ON SOLUTIONS OF ELLIPTIC EQUATIONS THAT DECAY RAPIDLY ON PATHS

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Abstract. Let $P(D)$ be an elliptic differential operator on $\mathbb{R}^n$ with constant coefficients. It is known that if $u$ is a solution of $P(D)u = 0$ on an unbounded domain and if $u$ decays uniformly and sufficiently rapidly, then $u = 0$. In this note it is shown that the same conclusion holds if $u$ decays rapidly, but not a priori uniformly, on a sufficiently large set of unbounded paths.

Throughout this note $\Omega$ is an unbounded domain in $\mathbb{R}^n$, where $n \geq 2$, and $P(D) = \sum_{|\alpha| \leq d} a_{\alpha}D^\alpha$ is an elliptic linear differential operator on $\mathbb{R}^n$ with constant complex coefficients. In response to a problem proposed for the harmonic case ($P(D) = \Delta$, the Laplacian operator) at a Durham Conference in 1983 [3, Problem 3.27], Armitage, Bagby and Gauthier [1] gave two proofs of the following result.

Theorem A. There exists a continuous function $\varepsilon: [0, +\infty) \to (0, 1]$ with the following property. If $u$ is a solution of $P(D)u = 0$ on $\Omega$ such that $|u(x)| \leq \varepsilon(||x||)$ for all $x \in \Omega$, then $u = 0$.

Some theorems for special domains $\Omega$ in the harmonic and holomorphic ($n = 2$ and $P(D) = \partial$) cases suggest that it may be possible to replace the condition $|u(x)| \leq \varepsilon(||x||)$ by a requirement that $u$ should decay rapidly, but not a priori uniformly, on a suitable set of unbounded paths; see, for example, Armitage and Goldstein [2]. Here we confirm that there is indeed a general result of this type.

We now fix some further notation. Let $M$ be an $(n - 1)$-dimensional manifold, and let $\Gamma: M \times [0, +\infty) \to \Omega$ be a continuous function such that (i) $\Gamma(\omega \times (0, +\infty))$ is open for each open subset $\omega$ of $M$, and (ii) for each $\xi \in M$ the set $\gamma_\xi = \{\Gamma(\xi, t): t \geq 0\}$ is closed and unbounded.

Theorem 1. There exists a continuous function $\eta: [0, +\infty) \to (0, 1]$ with the following property. If $u$ is a solution of $P(D)u = 0$ on $\Omega$ such that

\begin{equation}
(1) \quad u(x) = O(\eta(||x||)) \quad (||x|| \to +\infty, x \in \gamma_\xi)
\end{equation}

for a second category set of $\xi$ in $M$, then $u \equiv 0$.

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The category condition is indispensible, at least in the holomorphic case: if \( S \) is a first category subset of the unit circle and \( \eta: [0, +\infty) \rightarrow (0, 1] \) is continuous, then there exists a non-constant entire function \( f \) such that \( f(rz) = o(\eta(r)) \) as \( r \rightarrow +\infty \) for each \( z \in S \) (see Schneider [4, Example 10]).

In proving Theorem 1, we indicate first how \( \eta \) is chosen. Let \( \{B_1, B_2, \ldots \} \) be a countable base for the topology of \( M \). For the moment let \( k \) be a fixed positive integer. By hypothesis, \( \Gamma(B_k \times (0, +\infty)) \) is open. Moreover, this set has an unbounded connected component, since it contains an unbounded connected set of the form \( \Gamma(\{\xi\} \times (0, +\infty)) \). Let \( \Omega_k \) be an unbounded, connected, open subset of \( \Gamma(B_k \times (0, +\infty)) \) such that \( \overline{\Omega_k} \subset \Omega \). By Theorem A, there exists a continuous function \( \varepsilon_k: [0, +\infty) \rightarrow (0, 1] \) with the property that the zero function is the only solution of \( P(D)u = 0 \) on \( \Gamma_k \) satisfying \( |u(x)| \leq \varepsilon_k(\|x\|) \) for all \( x \in \Omega_k \). We take \( \eta: [0, +\infty) \rightarrow (0, 1] \) to be a continuous function such that \( \eta \leq \varepsilon_k \) on \( (k, +\infty) \) for each \( k \).

Now suppose that \( u \) is a solution of \( P(D)u = 0 \) on \( \Omega \) satisfying (1) for all \( \xi \) belonging to a second category subset \( E \) of \( M \), and define a function \( \Phi \) on \( M \) by

\[
\Phi(\xi) = \sup\{|u(x)|/\eta(\|x\|): x \in \gamma_\xi\}.
\]

We claim that \( \Phi \) is lower semi-continuous on \( M \). To prove this, suppose that \( \xi \in M \) and that \( A < \Phi(\xi) \). Then there exists \( x \in \gamma_\xi \), say \( x = \Gamma(\xi, t) \), such that \( |u(x)|/\eta(\|x\|) > A \). By the continuity of \( u \), \( \eta \) and \( \Gamma \), there exists \( \delta > 0 \) such that \( |u(y)|/\eta(\|y\|) > A \) whenever \( \|x - y\| < \delta \), and there exists an open neighbourhood \( N \) of \( \xi \) such that \( \|x - \Gamma(\xi, t)\| < \delta \) for all \( \xi \in N \). Hence

\[
\Phi(\xi) \geq |u(\Gamma(\xi, t))/\eta(\|\Gamma(\xi, t)\|) > A \quad (\xi \in N),
\]

so \( \Phi \) is lower semi-continuous at \( \xi \). Now define \( A_m = \{\xi \in M: \Phi(\xi) \leq m\} \) \((m = 1, 2, \ldots)\). By the lower semi-continuity of \( \Phi \), each \( A_m \) is closed. Clearly \( \Phi(\xi) < +\infty \) for each \( \xi \in E \), so that \( E \subseteq \bigcup_{m=1}^{\infty} A_m \). Since \( E \) is second category, some \( A_k \) has non-empty interior; this interior contains some \( B_q \). If \( x \in \Omega_q \), then \( x \in \gamma_\xi \) for some \( \xi \in B_q \subseteq A_k \), so that \( |u(x)|/\eta(\|x\|) \leq \Phi(\xi) \leq k \). Hence

\[
|u(x)| \leq k\eta(\|x\|) \leq k\varepsilon_q(\|x\|) \quad (x \in \Omega_q, \|x\| > q).
\]

Since, further, \( u \) is bounded on the compact set \( \{x \in \Omega_q: \|x\| \leq q\} \), there exists a positive constant \( C \) such that \( C|u(x)| \leq \varepsilon_q(\|x\|) \) for all \( x \in \Omega_q \). From our choice of \( \varepsilon_q \), it follows that \( u = 0 \) on \( \Omega_q \). Since \( u \) is real-analytic on \( \Omega \), we conclude that \( u = 0 \) on \( \Omega \).

REFERENCES


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