COMPLETENESS OF METRIZABLE PRE-IMAGES 
OF VAN DOUWEN-COMPLETE SPACES

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Abstract. We shall show the recurrence of complete metrizability of irreducible closed pre-images of van Douwen-complete spaces. As its corollary we shall show that every van Douwen-complete space is $G_δ$ in any Češněv space.

1. Introduction and preliminaries

All spaces in this paper are assumed to be Tychonoff, and all maps are assumed to be continuous. A space $Y$ is a Češněv (respectively, van Douwen-complete) space provided that there exists a closed onto map $f : X → Y$, where $X$ is a (respectively, completely) metrizable space. By the following fact, it is equivalent to also require that the map $f$ be irreducible, i.e., $f(A) ≠ Y$ for any proper closed subset $A$ of $X$.

Fact 0 [5]. For every closed mapping $f : X → Y$ from a metric space $X$ onto $Y$, there exists a closed set $A$ of $X$ such that $f|A : A → Y$ is a closed irreducible onto map.

It is known that van Douwen-complete spaces have some properties in common with Češ-complete spaces. For example, the Baire category theorem is valid for both of them [9, 13] (see also Remark 1(b)). On the other hand, a van Douwen-complete space cannot be Češ-complete unless it is metrizable (see also Example 2 below). Hence, Češ-completeness is not preserved by a closed irreducible mapping. However, our main theorem shows that it remembers completeness in the sense that arbitrary closed irreducible metrizable pre-images must be complete.

Theorem 1. Let $f : M → Y$ be a closed irreducible onto map from a complete (in particular, locally compact) metric space $M$, and suppose that $h : X → Y$ is an arbitrary closed irreducible map from a metric space $X$ onto $Y$. Then $X$ is completely (respectively, locally compact) metrizable.

The following three facts are fundamental tools for a proof of the above theorem.
Fact 1 [3, Lemma 4.4.16]. Let \( f : X \to Y \) be a closed map of a metric space onto a space \( Y \). Then, \( \text{Bdry } f^-(y) \) is compact if \( y \) has a countable neighborhood base.

Fact 2 [6]. Let \( f : X \to Y \) be a closed map from a paracompact space \( X \) onto a space \( Y \). Then, for any compact \( K \subset Y \), there exists a compact \( A \subset X \) such that \( f(A) = K \).

Fact 3 [3, Theorems 3.7.21, 3.7.24 and 3.9.10]. If \( X \) and \( Y \) are Tychonoff spaces and there exists a perfect mapping from \( X \) onto \( Y \), then \( X \) is Čech-complete (respectively, locally compact) if and only if \( Y \) is Čech-complete (respectively, locally compact).

2. Proofs of our results

Lemma 1. Let \( f : M \to Y \) be a closed irreducible onto map from a complete metric space \( M \), and suppose that \( h : X \to Y \) is a closed irreducible onto map from a metric space \( X \). Then, there exists a dense \( G_δ \)-subset \( Y_0 \) of \( Y \) such that

(a) both \( f|M_0 : M_0 \to Y_0 \) and \( h|X_0 : X_0 \to Y_0 \) are homeomorphisms,

where \( M_0 = f^-(Y_0) \) and \( X_0 = h^-(Y_0) \);

(b) \( y \in Y_0 \), whenever \( \text{Int}(h^-(y)) \neq \emptyset \).

Proof. For each \( n \geq 1 \) let \( G_n = \{ y \in Y : \text{diam } (f^-(y)) < 1/n \} \), and let \( H_n = \{ y \in Y : \text{diam } (h^-(y)) < 1/n \} \). Then, at first we shall show that \( H_n \) is open in \( Y \). Since \( h \) is a closed map, for a given \( y_0 \in H_n \), there exists an open neighborhood \( V \) in \( Y \) satisfying that

\[ h^-(V) \subset B_ε(h^-(y_0)) , \]

where \( ε = 1/2n - d/2 \), \( d = \text{diam } (h^-(y_0)) \), and \( B_ε(A) \) is the \( ε \)-neighborhood of the set \( A \). Then, \( V \subset H_n \) by the direct computation of the diameters of the fibers of each \( y \in V \).

Next we shall show that \( f^-(H_n) \) is dense in \( M \). Indeed, let \( U \) be an arbitrary non-empty open set of \( M \). Then, there exists a fiber \( f^-(y) \subset U \), since \( f \) is irreducible. Hence, there exists an open neighborhood \( V \) of \( y \) in \( Y \) such that \( f^-(V) \subset U \). For the open set \( h^-(V) \), take a non-empty open subset \( U' \) in \( X \) such that \( U' \subset h^-(V) \) and diam \( U' < 1/n \). Then, by a parallel argument applying for the open set \( U' \) and the map \( h \), we have a non-empty open set \( V' \) of \( Y \) such that \( h^-(V') \subset U' \). Note that \( V' \subset H_n \). Therefore,

\[ U \cap f^-(H_n) \supset f^-(V') \neq \emptyset . \]

By a parallel argument each \( f^-(G_n) \) is also open dense in \( M \) (see also [10]). Let

\[ M_0 = (\cap_n f^-(G_n)) \cap (\cap_n f^-(H_n)) \quad \text{and} \quad Y_0 = f(M_0) . \]

Then by the Baire category theorem, \( M_0 \) is a dense subset of \( M \), since \( M \) is complete. Therefore (a) holds by the definition of \( G_n \) and \( H_n \). The set \( Y_0 \) is a dense \( G_δ \) subset of \( Y \), since \( f \) is continuous and the following equality holds:

\[ Y_0 = (\cap_n G_n) \cap (\cap_n H_n) . \]

Suppose that \( \text{Int}(h^-(y)) \neq \emptyset \). Then, it is a one-point set by the irreducibility of \( h \). Hence, \( h^-(y) \) is an isolated point of \( X \) and \( y \) is also an isolated point in \( Y \) since \( h \) is a closed map. This completes (b), since \( Y_0 \) is dense in \( Y \).
Proof of Theorem 1. Let $X_0$, $M_0$, and $Y_0$ be the dense $G_δ$-subsets, satisfying Lemma 1. Put $Z_0 = \{(t, x) \in M \times X : f(t) = h(x) \in Y_0\}$ and $Z = \{(t, x) \in M \times X : f(t) = h(x)\}$. Let $\alpha : Z \to M$ and $\beta : Z \to X$ be the restrictions of natural projections (i.e., $\alpha(t, x) = t$ and $\beta(t, x) = x$). Then, by Fact 3 it suffices to show that both of $\alpha$ and $\beta$ are perfect onto maps.

(i) $\alpha$ is onto. Suppose that $t \in M \setminus M_0$. Since $M_0$ is dense in $M$, there exists a sequence $\{t_n\} \subset M_0$ converging to $t$. Put $y_n = f(t_n)$ and $y = f(t)$. Then by Fact 2 there exists a compact set $K \subset X$ such that

$$h(K) = \{y\} \cup \{y_n : n \in \omega\}.$$ 

Since $t \in M \setminus M_0$, we can assume that $\{y_n\}$ consists of infinitely many points. Let $x_n$ be the unique point of the one-point set $K \cap h^{-1}(y_n)$ for each $n$. Then there exists a subsequence $\{x_{n_k}\}$ converging to some $x \in K$. Hence, $(t, x) \in Z$ and $\alpha(t, x) = t$.

(ii) $\alpha$ is a closed map. Assume that $F$ is closed in $Z$, and suppose that $t \in \text{cl}(\alpha(F))$. Take a sequence $\{t_n\} \subset \alpha(F)$ converging to $t$. Then we can assume that $\{t_n\}$ consists of infinitely many distinct points, since otherwise $t = t_n \in \alpha(F)$ for some $n$. Hence, let

$$F = \bigcup_{n \in \omega} U_n.$$ 

For each $n \in \omega$, take a point $z_n \in F$ such that $t_n = \alpha(z_n)$. Let $V_n$ be the $1/2^n$-neighborhood of $\beta(z_n)$ with respect to some metric of $X$. Take a point

$$x_n = (s_n, x_n) \in Z_0 \cap (U_n \times V_n),$$ 

since $z_n \in \text{cl}(Z_0)$. Then, by Lemma 1(a) it holds that

$$s_n = f^{-1}(f(s_n)) \in U_n, \quad x_n = h^{-1}(h(x_n)) \in V_n, \quad \text{and} \quad f(s_n) = h(x_n).$$

Note also that $\{s_n\}$ also converges to $t$, and hence let $K$ be a compact subset of $X$ such that (1) holds, where $y = f(t)$ and $y_n = f(s_n) = h(x_n)$.

By (2) and (4), it holds that $\{y_n\}$ consists of infinitely many distinct points. Let $x_n$ be the unique point of the one-point set $K \cap h^{-1}(y_n)$ for each $n \in \omega$. Then, there exists a subsequence $\{x_{n_k}\}$ converging to some $x \in K$. Hence, $(t, x) \in Z$ and $(t, x) \in F$, since $\{w_n = (s_{n_k}, x_{n_k})\} \subset Z_0$ converges to $(t, x)$ and $\{z_{n_k}\}$ also converges to $(t, x)$ by (3) and (4). Hence, $t = \alpha(t, x) \in \alpha(F)$.

(iii) Each $\alpha^{-1}(t)$ is compact. We can assume that $t \notin M_0$, since $\alpha^{-1}(t) = f^{-1}(f(t)) \times h^{-1}(f(t))$ is a one-point set when $t \in M_0$. By Fact 1 it suffices to show that $\text{Int}(\alpha^{-1}(t)) = \emptyset$. Suppose that $U$ and $V$ are open sets of $M$ and $X$, respectively, such that

$$K = (U \times V) \cap \alpha^{-1}(t) \neq \emptyset.$$ 

Take a point $z \in K$. Then, $Z_0 \cap (U \times V) = U \times V \cap (Z_0 \setminus \alpha^{-1}(t)) \neq \emptyset$, since $z \in \text{cl}(Z_0)$ and $z \in U \times V$. Therefore, $\text{Int}(\alpha^{-1}(t)) = \emptyset$.

(iv) It can be shown by a parallel way that $\beta$ is a perfect onto map.

Corollary. Let $Y$ and $Z$ be a van Douwen-complete space and a Lašnev space, respectively. Suppose that $Y$ can be embedded as a subset of $Z$. Then, $Y$ is a $G_δ$-subset of $Z$.

Proof. Assume that $Y \subset Z$ and let $q : T \to Z$ be a closed (irreducible) mapping from a metric space $T$ onto $Z$. Then, for the restricted closed mapping $q|q^{-1}(Y) : q^{-1}(Y) \to Y$, let $h : X \to Y$ be an irreducible mapping given by Fact
0, where \( X \) is a closed subset of \( q^{-1}(Y) \). Hence, it holds that

\[
(5) \quad X = q^{-1}(Y) \cap (cl_T X).
\]

By Theorem 1 it holds that \( X \) is completely metrizable and hence the set \( cl_T X \setminus X \) is an \( F_\sigma \)-subset in \( T \). Put \( cl_T X \setminus X = \bigcup_i F_i \), where each \( F_i \) is a closed subset in \( T \). By (5) it holds that \( F_i \subset T \setminus q^{-1}(Y) \) for each \( i \). Since \( q \) is a closed map, it holds that \( q(cl_T X) = cl_Z Y \). Hence, \( W = q(cl_T X \setminus X) = \bigcup_i q(F_i) = cl_Z Y \setminus Y \) is an \( F_\sigma \)-set in \( Z \). Therefore, \( Y = (cl_Z Y) \cap (Z \setminus W) \) is a \( G_\delta \)-subset of \( Z \), since \( cl_Z Y \) is a \( G_\delta \)-subset of \( Z \).

### 3. Remarks and examples

**Remark 1.** (a) There is much flexibility for the topologies of the original spaces, which yield the same quotient space by suitable decompositions (e.g., there are several ways to make a topological torus, using non-homeomorphic original spaces). Even using the same space, there are many different decompositions which yield the same decomposition space. For example, collapsing a single compact set \( A \) in the Euclidian plane \( E^2 \), we have a quotient space \( Y = E^2/A \) which is homeomorphic to the original space \( E^2 \), when \( A \) has the trivial shape (i.e., it has the shape type of a point). On the other hand, there are infinitely many compact nowhere dense subsets \( A \) in \( E^2 \); all of them have the trivial shape, but are mutually non-homeomorphic. It can be said that our main theorem gives a property which every original space has even when the decomposition space is non-metrizable.

(b) Using the following decomposition theorem due to Lašnev [5], Van Doren [13] shows that the Baire category theorem is valid for every van Douwen-complete space.

**Fact 4.** Let \( f : X \to Y \) be an irreducible closed onto map from a metric space \( X \). Then, there exists a subspace \( M \) of \( Y \) such that \( f|f^{-1}(M) \) is a perfect map and \( Y \setminus M \) is \( \sigma \)-discrete in \( Y \).

One of the remarkable points of this decomposition of the quotient space \( Y \) is that the decomposition of Fact 4 does not depend on the maps \( f \) and \( h \) in Lemma 1 by Fact 1. Namely, we have the following equality:

\[
\{y \in Y : f^{-1}(y) \text{ is compact}\} = \{y \in Y : h^{-1}(y) \text{ is compact}\} = \{y \in Y : y \text{ has a countable neighborhood base in } Y\}.
\]

On the other hand, Stricklen Jr. [9] pointed out that the set \( S_0 = \{y \in Y : f^{-1}(y) \text{ is a one-point set}\} \) is a dense \( G_\delta \)-subset of \( Y \) so that the Baire category theorem for \( Y \) also follows from this fact. Unfortunately, the second example \( Y = E^2/A \) in (a) shows that \( S_0 \) is different from the set \( H_0 = \{y \in Y : h^{-1}(y) \text{ is a one-point set}\} \) for an irreducible closed map \( h \), in general. Lemma 1, however, shows that their intersection \( Y_0 = S_0 \cap H_0 \) remains a large set (i.e., it is a dense \( G_\delta \)-subset of \( Y \)). In our proof of Theorem 1 the set \( Y_0 \) plays a crucial role.

(c) A topological property \( P \) is said to be *perfect* provided it is both the *invariant* and *inverse invariant* of every perfect map. For example, completely metrizability and locally compactness are perfect topological properties. By the above proof of our Theorem 1 it holds that: Let \( f : M \to Y \) be a closed irreducible onto map from a complete metric space \( M \) which satisfies a perfect
property \( P \), and suppose that \( h : X \to Y \) is an arbitrary closed irreducible map from a metric space \( X \) onto \( Y \). Then \( X \) also satisfies \( P \).

**Remark 2.** The van Douwen-complete spaces must have some harmonious properties. For example, there exists a universal space \( D_\alpha \) for all van Douwen-complete spaces \( X \) of \( \dim X = 0 \) with the weight \( w(M) \leq \alpha \) [11] (see also [12] for more general results). It is also known [10] that there are no universal spaces without assumption of complete metrizability of \( M \). By virtue of our theorem we can show that \( X \) can be embedded in \( D_\alpha \) as a closed subset (see [12] for details).

**Remark 3.** By a method given in [7] we can show the following theorem.

**Theorem 2.** For any van Douwen-complete space \( X \) there exists a perfect map \( f \) from a strongly 0-dimensional van Douwen-complete space \( Z \) onto \( X \).

**Proof.** Oka [7] showed the following commutative diagram for every closed map \( f \) from a metric space \( M \) onto \( X \), where \( \dim Z = 0 \), \( h \) is closed onto map, and both \( p \) and \( g \) are perfect onto maps.

\[
\begin{array}{ccc}
M & \xrightarrow{f} & X \\
p & & g \\
\downarrow & & \downarrow \\
T & \xrightarrow{h} & Z
\end{array}
\]

In our case the space \( T \) is completely metrizable by Fact 3, since \( M \) is so and \( p \) is perfect. Hence, \( Z \) is a desired strongly 0-dimensional van Douwen-complete space.

The spaces \( T \) and \( Z \) used in the proof of Theorem 2 are constructed as the fiber products of some mappings (see [7] for details). The following example, however, shows that the fiber product \( S = \{(t, x) : f(t) = h(t)\} \) itself is too big for the proof of Theorem 1.

**Example 1.** If \( Y \) is the quotient space obtained from the Euclidean plane \( R^2 \) by identifying the set of all the points on the \( x \)-axis to a point \( y_0 \), then the natural quotient map \( \pi : R^2 \to Y \) is closed irreducible (note also that \( \pi^{-1}(y_0) \approx R \)). Let \( \alpha : S \to R^2 \) be the natural projection map from the fiber product \( S \) with respect to \( f = h = \pi \). Then, \( \alpha \) is not closed, since \( \alpha^{-1}(\pi^{-1}(y_0)) \approx R^2 \) and \( \alpha|\alpha^{-1}(\pi^{-1}(y_0)) \) is not a closed map.

The following example shows that without the assumption of metrizability perfect pre-images of a van Douwen-complete space are neither van Douwen-complete nor Čech-complete, in general.

**Example 2.** Let \( p : X \times C \to X \) be the natural projection map from the product space, where \( X \) is the van Douwen-complete space in Example 1 and \( C \) is a non-metrizable compact Hausdorff space. Then \( p \) is obviously perfect and \( X \times C \) is neither van Douwen-complete nor Čech-complete by Fact 3 and the following reason. Since every van Douwen-complete space has a \( G_\delta \)-diagonal, it is metrizable by Okuyama-Borges Theorem [1, 8] when it is assumed to be Čech-complete (note that in this case it is a paracompact \( p \)-space).

**Example 3.** Let \( X \) be the same van Douwen-complete space in Examples 1 and 2. Then, \( X \) is not \( G_\delta \) in its Stone-Čech compactification \( \beta X \). Hence, our corollary does not holds for non-Lašnev space \( Z \).
References

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