

LCM-STABILITY OF POWER SERIES EXTENSIONS CHARACTERIZES DEDEKIND DOMAINS

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ABSTRACT. In this paper we prove the following main result. A (commutative integral) domain R is a Dedekind domain if and only if $R[[X]] \subset T[[X]]$ is LCM-stable for each domain T containing R as a subring.

1. INTRODUCTION

Suppose R is a (commutative) domain with quotient field K . A result of F. Richman [12, Theorem 4] asserts that R is a Prüfer domain if and only if each overring of R (i.e., each R -subalgebra of K) is a R -flat. As a consequence, it is easy to see that R is a Prüfer domain if and only if each domain T containing R as a subring induces a flat extension, $T[X]$ over $R[X]$, of polynomial rings. In fact, there is an ideal-theoretic variant in terms of the LCM-stability concept. (As in [8], [9], we say an R -module M is LCM-stable if $aM \cap bM = (aR \cap bR)M$ for all $a, b \in R$.) Namely, R is Prüfer if and only if $T[X]$ is LCM-stable over $R[X]$ for each domain T containing R as a subring. Our main interest here is to study the analogous property for power series rings. In other words, we seek to characterize domains R such that $T[[x]]$ is LCM-stable over $R[[X]]$ for each domain T containing R as a subring. Our main result, Theorem 2.11, is that these R are precisely the Dedekind domains.

It is interesting to record additional motivation for the above problem. Since flatness implies LCM-stability and flatness is a universal property, a flat extension $R \subset T$ of domains entails LCM-stability of $R[X] \subset T[X]$. It is natural to ask whether LCM-stability of $R \subset T$ also entails LCM-stability of $R[X] \subset T[X]$. Several papers ([13], [15], [16]) have recently culminated in an affirmative answer [15, Corollary 3.7] in case R is a GCD-domain; later work gave an affirmative answer in case R is a Krull domain [16, Theorem 11]. These results naturally lead one to ask when LCM-stability of $R \subset T$ entails LCM-stability of $R[[X]] \subset T[[X]]$. The easiest case arises when all such T are LCM-stable over R ; and, by the remarks of the first paragraph, this happens

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precisely for Prüfer domains R . To a large extent, the details of §2 amount to determining that the Dedekind domains are precisely the (Prüfer) domains with the property $R[[X]] \subset T[[X]]$ is LCM-stable for each T containing R as a subring.

Theorem 2.5 establishes that if R is a Dedekind domain, then $T[[X]]$ is LCM-stable over $R[[X]]$ for each domain T containing R as a subring. With respect to the converse, namely determining which domains R satisfy $T[[X]]$ is LCM-stable over $R[[X]]$ for all domains T containing R as a subring, Corollary 2.2 reduces the study to a class of Prüfer domains. By considering the non-Archimedean domains (in the sense of [14]), we reduce the problem to studying one-dimensional Prüfer domains in Corollary 2.7. The local case leads to DVR's in Theorem 2.8, thanks to an appeal to work of M. Anderson and J. Watkins [1, Lemma, p. 191]. The route back from DVR's to the (global) answer is essayed by means of a globalization result and appeals to work on almost Dedekind domains (in the sense of [10]) due to Arnold [2], [4], and to Arnold-Brewer [3].

A closing remark presents partial results on related analogous questions about flatness.

Throughout, given a symbol representing a power series, the individual coefficients are denoted by subscripting the symbol. For example, if $\alpha \in R[[X]]$, then $\alpha = \sum_{i=0}^{\infty} \alpha_i X^i$. If $\{\alpha_j\}_{j=1}^n$ is a collection of power series, we write $\alpha_j = \sum_{i=0}^{\infty} \alpha_{j,i} X^i$ for each j .

2. RESULTS

Our goal is to classify the domains R such that $R[[X]] \subset T[[X]]$ is LCM-stable for all domains T containing R as a subring. We begin by showing in Proposition 2.2 that any such domain R must be a Prüfer domain. We first state a lemma, whose computational proof is left to the reader.

Lemma 2.1. *Suppose R is a subring of a domain S . If $R[[X]] \subset S[[X]]$ is LCM-stable, then $R \subset S$ is LCM-stable.*

Proposition 2.2. *Let R be a domain. If $R[[X]] \subset T[[X]]$ is LCM-stable for each domain T containing R as a subring, then R is a Prüfer domain.*

Proof. Lemma 2.1 shows $R \subset T$ is LCM-stable for all domains T containing R as a subring. In particular, $R \subset T$ is LCM-stable for all overrings T of R . By a result of Uda [15, Proposition 1.7] $R \subset T$ is flat for all overrings T of R . Hence R is a Prüfer domain, by the earlier-cited result of F. Richman [12, Theorem 4].

We shall soon show that among the Prüfer domains, (at least) the Dedekind domains have the desired property. First, we prove two lemmas.

Lemma 2.3. *Let R be a UFD and let T be a domain containing R as a subring. Then $R \subset T$ is LCM-stable if (and only if) $aT \cap bT = (aR \cap bR)T$ for all $a, b \in R$ such that $\text{g.c.d.}(a, b) = 1$.*

Proof. The "only if" assertion is trivial. For the converse, let $a, b \in R$ and set $d = \text{g.c.d.}(a, b)$. Put $a_1 = ad^{-1}$, $b_1 = bd^{-1}$; then $\text{g.c.d.}(a_1, b_1) = 1$.

$aT \cap bT = da_1T \cap db_1T = d(a_1T \cap b_1T)$. Since $\text{g.c.d.}(a_1, b_1) = 1$, $a_1T \cap$

$b_1T = (a_1R \cap b_1R)T$ by hypothesis. Therefore,

$$aT \cap bT = d((a_1R \cap b_1R)T) = (da_1R \cap db_1R)T = (aR \cap bR)T.$$

Lemma 2.4. *Suppose R and T are domains, with R a subring of T . Let S be a multiplicatively closed subset of $R[[X]]$ not containing X . Then $R[[X]]_S \subset T[[X]]_S$ is LCM-stable if (and only if)*

$$fT[[X]]_S \cap gT[[X]]_S = (fR[[X]]_S \cap gR[[X]]_S)T[[X]]_S$$

for all $f, g \in R[[X]]$ such that $f_0 \neq 0$ and $g_0 \neq 0$.

Proof. Once again, the “only if” assertion is trivial. For the converse, note that a typical element of $R[[X]]_S$ is of the form rs^{-1} where $r \in R[[X]]$ and $s \in S$. Since s is a unit of $R[[X]]_S$, $rR[[X]]_S = rs^{-1}R[[X]]_S$. Hence, to establish LCM-stability, it suffices to consider nonzero elements $f, g \in R[[X]]$. Then there exist integers $n, m \geq 0$ and $f^*, g^* \in R[[x]]$ with $f_0^* \neq 0$ and $g_0^* \neq 0$ such that $f = X^n f^*$ and $g = X^m g^*$. Now, suppose $\gamma \in fT[[X]]_S \cap gT[[X]]_S$. Then there exist $\alpha, \beta \in T[[X]]_S$ with $\gamma = f\alpha = g\beta$. So $X^n f^* \alpha = X^m g^* \beta$. Without loss of generality, $n \geq m$. Thus $X^{n-m} f^* \alpha = g^* \beta$. Since X is prime in $T[[X]]_S$ and X does not divide g^* , X^{n-m} divides β . So there exists $\beta^* \in T[[X]]_S$ such that $\beta = X^{n-m} \beta^*$. Hence, $\gamma X^{-n} = f^* \alpha = g^* \beta^*$. By assumption, there exist $r_j \in f^*R[[X]]_S \cap g^*R[[X]]_S$ and $\tau_j \in T[[x]]_S$ ($1 \leq j \leq n$) with $\gamma X^{-n} = \sum_{j=1}^n r_j \tau_j$. Therefore, $\gamma = \sum_{j=1}^n (r_j X^n) \tau_j$. Since $r_j X^n \in fR[[X]]_S \cap gR[[X]]_S$ for $1 \leq j \leq n$, $R[[X]]_S \subset T[[X]]_S$ is LCM-stable.

Theorem 2.5. *Let D be a Dedekind domain. Then $D[[X]] \subset T[[X]]$ is LCM-stable for every domain T containing D as a subring.*

Proof. By a result of Uda [15, Proposition 1.6], it is enough to prove that $D[[X]]_{(P, X)} \subset T[[x]]_{(P, X)}$ is LCM-stable for every nonzero prime ideal P of D . Fix a prime ideal P , and set $R = D[[X]]_{(P, X)}$ and $S = T[[X]]_{(P, X)}$.

Since D is regular, so is $D[[X]]$ (cf. [11, Exercise 5, p. 121]) and hence R is regular local. Hence, a theorem of Auslander-Buchsbaum shows R is a UFD (cf. [11, Theorem 184]). So, by Lemma 2.3, it is enough to show $fS \cap gS = (fR \cap gR)S$ for all $f, g \in R$ such that $\text{g.c.d.}(f, g) = 1$. By Lemma 2.4, we assume $f, g \in D[[X]]$ and $f_0 \neq 0$ and $g_0 \neq 0$.

Finally, we may assume that neither f nor g is a unit of R . To see this, note that if f , for example, is a unit of R , then $fR = R$ and $fS = S$, whence $fS \cap gS = gS = (gR)S = (fR \cap gR)S$.

This last assumption shows $(f, g) \subset (P, X)_{(P, X)}$, the maximal ideal of R . Since R is a UFD, the height 1 prime ideals of R are principal. Then, $\text{g.c.d.}(f, g) = 1$ and $\dim R = 2$ show that $(P, X)_{(P, X)}$ is the only prime ideal of R containing (f, g) . Therefore $\text{rad}(f, g) = (P, X)_{(P, X)}$. Hence, there exist $r_1, r_2 \in R$ and an integer $n \geq 1$ such that

$$X^n = r_1 f + r_2 g.$$

Consider $\gamma \in fS \cap gS$. There exist $\alpha, \beta \in S$ such that $\gamma = f\alpha = g\beta$. Multiplying both sides of the displayed equation by α yields $X^n \alpha = r_1(f\alpha) + r_2 g\alpha = r_1(g\beta) + r_2 g\alpha = g(r_1 \beta + r_2 \alpha)$. Now X is prime in S and $g_0 \neq 0$. Therefore, $\delta = (r_1 \beta + r_2 \alpha) X^{-n} \in S$. Then $\alpha = g\delta$ and $\gamma = f\alpha = fg\delta \in (fR \cap gR)S$, so that $fS \cap gS = (fR \cap gR)S$.

Results (2.6)–(2.10) will show that a Dedekind domain is the only type of (Prüfer) domain R having the property that $R[[X]] \subset T[[X]]$ is LCM-stable for each domain T containing R as a subring. The next proposition will be used to prove that no Prüfer domain R with $\dim R \geq 2$ can have this property. First, recall from [14] that a domain R is *Archimedean* if $\bigcap_{n=1}^{\infty} d^n R = 0$ for every nonzero nonunit $d \in R$; and recall (cf. [10]) that a valuation domain V is a Archimedean if and only if $\dim V \leq 1$.

Proposition 2.6. *Let R be a non-Archimedean domain. Then there exists an R -flat domain T containing R as a subring such that $R[[X]] \subset T[[X]]$ is not LCM-stable; it can be arranged that T is an overring of R .*

Proof. Since R is non-Archimedean, there is a nonzero nonunit $d \in R$ such that $\bigcap_{n=1}^{\infty} (d^n) \neq 0$. Choose $0 \neq a \in \bigcap_{n=1}^{\infty} (d^n R)$. Set $f = a + \sum_{n=1}^{\infty} ad^{-2n} X^n$. Set $T = R[d^{-1}] = R_{\{1, d, d^2, \dots\}}$; of course, T is R -flat.

By the choice of a , we have $ad^{-2n} \in R$ for each $n \geq 1$. Thus, $f \in R[[x]]$. Also, $fa^{-1} = 1 + \sum_{n=1}^{\infty} d^{-2n} X^n \in T[[X]]$. In fact, since its constant term is a unit of T , fa^{-1} is a unit of $T[[X]]$, and so $a = f(fa^{-1})^{-1} \in fT[[X]]$. Therefore, $a \in fT[[X]] \cap aT[[X]]$. We shall show that

$$a \notin (fR[[x]] \cap aR[[X]])T[[X]].$$

First, we claim that if a and f are as above, and $r, s \in R[[X]]$ are such that $rf = sa$, then $r_0 \in \bigcap_{n=1}^{\infty} (d^n R)$. To see this, observe first that equating coefficients of X^n on each side of $rf = sa$ gives:

$$ad^{-2n} r_0 + ad^{-2(n-1)} r_1 + \dots + ad^{-2} r_{n-1} + ar_n = as_n.$$

Next, multiplying both sides of this equation by $d^{2(n-1)}$ leads to:

$$ad^{-2(n-1)} r_0 = a(s_n d^{2(n-1)} - r_1 - \dots - r_{n-1} d^{2(n-1)-2} - r_n d^{2(n-1)}).$$

Thus, dividing by a yields:

$$d^{-2(n-1)} r_0 = s_n d^{2(n-1)} - r_1 - \dots - r_{n-1} d^{2(n-1)-2} - r_n d^{2(n-1)} \in R.$$

This holds for all $n \geq 1$. Hence $r_0 \in \bigcap_{n=1}^{\infty} (d^n R)$, as claimed.

Now suppose that $a \in (fR[[X]] \cap aR[[X]])T[[X]]$, contrary to what we wish to show. Then there exist $h_i \in fR[[X]] \cap aR[[X]]$ and $\gamma_i \in T[[X]]$ ($1 \leq i \leq m$) such that $a = \sum_{i=1}^m h_i \gamma_i$. Furthermore, for each i , there are $r_i, s_i \in R[[X]]$ with $h_i = r_i f = s_i a$. By the above claim, $r_{i,0} \in \bigcap_{n=1}^{\infty} (d^n R)$ for each i . Equating constant terms of $a = \sum_{i=1}^m h_i \gamma_i$ gives $a = \sum_{i=1}^m h_{i,0} \gamma_{i,0}$. Since $h_i = r_i f$ and $f_0 = a$, we have $a = \sum_{i=1}^m r_{i,0} a \gamma_{i,0}$. Dividing by a gives:

$$1 = \sum_{i=1}^m r_{i,0} \gamma_{i,0}.$$

Each $\gamma_{i,0} \in T = R[d^{-1}]$; so, for each i , there is an integer u_i such that $d^{u_i} \gamma_{i,0} \in R$. Let $u = \max_{1 \leq i \leq m} \{u_i\}$. Then $d^u \gamma_{i,0} \in R$ for each i . Multiplying the last displayed equation by d^u gives $d^u = \sum_{i=1}^m r_{i,0} (d^{u_i} \gamma_{i,0} d^{u-u_i})$. Since $r_{i,0} \in \bigcap_{n=1}^{\infty} (d^n R)$ and $\gamma_{i,0} d^u \in R$ for $1 \leq i \leq m$, we have $d^u \in \bigcap_{n=1}^{\infty} (d^n R)$, whence we have $d = 0$ or d is a unit of R , contradicting our choice of d . Thus, $a \notin (fR[[X]] \cap aR[[X]])T[[X]]$, and $R[[X]] \subset T[[X]]$ is not LCM-stable.

Corollary 2.7. *Let R be a Prüfer domain such that $\dim R \geq 2$. Then there is $c \in R$ such that $R[[X]] \subset R[c^{-1}][[X]]$ is not LCM-stable.*

Proof. There exists a chain of prime ideals $0 \subsetneq P \subsetneq M$ in R . Pick $c \in M \setminus P$. Since R is a Prüfer domain, R_M is a valuation domain. Therefore $P \subset cR_M$. Hence $P \subset \bigcap_{n=1}^{\infty} (c^n R_M)$. In particular, we may choose $0 \neq a \in [\bigcap_{n=1}^{\infty} c^n R_M] \cap P$.

By the choice of a , $ac^{-n} \in R_M$ for each n . So, for each $n \geq 1$, there exist $r_n \in R$ and $s_n \in R \setminus M$ such that $ac^{-2^n} = r_n s_n^{-1}$. Put $f = a + \sum_{n=1}^{\infty} r_n X^n \in R[[X]]$. Now $r_n = a s_n c^{-2^n}$, so that $f = a + \sum_{n=1}^{\infty} a s_n c^{-2^n} X^n$. Since $1 + \sum_{n=1}^{\infty} s_n c^{-2^n} X^n$ is a unit of $R[c^{-1}][[X]]$ and $f = a(1 + \sum_{n=1}^{\infty} s_n c^{-2^n} X^n)$, we have $fR[c^{-1}][[X]] \cap aR[c^{-1}][[X]] = aR[c^{-1}][[X]]$. However, since $f, a \in R_M[[X]]$ and R_M is non-Archimedean (R_M is a valuation domain of dimension at least two), we may apply the proof of Proposition 2.6 to get $a \notin (fR_M[[X]] \cap aR_M[[X]])R_M[c^{-1}][[X]]$. (To adapt the earlier proof, get each $r_{i,0} \in \bigcap_{n=1}^{\infty} (c^n R_M)$ since each $s_n \in R \setminus M$.) Since $R[[X]] \subset R_M[[X]]$ and $R[c^{-1}][[X]] \subset R_M[c^{-1}][[X]]$ we have

$$(fR[[X]] \cap aR[[X]])R[c^{-1}][[X]] \subset (fR_M[[X]] \cap aR_M[[X]])R_M[c^{-1}][[X]].$$

Thus, $a \notin (fR[[X]] \cap aR[[X]])R[c^{-1}][[X]]$, and so $R[[X]] \subset R[c^{-1}][[X]]$ is not LCM-stable.

It should be noted that the ‘‘Prüfer’’ hypothesis in (2.7) can be weakened to ‘‘locally divided’’ (in the sense of D. E. Dobbs [6]). The present formulation was chosen in order to clarify the route to our main goal, Theorem 2.11. (In this vein, we also note that the ‘‘UFD’’ hypothesis in Lemma 2.3 can be weakened to ‘‘GCD-domain’’.)

Corollary 2.7 shows that we need only consider those Prüfer domains R such that $\dim R \leq 1$. In order to prove that the domains with the desired property are Dedekind, we shall show they are locally DVRs and Noetherian. We first consider the local case.

Theorem 2.8. *Let (V, M) be a rank one nondiscrete valuation domain. Then there exists a domain D containing V as a subring such that $V[[X]] \subset D[[X]]$ is not LCM-stable.*

Proof. Since V is rank one nondiscrete, there is a (surjective) valuation $v: \text{q.f.}(V) \rightarrow G$ where G is a dense subgroup of (the additive group of) \mathbb{R} . By density, we can inductively construct a sequence $\{\epsilon_j\}_{j=1}^{\infty} \subset G$ such that $\lim_{j \rightarrow \infty} \epsilon_j = 0$, $\epsilon_j > \epsilon_{j+1} > 0$ for all $j \geq 1$, and $\epsilon_j - \epsilon_{j+1} > \epsilon_{j+1} - \epsilon_{j+2}$ for all j . (Perhaps the easiest way to do this is to insure that $0 < \epsilon_{j+1} < 2^{-1}\epsilon_j$.) For each $j \geq 1$, choose $t_j \in V$ such that $v(t_j) = \epsilon_j$. Put $g = \sum_{j=0}^{\infty} t_{j+1} X^j \in V[[X]]$. Let Y be an indeterminate over V , and set $D = V[Y, \{Yt_j^{-1}\}_{j=2}^{\infty}, \{Yt_j t_1^{-1}\}_{j=1}^{\infty}]$.

We claim $Yt_1^{-1} \notin D$. To see this, suppose, to the contrary, that $Yt_1^{-1} \in D$. Then Yt_1^{-1} can be expressed as a polynomial over V in $Y, \{Yt_j^{-1}\}_{j=2}^{\infty}$, and $\{Yt_j t_1^{-1}\}_{j=1}^{\infty}$. Working in $\text{q.f.}(V)[Y]$, we write this polynomial as $Yt_1^{-1} = c_0 + c_1 Y + c_2 Y^2 + \dots + c_n Y^n$, with $c_i \in \text{q.f.}(V)$ for all i . Since Y is an indeterminate, we have $c_i = 0$ if $i \neq 1$. Hence any element of D of the form $c_1 Y$ (with $c_1 \in \text{q.f.}(V)$) must be a linear combination of $Y, \{Yt_j^{-1}\}_{j=2}^{\infty}$, and $\{Yt_j t_1^{-1}\}_{j=1}^{\infty}$. Thus, $Yt_1^{-1} = \sum_{r=1}^a d_r Yt_r^{-1} + \sum_{s=1}^b e_s Yt_s t_1^{-1} + cY$ where

$d_r, e_s, c \in V$ for all r, s . (Note all $j_r \geq 2$.) Set $u = \max_{r,s} \{v(t_{j_r}), v(t_{j_s}t_1^{-1})\}$. Note that $0 \leq u < v(t_1)$ since $\{\varepsilon_j\}_{j=1}^\infty$ is a strictly decreasing sequence of positive numbers. Pick $t \in V$ such that $v(t) = u$. Then, multiplying the above linear combination by tY^{-1} leads to:

$$tt_1^{-1} = \sum_{r=1}^a d_r tt_{j_r}^{-1} + \sum_{s=1}^b e_s (tt_{j_s}t_1^{-1}) + ct.$$

By the choices of u and t , this is an element of V . But $v(tt_1^{-1}) = v(t) - v(t_1) < 0$, whence $tt_1^{-1} \notin V$, a contradiction. Hence, $Yt_1^{-1} \notin D$, as claimed.

Consider Yg . Clearly, $Yg \in gD[[X]]$. Also,

$$Yg = \sum_{j=0}^\infty Yt_{j+1}X^j = \sum_{j=0}^\infty t_1(Yt_{j+1}t_1^{-1})X^j = t_1 \sum_{j=0}^\infty (Yt_{j+1}t_1^{-1})X^j \in t_1D[[X]].$$

Hence, $Yg \in gD[[X]] \cap t_1D[[X]]$. We shall show that

$$Yg \notin (gV[[X]] \cap t_1V[[X]])D[[X]],$$

and therefore that $V[[X]] \subset D[[X]]$ is not LCM-stable.

Suppose $f \in gV[[X]] \cap t_1D[[X]]$. Then there exist $r, s \in V[[X]]$ with $f = gr = t_1s$. We can now apply [1, Lemma, p. 191] which seems tailor-made for such series. The result is $v(r_i) \geq v(t_1) - v(t_{j+1})$ for each i and for each j . Since $\lim_{j \rightarrow \infty} v(t_{j+1}) = 0$, we have $v(r_i) \geq v(t_1)$ for each i . Thus $rit_1^{-1} \in V$ for each i , so that $rt_1^{-1} \in V[[X]]$. Next, write $f = gr = gt_1(rt_1^{-1})$ and equate constant terms. Since $g_0 = t_1$, this gives $f_0 = g_0t_1(r_0t_1^{-1}) = t_1^2r_0t_1^{-1} = t_1r_0$. Thus, $v(f_0) = v(t_1r_0) = v(t_1) + v(r_0) \geq v(t_1) + v(t_1) = v(t_1^2)$. Hence, there is $f_0^* \in V$ such that $f_0 = t_1^2f_0^*$.

Now, if $Yg \in (gV[[X]] \cap t_1V[[X]])D[[X]]$, then $Yg = \sum_{k=1}^m f_k\alpha_k$ for some $f_k \in gV[[X]] \cap t_1V[[X]]$ and $\alpha_k \in D[[X]]$. Equating constant terms and using the result of the preceding paragraph give $f_{k,0}^* \in V$ (for $1 \leq k \leq m$) such that $Yg_0 = Yt_1 = \sum_{k=1}^m f_{k,0}\alpha_{k,0} = \sum_{k=1}^m (t_1^2f_{k,0}^*)\alpha_{k,0} = t_1^2 \sum_{k=1}^m f_{k,0}^*\alpha_{k,0}$. Thus, $Yt_1 = t_1^2 \sum_{k=1}^m f_{k,0}^*\alpha_{k,0}$. Dividing by t_1^2 yields $Yt_1^{-1} = \sum_{k=1}^m f_{k,0}^*\alpha_{k,0} \in D$. But this contradicts the claim that was established above. Hence, $Yg \notin (gV[[X]] \cap t_1V[[X]])D[[X]]$ and $V[[X]] \subset D[[X]]$ is not LCM-stable.

The local case that was considered in Theorem 2.8 is next used to eliminate any one-dimensional Prüfer domain which is not almost Dedekind. (As in [10], a domain R is said to be *almost Dedekind* if R_M is a DVR for each maximal ideal M of R . Each almost Dedekind domain is Prüfer and of dimension at most one. The converse is false.)

Corollary 2.9. *Let R be a Prüfer domain. If $\dim R = 1$ and R is not almost Dedekind, then there is a domain D containing R as a subring such that $R[[X]] \subset D[[X]]$ is not LCM-stable.*

Proof. Since R is a Prüfer domain, R_M is a valuation domain for each prime ideal M of R . Moreover, $\dim R = 1$ shows R_M is a rank one valuation domain for each nonzero prime ideal M . Also, since R is not almost Dedekind, there is at least one prime ideal M such that R_M is nondiscrete. Fix a prime M such that R_M is a nondiscrete valuation domain.

As in the proof of Theorem 2.8, choose $t_j \in R_M$ such that $v(t_j) = \varepsilon_j$ for all $j \geq 1$, where the sequence $\{\varepsilon_j\}_{j=1}^\infty$ is as in the earlier proof. Now, for each j , write $t_j = r_j s_j^{-1}$, with $r_j \in R$, $s_j \in R \setminus M$. Since s_j is a unit of R_M , we have $v(s_j) = 0$ for $j \geq 1$. Thus $v(t_j) = v(r_j s_j^{-1}) = v(r_j) - v(s_j) = v(r_j)$ for all $j \geq 1$. So, by putting $g = \sum_{j=0}^\infty r_{j+1} X^j$, the proof of Theorem 2.8 shows there exists a domain D containing R_M (and hence R) as a subring such that $gD[[X]] \cap r_1 D[[X]] \not\supseteq (gR_M[[X]] \cap r_1 R_M[[X]])D[[X]]$. But $g, r_1 \in R[[X]]$ and $gR_M[[X]] \cap r_1 R_M[[X]] \supset gR[[X]] \cap r_1 R[[X]]$. Thus, $gD[[X]] \cap r_1 D[[X]] \not\supseteq (gR[[X]] \cap r_1 R[[X]])D[[X]]$, and so the domain D has the required properties.

Finally, we show the almost Dedekind, but not Dedekind, domains do not have the desired property. It is convenient to recall that a domain R is an *SFT-ring* if, for each ideal I of R , there is a finitely generated ideal J of R such that $J \subset I$ and there exists an integer $k \geq 1$ such that $i^k \in J$ for each $i \in I$.

Theorem 2.10. *Let R be an almost Dedekind, but not Dedekind, domain. Then there exists a domain D containing R as a subring such that $R[[X]] \subset D[[X]]$ is not LCM-stable; it can be arranged that D is an overring of R .*

Proof. By a remark of Arnold [3, §3, p. 4], R is not an SFT ring. Hence, by [2, Theorem 1], there exists a prime ideal P of R such that $P[[X]] \neq \text{rad}(PR[[X]])$. Then [4, Proposition 1] shows $\text{height}(P[[X]]) \geq 2$. We claim $R[[X]]_{P[[X]]}$ is not a valuation domain; indeed, since $\dim(R[[X]]_{P[[X]]) \geq 2$, $R[[X]]_{P[[X]]}$ is not a DVR, and so the claim follows from [4, Theorem 1]. Since R_P is a DVR and $R[[X]]_{P[[X]]}$ is not a valuation domain, [4, Theorem 3] shows $R_P[[X]] \cap \text{q.f.}(R[[X]]) \not\subset R[[X]]_{P[[X]]}$.

Pick $\gamma \in R_P[[X]] \cap \text{q.f.}(R[[X]]) \setminus R[[X]]_{P[[X]]}$. Since $\gamma \in \text{q.f.}(R[[X]])$, we may write $\gamma = ab^{-1}$ with $0 \neq a, b \in R[[X]]$. Thus $a = b\gamma$ and hence $aR_P[[X]] \cap bR_P[[X]] = aR_P[[X]] = aR_P[[X]]$. We shall show if $R[[X]] \subset R_P[[X]]$ is LCM-stable, then $\gamma \in R[[X]]_{P[[X]]}$, contradicting the choice of γ .

Now, if we assume $R[[X]] \subset R_P[[X]]$ is LCM-stable then

$$a \in (aR[[X]] \cap bR[[X]])R_P[[X]].$$

Hence, there exist $h_i \in aR[[X]] \cap bR[[X]]$ and $\tau_i \in R_P[[X]]$ (for $1 \leq i \leq n$) such that $a = \sum_{i=1}^n h_i \tau_i$. Also, there exist $r_i, s_i \in R[[X]]$ (for $1 \leq i \leq n$) with $h_i = ar_i = bs_i$. Thus $a = \sum_{i=1}^n h_i \tau_i = \sum_{i=1}^n (ar_i) \tau_i = a \sum_{i=1}^n r_i \tau_i$. Therefore $1 = \sum_{i=1}^n r_i \tau_i$. Since $R_P[[X]]$ is a local ring, there exists i_0 such that $r_{i_0} \tau_{i_0}$ is a unit in $R_P[[X]]$. Without loss of generality, $i_0 = 1$. But then r_1 is a unit in $R_P[[X]]$ and hence $r_{1,0} \in R \setminus P$. Since $r_1 \in R[[X]]$, r_1 is a unit in $R[[X]]_{P[[X]]}$. Thus, $ar_1 = bs_1$ leads to $\gamma = ab^{-1} = s_1 r_1^{-1} \in R[[X]]_{P[[X]]}$, the desired contradiction. Hence $R[[X]] \subset R_P[[X]]$ is not LCM-stable and $D = R_P$ satisfies the conclusion of the theorem.

Hence, (2.1)–(2.10) combine to establish our main result.

Theorem 2.11. *A domain R is a Dedekind domain if and only if $R[[X]] \subset T[[X]]$ is LCM-stable for all domains T containing R as a subring.*

The question whether the domains T in Theorem 2.11 can be restricted to be overrings of R is open. By our methods of proof, one need only settle this issue in the context of Theorem 2.8.

Remark 2.12 sketches some related work on flatness. First, we give some relevant background.

In [7], Dobbs introduced the notion of n -flatness, where n is a positive integer. It is classical [5] that an R -module M is R -flat if and only if M is n -flat over R for each integer n . Dobbs [8, Theorem 3.3] shows that LCM-stability is equivalent to 2-flatness. Hence, Theorem 2.11 may be rephrased as follows. The domain R is a Dedekind domain if and only if $T[[X]]$ is 2-flat over $R[[X]]$ for each domain T containing R as a subring. Replacing “2-flat” with “flat”, we feel it is natural to ask which domains have the property (*) that $T[[x]]$ is a flat $R[[X]]$ -module for all domains T containing R as a subring. In view of the above remarks, Theorem 2.11 shows that any domain having the property (*) must be a Dedekind domain. That *some* domain does in fact have property (*) is included in Remark 2.12(a).

Remark 2.12. (a) If A is a subring of the domain B such that $\text{q.f.}(A) \subset B$, then $B[[X]]$ is $A[[X]]$ -flat. (The proof is a lengthy calculation using the characterization of flatness in terms of conductors [5, Exercise 2.2, p. 47].)

(b) One motivation for (a) is the trivium that fields satisfy (*). In fact, by using [5, Exercise 17, p. 250], we see that $B[[X_1, \dots, X_n]]$ is $K[[X_1, \dots, X_n]]$ -flat for all $n \geq 1$ whenever B contains the field K as a subring. We next state a result leading to a partial result involving arbitrary Dedekind domains: If R and T are Noetherian domains such that R is a subring of T and T is R -flat, then $T[[X]]$ is $R[[X]]$ -flat. (The proof applies [5, Theorem 3, p. 203] to completion in the X -adic topology, to see that $T[[X]]$ is $T[X]$ -flat; and it concludes via transitivity of flatness and [5, Proposition 4, p. 231].)

(c) If R is Dedekind, each domain containing R as a subring is R -flat. Hence (b) yields: if R is a Dedekind domain, then $T[[X]]$ is $R[[X]]$ -flat for each Noetherian domain T containing R as a subring.

The question of characterizing domains having property (*) now reduces to determining whether “Noetherian” may be deleted from the preceding statement.

REFERENCES

1. M. Anderson and J. Watkins, *Coherence of power series rings*, J. Algebra **107** (1987), 187–194.
2. J. T. Arnold, *Krull dimension in power series rings*, Trans. Amer. Math. Soc. **177** (1973), 299–304.
3. ———, *Power series rings over Prüfer domains*, Pacific J. Math. **44** (1973), 1–11.
4. J. T. Arnold and J. Brewer, *When $(D[[X]])_{P[[X]]}$ is a valuation ring*, Proc. Amer. Math. Soc. **37** (1973), 326–332.
5. N. Bourbaki, *Commutative algebra*, Addison-Wesley, Reading, MA, 1972.
6. D. E. Dobbs, *On locally divided integral domains and CPI-overrings*, Internat. J. Math. Math. Sci. **4** (1981), 119–135.
7. ———, *On n -flat modules over a commutative ring*, Bull. Austral. Math. Soc. **43** (1991), 491–498.
8. ———, *On the criteria of D. D. Anderson for invertible and flat ideals*, Canad. Math. Bull. **29** (1986), 25–32.
9. R. Gilmer, *Finite element factorization in group rings*, Lecture Notes in Pure and Appl. Math., vol. 7, Dekker, New York, 1974.
10. ———, *Multiplicative ideal theory*, Dekker, New York, 1972.

11. I. Kaplansky, *Commutative rings*, 2nd ed., Univ. of Chicago Press, Chicago and London, 1974.
12. F. Richman, *Generalized quotient rings*, Proc. Amer. Math. Soc. **16** (1965), 794–799.
13. J. Sato and K. Yoshida, *The LCM-stability on polynomial extensions*, Math. Rep. Toyama Univ. **10** (1987), 75–84.
14. P. Sheldon, *How changing $D[[X]]$ changes its quotient field*, Trans. Amer. Math. Soc. **159** (1971), 223–244.
15. H. Uda, *LCM-stableness in ring extensions*, Hiroshima Math. J. **13** (1983), 357–377.
16. ———, *G_2 -stableness and LCM-stableness*, Hiroshima Math. J. **18** (1988), 47–52.

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