INFINITE DIFFERENTIABILITY
IN POLYNOMIALLY BOUNDED O-MINIMAL STRUCTURES

CHRIS MILLER

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Abstract. Infinitely differentiable functions definable in a polynomially bounded o-minimal expansion \( R \) of the ordered field of real numbers are shown to have some of the nice properties of real analytic functions. In particular, if a definable function \( f : \mathbb{R}^n \to \mathbb{R} \) is \( C^N \) at \( a \in \mathbb{R}^n \) for all \( N \in \mathbb{N} \) and all partial derivatives of \( f \) vanish at \( a \), then \( f \) vanishes identically on some open neighborhood of \( a \). Combining this with the Abhyankar-Moh theorem on convergence of power series, it is shown that if \( R \) is a polynomially bounded o-minimal expansion of the field of real numbers with restricted analytic functions, then all \( C^\infty \) functions definable in \( R \) are real analytic, provided that this is true for all definable functions of one variable.

Introduction

Throughout this note, \( R \) denotes a fixed (but arbitrary) expansion of the structure \( (\mathbb{R}, +, \cdot) \) in a first-order language extending \( \{+, \cdot\} \). Definable means first-order definable in \( R \) with parameters from \( \mathbb{R} \). A function \( f : X \to \mathbb{R} \), \( X \subseteq \mathbb{R}^n \), is said to be definable if its graph is definable. Whenever convenient, we may assume that we deal with totally defined functions by setting a definable function equal to 0 off its domain of definition. We say that \( R \) is polynomially bounded if for every definable function \( f : \mathbb{R} \to \mathbb{R} \) there exists \( N \in \mathbb{N} \) such that \( |f(t)| \leq t^N \) for all sufficiently large positive \( t \). We say that \( R \) is o-minimal if the definable subsets of \( \mathbb{R} \) are just the finite unions of intervals of all kinds, including singletons.

The structure \( (\mathbb{R}, <, 0, 1, +, -, \cdot) \) is polynomially bounded and o-minimal (by Tarski-Seidenberg); the sets definable in this structure are precisely the semialgebraic sets. (See [BCR] for a thorough treatment of semialgebraic sets.)

A polynomially bounded o-minimal structure in which non-semialgebraic sets are definable, due to Denef and van den Dries (see [DD] and [D]), is the ordered field of real numbers with restricted analytic functions

\[ \mathbb{R}_{\text{an}} := (\mathbb{R}, <, 0, 1, +, -, \cdot, (f^m)_{f \in \mathbb{R}(X, m), m \in \mathbb{N}}), \]

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where \( \mathbb{R}\{X, m\} \) denotes the ring of all power series in \( X_1, \ldots, X_m \) over \( \mathbb{R} \) that converge in a neighborhood of \([-1, 1]^m\), and where for each \( f \in \mathbb{R}\{X, m\} \) we define \( \hat{f} : \mathbb{R}^m \to \mathbb{R} \) by

\[
\hat{f}(x) := \begin{cases} 
  f(x), & x \in [-1, 1]^m, \\
  0, & x \in \mathbb{R}^m \setminus [-1, 1]^m.
\end{cases}
\]

The sets definable in \( \mathbb{R}_{an} \) are the finitely subanalytic sets introduced in [D]; these are locally just like subanalytic sets, but have nicer global properties and behave better from a logical viewpoint. (See [BM] for general facts about subanalytic sets.)

At present, the largest known polynomially bounded o-minimal expansion of \( (\mathbb{R}, +, \cdot) \) is the structure \( (\mathbb{R}_{an}, (x \mapsto x^r)_{r \in \mathbb{R}}) \), where we set \( x^r := 0 \) for \( x \leq 0 \) (see [M2]). The class of sets definable in this structure properly contains the class of finitely subanalytic sets; by [D] or [M2], the function \( x \mapsto x^r : (0, +\infty) \to \mathbb{R} \) is definable in \( \mathbb{R}_{an} \) if and only if \( r \) is rational.

Polynomially bounded o-minimal expansions of \( (\mathbb{R}, +, \cdot) \) are becoming increasingly important objects of study. In this note, we establish some basic differentiability properties of functions definable in such structures.

**Uniform bounds on orders of vanishing**

Before we can state the main result, we need some further definitions and notational conventions.

Let \( U \) be an open subset of \( \mathbb{R}^n \) \( (n \geq 1) \), and let \( f : U \to \mathbb{R} \) be given. We say that \( f \) is \( C^\infty \) at \( a \in U \) if \( f \) is \( C^\infty \) on some open neighborhood of \( a \), and that \( f \) is analytic at \( a \) if \( f \) is (real) analytic on an open neighborhood of \( a \). We say that \( f \) is weak-\( C^\infty \) at \( a \in U \) if for every \( N \in \mathbb{N} \), \( f \) is \( C^N \) at \( a \) (i.e., there exists an open neighborhood \( U_N \) of \( a \), \( U_N \subseteq U \), such that \( f \upharpoonright U_N \) is \( C^N \)). If \( f \) is \( C^N \) at some \( a \in U \) and all partial derivatives of \( f \) of order less than or equal to \( N \) (including \( f \) itself) vanish at \( a \), we say that \( f \) is \( N \)-flat at \( a \). If \( f \) is \( N \)-flat at \( a \) for all \( N \in \mathbb{N} \), then \( f \) is said to be flat at \( a \).

Let \( U \subseteq \mathbb{R}^n \) be a definable open set. Then \( C^\infty_{df}(U) \) denotes the ring of definable functions \( f : U \to \mathbb{R} \) that are \( C^\infty \) on \( U \). (Note that \( f \) is \( C^\infty \) on \( U \) if and only if \( f \) is weak-\( C^\infty \) at each \( a \in U \).)

For \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \), \( |x| \) denotes \( \max\{|x_1|, \ldots, |x_n|\} \).

Given a set \( A \subseteq \mathbb{R}^{m+n} \) \( (m, n \geq 1) \) and \( x \in \mathbb{R}^m \), we put \( A_x := \{ y \in \mathbb{R}^n : (x, y) \in A \} \). For a function \( f : A \to \mathbb{R} \) and \( x \in \mathbb{R}^m \), if \( A_x \neq \emptyset \), then \( f(x, -) \) denotes the function \( y \mapsto f(x, y) : A_x \to \mathbb{R} \). For convenience, we also allow the possibility that \( m = 0 \), in which case the obvious interpretations apply.

We come now to the main technical result of this paper:

**Theorem.** Assume that \( \mathcal{R} \) is polynomially bounded and o-minimal. Let \( f : A \to \mathbb{R} \) be definable, with \( A \subseteq \mathbb{R}^{m+n} \) \( (m \geq 0 \text{ and } n \geq 1) \). Then there exists \( N \in \mathbb{N} \) such that for all \( (x, y) \in A \), if \( y \) is in the interior of \( A_x \) and \( f(x, -) \) is \( N \)-flat at \( y \), then \( f(x, -) \) vanishes identically in a neighborhood of \( y \).

Note that in the special case that \( m = 0 \) and \( A \) is open, we have that for all \( y \in A \), if \( f \) is flat at \( y \), then \( f \) vanishes identically in a neighborhood of \( y \).

We will require for the proof the following result from [M2]:

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Let $g : B \times \mathbb{R} \to \mathbb{R}$ be definable, $B \subseteq \mathbb{R}^p \ (p \geq 0)$. Then there exist $r_1, \ldots, r_l \in \mathbb{R}$ such that for all $b \in \mathbb{R}^p$, either $g(b, t) = 0$ for all sufficiently large (depending on $b$) positive $t$, or $\lim_{t \to +\infty} g(b, t)/t^{r_i} = c$, for some $i \in \{1, \ldots, l\}$ and $c \in \mathbb{R} \setminus \{0\}$.

**Proof of the Theorem.** Let $F : A \times (0, \infty) \to \mathbb{R}$ be the definable function given by

$$F(x, y, t) := \begin{cases} \max_{z \in A_x, |y-z|=t} |f(x, z)| & \text{if } \{z \in A_x : |y-z|=t\} \neq \emptyset, \\ 1 & \text{otherwise.} \end{cases}$$

Applying (*) to $(x, y, t) \to F(x, y, 1/t)$, there exist $r_1, \ldots, r_l \in \mathbb{R}$ such that for all $(x, y) \in A$, either $F(x, y, t) = 0$ for all sufficiently small (depending on $(x, y)$) positive $t$, or $\lim_{t \to 0^+} F(x, y, t)/t^{r_i} = c$ for some $i \in \{1, \ldots, l\}$. Choose $N \in \mathbb{N}$ with $N > \max\{r_1, \ldots, r_l\}$. Suppose that $(x, y) \in A$ is such that $y$ is in the interior of $A_x$ and $f(x, -)$ is $N$-flat at $y$. By Taylor's formula, $|f(x, z)| = O(|y-z|^N)$ as $|y-z| \to 0^+$; i.e., $F(x, y, t) = O(t^N)$ as $t \to 0^+$. Since $N > \max\{r_1, \ldots, r_l\}$, we must have $\lim_{t \to 0^+} F(x, y, t)/t^{r_i} = 0$ for $i \in \{1, \ldots, l\}$. Thus, $F(x, y, t) = 0$ for all sufficiently small positive $t$. It follows then (from the definition of $F$) that $f(x, -)$ vanishes identically in a neighborhood of $y$. \hfill \Box

**Remark.** The assumption that $\mathcal{R}$ be polynomially bounded is necessary. By [M1], if $\mathcal{R}$ is o-minimal and not polynomially bounded, then the exponential function $e^x$ is definable. Thus, the conclusion of the theorem fails by the classic counterexample

$$f(t) = \begin{cases} e^{-1/t}, & t > 0, \\ 0, & t \leq 0. \end{cases}$$

**Corollary 1.** Assume that $\mathcal{R}$ is polynomially bounded and o-minimal. Let $U \subseteq \mathbb{R}^n$ be an open connected definable set.

1. If $f \in C^\infty_{df}(U)$ is flat at some $a_0 \in U$, then $f = 0$; i.e., $C^\infty_{df}(U)$ is a quasianalytic class.
2. $C^\infty_{df}(U)$ is an integral domain.

**Proof.** (1) Consider the definable open set $A$ consisting of all $a \in U$ such that $f \upharpoonright V = 0$ for some open $V \subseteq U$ with $a \in V$. By the Theorem (with $m = 0$), $a_0 \in A$. Let $a \in Cl(A) \cap U$. All partials of $f$ are continuous on $U$ and vanish identically on $A$. Then all partials of $f$ vanish at $a$; i.e., $f$ is flat at $a$. By the Theorem, $a \in A$. Thus, $A$ is both open and closed in $U$, so $A = U$.

(2) Let $f, g \in C^\infty_{df}(U)$ with $fg = 0$. If $g(a) \neq 0$ for some $a \in U$, then $f$ vanishes identically in a neighborhood of $a$; hence $f = 0$ by (1). \hfill \Box

**Definable germs**

Given $a \in \mathbb{R}^n$ we define an equivalence relation $\sim$ on the set of real-valued functions whose domain contains a neighborhood of $a$ by $f \sim g$ if there is a neighborhood $V$ of $a$, $V \subseteq \text{dom}(f) \cap \text{dom}(g)$, such that $f \upharpoonright V = g \upharpoonright V$. The equivalence classes are called germs at $a$. The equivalence classes of definable functions that are weak-$C^\infty$ at $a$ are called definable weak-$C^\infty$ germs at $a$.

These germs can be added and multiplied in the usual way and are easily seen to
form a local ring with maximal ideal the germs vanishing at $a$. We denote it by $D^w_a$. We also let $D^\omega_a$ (respectively, $D^\infty_a$) denote the local rings of definable $C^\infty$ (respectively, definable analytic) germs at $a$. For $a = 0 \in \mathbb{R}^n$, we write $D^w(n)$, $D^\infty(n)$, and $D^\omega(n)$ as appropriate. Clearly, $D^\omega_a \subseteq D^\infty_a \subseteq D^w_a$ for all $a \in \mathbb{R}^n$.

**Proposition 1.** If $\mathfrak{R}$ is a polynomially bounded o-minimal expansion of $\mathbb{R}$ such that $D^w(1) = D^\omega(1)$, then $D^w(n) = D^\omega(n)$ for all $n \geq 1$.

**Proof.** By induction on $n$. The base case holds by assumption, so suppose that the result holds for $n$. Let $f : \mathbb{R}^{n+1} \to \mathbb{R}$ be definable and weak-$C^\infty$ at $0 \in \mathbb{R}^{n+1}$. Then for every $r \in \mathbb{R}$ the definable function $(x_1, \ldots, x_n) \mapsto f(x_1, \ldots, x_n, rx_n) : \mathbb{R}^n \to \mathbb{R}$ is weak-$C^\infty$ at $0 \in \mathbb{R}^n$. By the inductive assumption, each such function is analytic at $0 \in \mathbb{R}^n$. By Abhyankar-Moh [AM], the Taylor series of $f$ at $0 \in \mathbb{R}^{n+1}$ then converges on some neighborhood $U$ of $0 \in \mathbb{R}^{n+1}$ to an analytic function $g : U \to \mathbb{R}$. Now for some $r > 0$, $g \mid [-r, r]^{n+1}$ is definable in $\mathfrak{R}$. By Corollary 1(1) we have $f \mid (-r, r)^{n+1} = g \mid (-r, r)^{n+1}$. □

**Remark.** Clearly, in the above one can replace "$D^w_a$" and "weak-$C^\infty$" by "$D^\infty_a$" and "$C^\infty$", respectively.

In [M2], it is shown that $D^w(1) = D^\omega(1)$ for $(\mathbb{R}, (x \mapsto x^r)_{r \in \mathbb{R}})$. Thus, given any function $f : U \to \mathbb{R}$ definable in $(\mathbb{R}, (x \mapsto x^r)_{r \in \mathbb{R}})$, $U$ open in $\mathbb{R}^n$, if $f$ is weak-$C^\infty$ at $a$ then $f$ is analytic at $a$.

**Corollary 2.** Let $\mathfrak{R}$ be polynomially bounded and o-minimal. Then the function $T : D^w(\mathfrak{R}) \to \mathbb{R}$ sending the germ at $0$ of a definable function $f : \mathbb{R}^n \to \mathbb{R}$, weak-$C^\infty$ at $0$, to its formal Taylor expansion at $0$, is an injective ring homomorphism.

**Proof.** That $T$ is a ring homomorphism is routine. By the Theorem, the kernel of $T$ is the germ of the zero map $0 : \mathbb{R}^n \to \mathbb{R}$. □

Corollary 2 also holds with "$D^\infty(\mathfrak{R})$" in place of "$D^w(\mathfrak{R})". It thus follows that if $\mathfrak{R}$ is polynomially bounded and o-minimal, then $D^w(\mathfrak{R})$ and $D^\infty(\mathfrak{R})$ are integral domains. (Of course, this is true for $D^\omega(\mathfrak{R})$ without assumptions that $\mathfrak{R}$ be polynomially bounded or o-minimal.)

**Proposition 2.** The maximal ideals of $D^w(\mathfrak{R})$, $D^\infty(\mathfrak{R})$, and $D^\omega(\mathfrak{R})$ are each generated by the germs at $0$ of the coordinate functions $x \mapsto x_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, n$.

**Proof.** We do only the case of $D^w(\mathfrak{R})$; the others are similar.

First, suppose that $f : U \times V \to \mathbb{R}$ is definable and $C^{N+1}$ for some $N \in \mathbb{N}$, with $U \times V \subseteq \mathbb{R}^m \times \mathbb{R}$ an open box neighborhood of $0 \in \mathbb{R}^{m+1}$, $m \geq 0$. Consider the definable function $g : U \times V \to \mathbb{R}$ with

$$g(x, y) = \begin{cases} f(x, y) - f(x, 0), & y \neq 0, \\ \frac{\partial f}{\partial y}(x, 0), & y = 0. \end{cases}$$

Given $(x, y) \in U \times V$ we have

$$f(x, y) - f(x, 0) = \int_0^1 \frac{d}{dt}(f(x, ty)) \, dt = y \int_0^1 \frac{\partial f}{\partial y}(x, ty) \, dt.$$
Thus, $g$ is $C^N$ on $U \times V$. Note that $f(x, y) = f(x, 0) + yg(x, y)$ for all $(x, y) \in U \times V$. Using this fact, an easy induction on $n$ yields the result. □

A local ring $R$ with maximal ideal $M$ is called Henselian if given $P \in R[T]$ and $a \in R$ with $P(a) \in M$ and $P'(a)$ invertible, there exists $b \in R$ with $P(b) = 0$ and $a \equiv b \bmod M$. It is easy to see that the implicit function theorems ($C^N$, $C^\infty$ and analytic versions) yield definable functions when the data are definable; thus $D^{wk}(n)$, $D^\infty(n)$ and $D^\omega(n)$ are Henselian rings.

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Department of Mathematics, University of Illinois, Urbana, Illinois 61801

*Current address*: Department of Mathematics, Statistics and Computer Science, University of Illinois, Chicago, Illinois 60607-7045

*E-mail address*: miller@math.uiuc.edu