ON THE NUMBERS OF 2-WEIGHTS, UNIPOTENT CONJUGACY CLASSES, AND IRREDUCIBLE BRAUER 2-CHARACTERS OF FINITE CLASSICAL GROUPS

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(Communicated by Ronald M. Solomon)

Abstract. A closed expression for the number of unipotent conjugacy classes of a classical group $G$ is given and Alperin's weight conjecture is confirmed for $G$ globally and for a symplectic or odd-dimensional orthogonal group block by block.

INTRODUCTION

Let $G$ be a classical symplectic or orthogonal group defined over a field of odd characteristic. A 2-block $B$ of $G$ is labelled by a semisimple 2'-element $s^*$ in $G$ by results of Cabanes and Enguehard [7] and the first author [3]. Let $\mathcal{W}(B)$ be the number of 2-weights in $G$ associated with $B$ and $\mathcal{U}_G(s^*)$ the number of unipotent conjugacy classes of $C_G(s^*)$, and let $l(B)$ be the number of irreducible Brauer 2-characters in $B$. In this paper we prove that $\mathcal{W}(B) = \mathcal{U}_G(s^*) = l(B)$ for each block $B$ of a symplectic or odd-dimensional orthogonal group $G$. In addition, $\mathcal{W}(B) = \mathcal{U}_G(s^*)$ and $\mathcal{U}_{G_0}(s^*) = l(B')$ when $G$ is an even-dimensional orthogonal group, where $G_0$ is the special orthogonal group, and $\mathcal{U}_{G_0}(s^*)$ is the number of unipotent conjugacy classes in $C_{G_0}(s^*)$, and $B'$ is a block of $G_0$ covered by $B$. In the latter case, we could not get the equation $\mathcal{U}_G(s^*) = l(B)$ because we do not know how to get $l(B)$. We give as corollaries a closed expression for the number of unipotent conjugacy classes of $G$, and get an affirmative answer for Alperin's weight conjecture globally for $G$ and block by block for a symplectic or odd-dimensional orthogonal group $G$. Notice that the three numbers $\mathcal{W}(B), \mathcal{U}_G(s^*),$ and $l(B)$ are also the same for a 2-block $B$ of a general linear or unitary group by results of [1-2], [4], and [10].

In §1 we use the generating function given by Wall [14] for unipotent conjugacy classes of a symplectic or orthogonal group to show that $\mathcal{W}(B) = \mathcal{U}_G(s^*)$, and we give a closed formula for the number of unipotent conjugacy classes. In §2 we use the results of Broué [5] to show that $\mathcal{U}_{G_0}(s^*) = l(B')$. 
1. THE NUMBERS OF WEIGHTS FOR 2-BLOCKS

Let \( \mathbb{F}_q \) be a field of \( q \) elements with odd characteristic, and let \( V \) be a non-degenerate finite-dimensional symplectic or orthogonal space over \( \mathbb{F}_q \). In addition, let \( I(V) \) be the group of all isometries of \( V \) and \( I_0(V) \) the subgroup of \( I(V) \) of isometries of determinant 1. Thus \( I(V) = I_0(V) \) if \( V \) is symplectic. Let \( G = I_0(V) \), and let \( G^* \) be the dual group of \( G \). If \( s \) is a semisimple element of \( G^* \), then let \( (s) \) be the conjugacy class of \( G^* \) containing \( s \), and let \( \mathcal{E}(G,(s)) \) be the set of the irreducible constituents of Deligne-Lusztig generalized characters associated with \( (s) \) (see [6, p. 57]). If \( s \) is a semisimple 2'-element of \( G^* \), let

\[
\mathcal{E}_2(G,(s)) = \bigcup_u \mathcal{E}(G,(su)),
\]

where \( u \) runs over all 2-elements of \( C_{G^*}(s) \). By [7, Theorem 13] and [5, Theorem 3.2], \( \mathcal{E}_2(G,(s)) \) is a 2-block, so that a 2-block \( B \) of \( I(V) \) covers a 2-block \( \mathcal{E}_2(G,(s)) \). By [3, (5B)(c)], \( B \) covers another block \( \mathcal{E}_2(G,(s')) \) if and only if \( s \) and \( s' \) are conjugate in \( I(V^*) \), where \( V^* \) is the underlying space of \( G^* \). For each semisimple 2'-element \( s \) in \( I_0(V)^* \) a dual element \( s^* \) in \( I_0(V) \) of \( s \) is defined by [3, (4A)] and \( s^* \) is determined uniquely by \( s \) up to conjugacy in \( I(V) \). We shall say that \( s^* \) is a semisimple label of \( B \). Thus a semisimple label of \( B \) is determined uniquely up to conjugacy in \( I(V) \). In this section, we shall show that the number of weights for a 2-block of \( I(V) \) with semisimple label \( s^* \) is the number \( \mathcal{Z}_{I(V)}(s^*) \) of unipotent conjugacy classes of \( C_{I(V)}(s^*) \). In particular, we shall get a closed formula for the number of unipotent conjugacy classes of \( I(V) \).

First of all, we consider the symplectic group \( \text{Sp}(V) \). We shall need the following lemma.

**Lemma (1A).** The following identity holds:

\[
\prod_{j=1}^{\infty} (1 - t^{2j}) \prod_{j=1}^{\infty} (1 + t^j) = \sum_{i=1}^{\infty} t^{i(i-1)/2}.
\]

**Proof.** If \( C(2, m) \) is the number of 2-cores of rank \( m \), then

\[
C(2, m) = \begin{cases} 1 & \text{if } m = \frac{i(i-1)}{2} \text{ for some } i \in \mathbb{N}, \\ 0 & \text{otherwise}, \end{cases}
\]

where \( \mathbb{N} \) is the set of all natural numbers. Now let \( P(t) \) and \( F_2(t) \) be the generating functions of partitions and 2-cores, respectively. Then

\[
P(t) = \prod_{i=1}^{\infty} \frac{1}{1 - t^i} \quad \text{and} \quad F_2(t) = \sum_{i=1}^{\infty} t^{i(i-1)/2},
\]

and by [12, Proposition 3.3]

\[
F_2(t) = (P(t^2))^{-2} P(t),
\]

therefore

\[
F_2(t) = \prod_{j=1}^{\infty} (1 - t^{2j})^2 \prod_{i=1}^{\infty} \frac{1}{1 - t^i} = \prod_{j=1}^{\infty} (1 - t^{2j}) \prod_{i=1}^{\infty} (1 + t^i).
\]

Thus (1.1) holds.
Corollary (1B). Let \( \pi_o(m) \) and \( \pi_e(m) \) be the number of odd and even partitions of rank \( m \), respectively, where an odd partition is a partition with odd parts and an even partition is a partition with even parts. Then

\[
\pi_o(m) = \sum_{j=1}^{m} \pi_e(j)C(2, m - j) = \sum_{i=1}^{m} \pi_e(m - i)C(2, i),
\]

where \( C(2, l) \) is the number of 2-cores of rank \( l \) given by (1.2).

Proof. Let \( g_o(t) \) and \( g_e(t) \) be the generating functions of odd and even partitions, respectively. Then

\[
g_o(t) = \prod_{l=1}^{\infty} (1 + t^l) \quad \text{and} \quad g_e(t) = \prod_{l=1}^{\infty} \frac{1}{1 - t^{2l}}.
\]

By (1A) \( g_o(t) = g_e(t)F_2(t) \), and so (1.5) holds.

Proposition (1C). Let \( G \) be the symplectic group \( \text{Sp}(V) = \text{Sp}(2n, q) \) and \( B_0 \) the principal 2-block of \( G \), and let \( \mathcal{W}(B_0) \) be the number of \( B_0 \)-weights. Then \( \mathcal{W}(B_0) \) is the number \( \mathcal{U}_G(1) \) of unipotent conjugacy classes of \( G \). In particular,

\[
\mathcal{U}_G(1) = \sum_{\kappa} f_{X-1, \kappa},
\]

where \( \kappa \) runs over all 2-cores with \( |\kappa| \leq n \) and \( f_{X-1, \kappa} \) is the number of pairs \( (\lambda_1, \lambda_2) \) of partitions \( \lambda_i \) such that \( |\lambda_1| + |\lambda_2| = n - |\kappa| \).

Proof. Let

\[
f_{X-1} = \sum_{\kappa} f_{X-1, \kappa},
\]

where \( \kappa \) runs over all 2-cores with \( |\kappa| \leq n \). By [3, (6D)(d)] \( \mathcal{W}(B_0) = f_{X-1} \).

Let \( k(m, l) \) be the number of \( m \)-tuples \( (\lambda_1, \lambda_2, \ldots, \lambda_m) \) of partitions \( \lambda_i \) such that

\[
\sum_{i=1}^{m} |\lambda_i| = l.
\]

Then by [13, p. 237],

\[
P(t)^m = \sum_{l \geq 0} k(m, l) t^l.
\]

The generating function of \( \mathcal{W}(B_0) \) is

\[
f(t) = \left( \sum_{l=0}^{\infty} k(2, l) t^{2l} \right) F_2(t^2) = \left( \prod_{i=1}^{\infty} \frac{1}{1 - t^{2i}} \right)^2 F_2(t^2),
\]

where \( F_2(t) \) is the generating function of 2-cores given by (1.3). By [14, p. 38], the generating function of the number of unipotent conjugacy classes in \( G \) is

\[
F_-(t) = \prod_{i=1}^{\infty} \frac{(1 + t^{2i})^2}{1 - t^{2i}}.
\]
Now set $x = t^2$. It suffices to show that

\[(1.8) \quad \prod_{i=1}^{\infty} \frac{(1 + x^i)^2}{(1 - x^i)^2} = \left( \prod_{i=1}^{\infty} \frac{1}{1 - x^i} \right)^2 F_2(x).\]

By (1A) $F_2(x) = \prod_{i=1}^{\infty} (1 - x^{2i})(1 + x^i)$, so that $F_2(x) = \prod_{i=1}^{\infty} (1 - x^{2i})(1 + x^i)^2$, and (1.8) follows. This completes the proof.

We now consider an orthogonal group $O(V)$. Let $B_0$ be the principal 2-block of $O(V)$ and $\mathcal{W}(B_0)$ the number of $B_0$-weights. Then $\mathcal{W}(B_0)$ is given by [3, (6D)]. If $\dim V$ is odd, then in the notation of [3, (6D)], $\mathcal{W}(B_0)$ is the number

\[(1.9) \quad f_{x-1} = \sum_{\kappa_1, \kappa_2, \kappa} f_{x-1, \kappa_1, \kappa_2, \kappa},\]

where $\kappa_1$ and $\kappa_2$ run over all 2-cores such that $|\kappa_1|$ and $|\kappa_2|$ are odd and even, respectively, $\kappa$ runs over all 2-cores and $f_{x-1, \kappa_1, \kappa_2, \kappa}$ is the number of 4-tuples $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ of partitions $\lambda_i$ such that

\[|\lambda_1| + |\lambda_2| + |\lambda_3| + |\lambda_4| = \frac{1}{4}(\dim V - |\kappa_1| - |\kappa_2| - 2|\kappa|).\]

Let $D(V)$ be the discriminant of $V$ and $\sigma$ a non-square element of $\mathbb{F}_q$. If $\dim V$ is even, then $\mathcal{W}(B_0)$ is also given by (1.9), where $\kappa_1$ and $\kappa_2$ run over all 2-cores such that $D(V) = \sigma|\kappa_1|$ and $|\kappa_1|, |\kappa_2|$ are either both odd or both even, $\kappa$ runs over all 2-cores and $f_{x-1, \kappa_1, \kappa_2, \kappa}$ is the number of 4-tuples $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ of partitions $\lambda_i$ such that

\[|\lambda_1| + |\lambda_2| + |\lambda_3| + |\lambda_4| = \frac{1}{4}(\dim V - |\kappa_1| - |\kappa_2| - 2|\kappa|).\]

If $\dim V = w$ and the type $\eta(V)$ of $V$ is $\pm$, then we denote by $\mathcal{W}_w^\pm$ the number $\mathcal{W}(B_0)$. Thus $\mathcal{W}_w^+ = \mathcal{W}_w^-$ if $w$ is odd. Let $g(t)$ be the generating function of $\mathcal{W}_w^+ + \mathcal{W}_w^-$, so that

\[g(t) = \left( \sum_{w=1}^{\infty} k(4, w)t^{4w} \right) F_2(t^2)F_2(t)^2.\]

By (1.4) and (1.7)

\[g(t) = P(t^4)^4 P(t^4)^{-2} P(t^2)^2 P(t^2)^{-4} P(t)^2 = P(t^4)^2 P(t^2)^{-3} P(t)^2 \]

\[= \prod_{k=1}^{\infty} \frac{1 + t^k}{(1 - t^k)(1 + t^k)^2}.\]

On the other hand, if $\mathcal{Z}_w^\pm$ is the number of unipotent conjugacy classes of $O(V)$ such that $\dim V = w$ and $\eta(V) = \pm$, then by [14, (2.6.17)], the generating function of $\mathcal{Z}_w^+ + \mathcal{Z}_w^-$ is

\[F_+^+(t) = \prod_{k=1}^{\infty} \frac{(1 + t^{2k-1})^2}{1 - t^{2k}}.\]

But

\[(1.10) \quad \prod_{k=1}^{\infty} (1 + t^k)^2 = \prod_{k=1}^{\infty} (1 + t^{2k-1})^2(1 + t^{2k})^2,\]
so it follows that \( g(t) = F_+^+(t) \). In particular, if \( w \) is odd, then \( \mathcal{W}(B_0) = \mathcal{W}^+_w = \mathcal{W}^-_w \) is the number of unipotent conjugacy classes of \( O(V) \).

Suppose \( w = 2n \) for some integer \( n \). Then
\[
F_2(t) + F_2(-t) = \sum_{\kappa} 2t^{|\kappa|} \quad F_2(t) - F_2(-t) = \sum_{\kappa'} 2t^{|\kappa'|},
\]
where \( \kappa \) and \( \kappa' \) run over all 2-cores such that \(|\kappa|\) and \(|\kappa'|\) are even and odd, respectively. Thus the generating function \( h(t) \) of \( \mathcal{W}_2^+ - \mathcal{W}_2^- \) is given by
\[
\left( \sum_{j=1}^{\infty} k(4,j) t^{4j} \right) F_2(t^2) \left[ \left( \frac{1}{2} (F_2(t) + F_2(-t)) \right)^2 - \left( \frac{1}{2} (F_2(t) - F_2(-t)) \right)^2 \right],
\]
so that \( h(t) = P(t^4) F_2(t^2) F_2(t) F_2(-t) \). By (1.4) and (1.7)
\[
h(t) = P(t^4) P(t^2)^{-2} P(t^2)^{-2} P(t) P((-t)^2)^{-2} P(-t)
\]
\[
= P(t^4)^2 P(t^2)^{-3} P(t) P(-t)
\]
\[
\prod_{k=1}^{\infty} \frac{1}{(1 - t^{4k})^2} \prod_{k=1}^{\infty} \frac{1}{(1 - t^{2k})^3} \prod_{k=1}^{\infty} \frac{1}{(1 - t^k)(1 - (-t)^k)}
\]
\[
= \prod_{k=1}^{\infty} \frac{1}{(1 - t^{4k})^2} \prod_{k=1}^{\infty} (1 - t^{2k})^3 \prod_{k=1}^{\infty} \frac{1}{(1 - t^{2k})^2} \prod_{k=1}^{\infty} \frac{1}{(1 - t^{4k})}.
\]
But \( \prod_{k=1}^{\infty} (1 - t^{4k})^{-1} = \prod_{k=1}^{\infty} (1 + t^{2k}) \) (cf. [14, p. 42]), so
\[
h(t) = \prod_{k=1}^{\infty} \frac{1}{(1 - t^{4k})^2} \prod_{k=1}^{\infty} (1 - t^{4k}) = P(t^4).
\]
By [14, (2.6.18)] \( P(t^4) \) is the generating function \( F_+^-(t) \) of \( \mathcal{W}_2^+ - \mathcal{W}_2^- \). Thus \( h(t) = F_+^-(t) \) and then \( \mathcal{W}(B_0) \) is the number of unipotent conjugacy classes of \( O(V) \). So we have proved the following proposition.

**Proposition (1D).** Let \( B_0 \) be the principal 2-block of \( O(V) \), and let \( \mathcal{W}(B_0) \) be the number of \( B_0 \)-weights. Then \( \mathcal{W}(B_0) \) is the number \( \mathcal{W}_0(V)(1) \) of unipotent conjugacy classes of \( O(V) \). In particular, if \( D(V) \) is the discriminant of \( V \) and \( \sigma \) is a non-square element of \( \mathbb{F}_q \), then
\[
\mathcal{W}_0(V)(1) = \sum_{\kappa_1, \kappa_2, \kappa} f_{x-1, \kappa_1, \kappa_2, \kappa},
\]
where \( \kappa_1, \kappa_2, \kappa \) run over all 2-cores such that \( D(V) = \sigma^{|\kappa_1|} \) and \( f_{x-1, \kappa_1, \kappa_2, \kappa} \) is the number of 4-tuples \( (\lambda_1, \lambda_2, \lambda_3, \lambda_4) \) of partitions \( \lambda_i \) such that
\[
|\lambda_1| + |\lambda_2| + |\lambda_3| + |\lambda_4| = \frac{1}{4} (\dim V - |\kappa_1| - |\kappa_2| - 2|\kappa|).
\]
Now let \( B \) be a 2-block of \( I(V) \) covering \( \mathcal{E}_2(I_0(V), (s)) \) for some semisimple 2'-element \( s \) of \( I_0(V)^* \), and let \( s = \prod_{\Gamma} s_{\Gamma} \) be the primary decomposition of \( s \) in \( G^* \) in the sense of [11, p. 125]. In addition, let \( m_{\Gamma}(s) \) be the multiplicity of \( \Gamma \) in \( s \), let \( V_{x-1}^* \) be the underlying space of \( s_{x-1}^* \), and let \( V_{x-1} \) be the
space dual of \( V_{X-1}^* \) in the sense of [11, (3.1)]. By [3, (6E)] the number \( \mathcal{W}(B) \) of \( B \)-weights is \( \prod \Gamma f_\Gamma \), where \( f_\Gamma \) is the number of partitions of rank \( m_\Gamma(s) \) except when \( \Gamma = X - 1 \), in which case \( f_{X-1} \) is given by (1.6) or (1.9) with \( \dim V \) replaced by \( \dim V_{X-1} \) according as \( V \) is symplectic or orthogonal. Thus \( f_{X-1} \) is the number of unipotent conjugacy classes of \( I(V_{X-1}) \) by (1C) and (1D). By [11, (1.13)]

\[
C_{I_0(V)^*}(s^*) \simeq \prod_{\Gamma \neq X-1} \GL(m_\Gamma(s), \varepsilon_\Gamma q^{\delta_\Gamma}) \times I_0(V_{X-1}),
\]

where \( \varepsilon_\Gamma \) and \( \delta_\Gamma \) are defined by [11, (1.8) and (1.9)] and \( \GL(m, -q^\delta) = U(m, q^\delta) \) for all \( \delta \geq 1 \). Let \( s^* \) be a dual of \( s \) in \( I_0(V) \) defined by [3, (4A)] with the primary decomposition \( \prod \Gamma s^*_\Gamma \). By [3, (4.1)]

\[
C_{I_0(V^*)}(s^*) \simeq C_{I_0(V)}(s^*)
\]

and by definition, \( m_\Gamma(s) = m_\Gamma(s) \) for \( \Gamma \neq X - 1 \) and \( V_{X-1} \) is the underlying space of \( s^*_{X-1} \), therefore

\[
C_{I(V)}(s^*) \simeq \prod_{\Gamma \neq X-1} \GL(m_\Gamma(s^*), \varepsilon_\Gamma q^{\delta_\Gamma}) \times I(V_{X-1}).
\]

But the number of unipotent conjugacy classes of \( \GL(m, \varepsilon q^\delta) \) for any sign \( \varepsilon = \pm \) and \( \delta \geq 1 \) is the number of partitions of rank \( m \), so \( \mathcal{W}(B) \) is the number of unipotent conjugacy classes of \( C_{I(V)}(s^*) \). Thus we have proved the following.

**Proposition (IE).** Let \( B \) be a 2-block of \( I(V) \) with a semisimple label \( s^* \) for some semisimple 2'-element \( s^* \) of \( I_0(V) \). Then the number \( \mathcal{W}(B) \) of \( B \)-weights is the number of unipotent conjugacy classes \( \mathcal{U}_{I(V)}(s^*) \) of \( C_{I(V)}(s^*) \). In particular, if \( \prod \Gamma s^*_\Gamma \) is the primary decomposition of \( s^* \) in \( I(V) \) in the sense of [11, p. 125], then

\[
\mathcal{U}_{I(V)}(s^*) = \prod \Gamma f_\Gamma,
\]

where \( f_\Gamma \) is the number of partitions of multiplicity \( m_\Gamma(s^*) \) of the elementary divisor \( \Gamma \) in \( s^* \) except when \( \Gamma = X - 1 \), in which case \( f_{X-1} \) is given by (1.6) or (1.9) with \( V \) replaced by the underlying space of \( s^*_{X-1} \) according as \( V \) is symplectic or orthogonal.

**Remark (IF).** As a corollary of (IE), we can get an affirmative answer for Alperin's weight conjecture for \( I(V) \). Indeed, if \( \mathcal{W}(I(V)) \) is the number of weights of \( I(V) \), then by (1E),

\[
\mathcal{W}(I(V)) = \sum_{s^*} \mathcal{U}_{I(V)}(s^*),
\]

where \( s^* \) runs over all semisimple 2'-elements of \( I(V) \). Now the right-hand side of (1.11) is the number of conjugacy classes of 2-regular elements in \( G \) and it is the number of irreducible Brauer characters \( l(I(V)) \) of \( I(V) \) by a result of Brauer. Thus \( \mathcal{W}(I(V)) = l(I(V)) \) and the remark follows.

2. **The number of irreducible Brauer characters**

The notation and terminology of §1 are continued in this section. The number of irreducible Brauer characters in a 2-block of a symplectic or special orthog-
nal group will be given and the weight conjecture of Alperin will be confirmed block by block for a symplectic or odd-dimensional orthogonal group.

The proof of the following proposition was pointed out by the referee of [3].

**Proposition (2A).** Let q be a power of an odd prime, \( G = I_0(V) \), \( B \) a 2-block of \( G \), and \( l(B) \) the number of irreducible Brauer characters in \( B \). If \( B = \mathcal{E}_2(G, (s)) \) for some semisimple 2’-element \( s \) of the dual group \( G^* = I_0(V)^* \), then \( l(B) \) is the number of unipotent conjugacy classes of \( C_{G^*}(s)^* \).

**Proof.** Let \( t^* \) be a semisimple 2’-element of \( G \), and let \( t \) be its dual given by [3, (4A)], so that \( C_G(t) \simeq C_{G^*}(t^*) \). Let \( \mathcal{Z}(t^*) \) be the number of unipotent conjugacy classes of \( C_G(t^*) \). If \( \dim V \leq 2 \), then it is trivial to check that \( l(B) = \mathcal{Z}(s^*) \). Suppose \( s \neq 1 \). Then \( C_G(s) \) is a proper regular subgroup of \( G^* \) and \( C_G(s^*) \) is its dual group. By Broué [5, Theorem 2.3] there is a perfect isometry between \( B \) and \( \mathcal{E}_2(C_G(s^*), (1)) \). It follows that \( l(B) \) is the number of irreducible Brauer characters of \( \mathcal{E}_2(C_G(s^*), (1)) \). Let \( \prod \mathfrak{s}_r \) be the primary decomposition of \( s^* \) in \( G \) and \( V_{X-1} \) the underlying space of \( s_{X-1}^* \). Then

\[
C_G(s^*) \simeq \left( \prod_{r \neq X-1} \text{GL}(m_r(s), q_{\mathfrak{r}\mathfrak{q}^r}) \right) \times I_0(V_{X-1})
\]

and \( \dim V_{X-1} < \dim V \), so by induction \( l(\mathcal{E}_2(I_0(V_{X-1})), (1)) \) is the number of unipotent conjugacy classes of \( I_0(V_{X-1}) \). By [10, §8] \( I(I_0(V), q_{\mathfrak{r}\mathfrak{q}^r}) \) is the number of partitions of rank \( m_r(s) \). Thus \( l(B) = \mathcal{Z}(s^*) \). Now let \( s = 1 \). The number of irreducible Brauer characters \( l(G) \) of \( G \) is

\[
l(B_0) + \sum_{t^* \neq 1} \mathcal{Z}(t^*),
\]

where \( t^* \) runs over all representatives for the semisimple conjugacy 2’-classes of \( G \) with \( t^* \neq 1 \). By a result of Brauer \( l(G) \) is the number of conjugacy 2’-classes in \( G \), so that

\[
l(G) = \mathcal{Z}(1) + \sum_{t^* \neq 1} \mathcal{Z}(t^*).
\]

It follows that \( l(B_0) = \mathcal{Z}(1) \) and this proves (2A).

**Proposition (2B).** Let \( V \) be a symplectic or odd-dimensional orthogonal space and \( B \) a 2-block of \( I(V) \) such that \( B \) covers a 2-block \( B' = \mathcal{E}_2(I_0(V), (s)) \) of \( I_0(V) \), where \( s \) is a semisimple 2’-element of the dual group \( I_0(V)^* \) of \( I_0(V) \). Let \( s^* \) be a dual of \( s \) in \( I_0(V) \) given by [3, (4A)], and let \( \mathcal{W}(s^*) \) be the number of unipotent conjugacy classes of \( C_{I(V)}(s^*) \). In addition, let \( \mathcal{W}(B) \) be the number of B-weights. If \( l(B) \) is the number of irreducible Brauer characters in \( B \), then \( l(B) = \mathcal{W}(B) = \mathcal{W}(s^*) \). In particular, Alperin’s weight conjecture holds for \( B \) block by block.

**Proof.** Let \( \mathcal{W}(s^*) \) be the number of unipotent conjugacy classes of \( C_{I(V)}(s^*) \). If \( V \) is symplectic, then \( I(V) = I_0(V) \) and \( C_{I(V)}(s^*) \simeq C_{I_0(V)}(s^*) \).

By (2A) and (1E) \( l(B) = \mathcal{W}(s^*) = \mathcal{W}(B) \). If \( V \) is odd-dimensional orthogonal, then \( I(V) = \langle -1_V \rangle \times G \) and \( C_{I(V)}(s^*) = \langle -1_V \rangle \times C_{I_0(V)}(s^*) \), where \( 1_V \) is the identity of \( I(V) \). Thus \( \mathcal{W}(s^*) = \mathcal{W}(s^*) \) and \( l(B) = l(B') \), so that (2B) follows from (2A) and (1E).
Remark (2C). In the notation of (2B), suppose $V$ is even-dimensional orthogonal. If the multiplicity $m_{X-1}(s^*)$ of $X-1$ in $s^*$ is zero, then $\mathcal{Z}_{I(V)}(s^*) = l(B)$ and so Alperin’s weight conjecture has an affirmative answer for $B$. Indeed, in this case $C_{I(V)}(s^*) = C_{I_0(V)}(s^*)$, so that $\mathcal{Z}_{I(V)}(s^*) = \mathcal{Z}_{I_0(V)}(s^*) = l(B')$. Let $\tau$ be an involution of $I(V^*)$ with determinant $-1$, where $V^*$ is the underlying space of $I_0(V)^*$. Then $\mathcal{Z}_2(I_0(V), (s^*))$ is a block $B''$ of $I_0(V)$ and $B$ covers $B''$ by [3, (5B)]. Since $s$ is not conjugate with $s^\tau$ in $I_0(V)^*$, it follows that $B' \neq B''$, so that $B'' \neq B''$ and $\tau \notin N(B')$, where $N(B')$ is the stabilizer of $B'$ in $I(V)$. Thus $N(B') = I_0(V)$ and $l(B) = l(B')$ by a result of Fong and Reynolds [9, Theorem V.2.5]. It follows that $\mathcal{Z}_{I(V)}(s^*) = l(B)$ and the remark follows.

References

4. M. Broué, Les $l$-blocs des groupes GL$(n, q)$ et U$(n, q^2)$ et leurs structures locales, Séminaire Bourbaki Astérisque 640 (1986), 159–188.