s = \sigma(xds) = \sigma(xcr) = ycr \) if we observe that \( E(N) \) is a non-singular \( R \)-module. If we set \( \tau(x) = y \), then \( \tau \) extends \( \sigma \). It remains to show that \( \tau \) is \( R \)-homogeneous.

If \( x \in E(M) \) and \( r \in R \), then there are regular elements \( c, d \in R \) with \( xc, xrd \in M \). Choose \( s, t \in R \) with \( t \) regular such that \( c^{-1}xrd = st^{-1} \). The equation \( rdt = cs \) yields \( \tau(xr)dt = \sigma(xrdt) = \sigma(xcs) = \sigma(xc)s = \tau(x)cs \). Since \( dt \) is regular, \( \tau(xr) = \tau(x)r \).

To show that \( \sigma \) is additive, it is enough to show this for \( \tau \). As in the proof of Theorem 3.4, we view \( \tau \) as a \( Q(R) \)-homogeneous map from \( E(M) \) to \( E(N) \). Define an \( R \)-homogeneous map \( \lambda: E(M) \otimes E(N) \to E(M) \otimes E(N) \) by \( \lambda(x, y) = (0, \tau(x)) \). Since all \( Q(R) \)-modules are endomorphical by [1, Theorem II.8], \( \lambda \) is additive, and the same holds for \( \tau \).

**Corollary 3.6.** There exists a semi-prime right Goldie-ring \( R \) such that not all \( R \)-modules are endomorphical, although all non-singular \( R \)-modules are endomorphical.

**Proof.** Let \( R = \left[ \begin{array}{cc} \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{array} \right] \). Since \( \mathbb{Q} \otimes_{\mathbb{Z}} R = \text{Mat}_2(\mathbb{Q}) \) is the classical right ring of quotient of the semi-prime Goldie-ring \( R \), every non-singular right \( R \)-module is endomorphical by the last theorem.

Assume that all \( R \)-modules are endomorphical. Observe that \( I = \left[ \begin{array}{cc} \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{array} \right] \) is a two-sided ideal of \( R \). It is easy to see that

\[
\frac{R}{I} \cong \left[ \begin{array}{cc} \mathbb{Z}/2\mathbb{Z} & 0 \\ \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} \end{array} \right].
\]

If \( N(R/I) \) denotes the nilradical of \( R/I \), then \( S = (R/I)/N(R/I) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \) is a ring epimorphic image of \( R \). By [1, Theorem II.2], every \( S \)-module \( M \) is endomorphical. Since \( S \) is semi-simple Artinian, \( M \) is injective, and hence an \( E(M) \)-ray. By Theorem 3.4, \( M \) is isomorphic to a submodule of \( Q(S) = S \), which yields a contradiction.

It is furthermore possible to characterize the right non-singular rings \( R \) for which the non-singular endomorphical modules are precisely the non-singular modules of Goldie-dimension 1.

**Theorem 3.7.** The following conditions are equivalent for a right non-singular ring \( R \):

(a) \( R \) has Goldie-dimension at most 1.
(b) The following conditions are equivalent for a non-singular right \( R \)-module \( M \):
   (i) \( M \) is endomorphical.
   (ii) \( M \) is a ray for the class of all non-singular modules.
   (iii) \( M \) has Goldie-dimension at most 1.

**Proof.** (a) \( \Rightarrow \) (b): Observe that a right non-singular ring of right Goldie-dimension 1 has to be a semi-prime Goldi-ring. It remains to show that (i) \( \Rightarrow \) (iii).

Consider a non-singular endomorphical \( R \)-module \( M \), and assume that \( U \) and \( V \) are non-zero submodules of \( M \) with \( U \cap V = 0 \). Let \( \tilde{U} \) and \( \tilde{V} \) be the
\( \mathcal{P} \)-closures of \( U \) and \( V \) in \( M \) respectively. Observe that \( \tilde{U}/U = Z(M/U) \).

Suppose that there are \( x \in M \) and \( r \in R \) such that \( 0 \neq xr \in \tilde{U} \). There is an essential right ideal \( I \) of \( R \) with \( xrI \subseteq U \). If \( s \in I \), then \( xrs \in U \), and \( sR \) is an essential right ideal of \( R \) since \( R \) has right Goldie-dimension 1. Since \( R \) is a right non-singular ring, we have \( rs \neq 0 \). Therefore, \( rsR \) is an essential right ideal of \( R \) with \( xrsR \subseteq U \). Therefore, \( x \in \tilde{U} \).

By [3], the map \( \phi: M \to M \) which is defined by

\[
\phi(x) = \begin{cases} 
  x & \text{if } x \in \tilde{U} \cup \tilde{V}, \\
  0 & \text{otherwise}
\end{cases}
\]

is \( R \)-homogeneous. Choose non-zero elements \( a \in U \) and \( b \in V \). If \( a+b \in \tilde{U} \), then there is a non-zero \( r \in R \) such that \( (a+b)r \in U \). Then, \( br \in U \cap V = 0 \). Since \( rR \) is essential in \( R \), we have \( b \in Z(M) = 0 \), which is not possible. Therefore, \( a+b \notin \tilde{U} \cup \tilde{V} \). This shows

\[
\phi(a+b) = 0 \neq a + b = \phi(a) + \phi(b).
\]

Thus, \( M \) is not endomorphal, a contradiction.

\( (b) \Rightarrow (a) \): Observe that \( R \) is a non-singular endomorphal module. Thus, \( R \) has Goldie-dimension at most 1 by (b).

**Corollary 3.8.** Let \( R \) be a right non-singular ring of Goldie-dimension 1.

(a) The class of endomorphal, non-singular \( R \)-modules is closed under submodules and essential extensions.

(b) An endomorphal module \( M \) is either non-singular or satisfies \( Z_2(M) = M \).

**Proof.** (a) Observe that the class of non-singular modules of Goldie-dimension at most 1 is closed under submodules and essential extensions.

(b) Let \( M \) be an endomorphal module with \( 0 \neq Z(M) \neq M \). Suppose that \( 0 \neq xr \in Z(M) \) for some \( x \in M \) and \( r \in R \). There is an essential right ideal \( J \) of \( R \) with \( xrJ = 0 \). Since \( R \) has Goldie-dimension 1 and is non-singular, we have that the non-zero right ideal \( rJ \) of \( R \) is essential. Thus, \( x \in Z(M) \). Therefore, the map \( \phi: M \to M \) which is defined by

\[
\phi(x) = \begin{cases} 
  x & \text{if } x \in Z(M), \\
  0 & \text{otherwise}
\end{cases}
\]

is \( R \)-homogeneous. Choose \( a \in M \setminus Z(M) \) and \( 0 \neq b \in Z(M) \). Suppose that there is an essential right ideal \( I \) of \( R \) with \( (a+b)I = 0 \). Without loss of generality, we can assume that \( I \) satisfies \( bI = 0 \) too. Then, \( aI = 0 \), which is not possible. Therefore, \( \phi(a+b) = 0 \neq b = \phi(a) + \phi(b) \).

We conclude with a few results concerning singular modules.

**Corollary 3.9.** Let \( R \) be a semi-prime right and left Goldie-ring. Then, \( Q(R)/R \) is a \( Q(R)/R \)-ray.

**Proof.** By Theorem 2.2, \( Q(R) \) is a locally cyclic \( R \)-module. Since the class of locally cyclic modules is closed under epimorphic images, the same holds for \( Q(R)/R \). Now apply Proposition 3.1.
**Corollary 3.10.** Let $R$ be a semi-prime right hereditary right and left Goldie-ring. Every uniform injective module is endomorphical.

**Proof.** Let $M$ be a non-zero uniform injective module, and $\phi: R \to M$ a non-zero map. Since $M$ is injective, $\phi$ extends to a map $\psi: Q(R) \to M$. The fact that $M$ is uniform implies $\psi(Q(R))$ is an essential submodule of $M$. Moreover, since $R$ is right hereditary, $\psi(Q(R))$ is injective too. This is only possible if $\psi(Q(R)) = M$. Since $Z(R) = 0$, we obtain that $Q(R)$ is locally cyclic by Theorem 2.2. The observation that the class of locally cyclic modules is closed with respect to homomorphic images yields that $M$ is locally cyclic. In particular, $M$ is endomorphical.

We want to point out that some of the results of this paper can be extended by considering conditions on the ring $R/Z_2(R)$ instead of assuming that $R$ is right non-singular. However, since the categories of non-singular right $R$- and non-singular right $R/Z_2(R)$-modules coincide, we have abstained from this generalization for the sake of a less complicated exposition.

**REFERENCES**


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