TWO-DIMENSIONAL REPRESENTATIONS
OF UNIFORM ALGEBRAS

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Abstract. It is shown that every two-dimensional representation of a uniform algebra has a dilation, which extends the result by Paulsen. We also prove some dilation result for a representation of the disk algebra.

1. Introduction

Let $C(X)$ be the algebra of complex-valued continuous functions on a compact Hausdorff space $X$, and let $A$ be a uniform algebra on $X$. Let $L(H)$ denote the algebra of all bounded linear operators on a separable Hilbert space $H$. An algebra homomorphism $\Phi : A \rightarrow L(H)$ is called a representation of $A$ on $H$ if $\Phi(1) = I_H$ and $\Phi$ is contractive, i.e., $\|\Phi(f)\| \leq \|f\|$ for all $f \in A$. Two representations $\Phi_1 : A \rightarrow L(H_1)$ and $\Phi_2 : A \rightarrow L(H_2)$ are said to be unitarily equivalent if there exists a unitary operator $U : H_1 \rightarrow H_2$ such that $U\Phi_1(f) = \Phi_2(f)U$ for all $f \in A$. For a representation $\Phi$ of $A$ on $H$, a representation $\tilde{\Phi} : C(X) \rightarrow L(K)$ is called a dilation of $\Phi$ if $H \subseteq K$ and $\tilde{\Phi}(f) = P_H\Phi(f)|_H$ for all $f \in A$, where $P_H$ is the orthogonal projection of $K$ onto $H$. Paulsen [6] showed that every two-dimensional representation of $A$ has a dilation in the case where $A$ is the algebra of all functions uniformly approximated on a compact subset $X$ of the complex plane by rational functions with poles off $X$ (see also [5]). In this note we give another proof of the above dilation result (for a general uniform algebra $A$).

B. Cole (see [1]) showed that for any closed ideal $J$ in a uniform algebra $A$, the quotient algebra $A/J$ is isometrically isomorphic to an algebra of bounded operators on a Hilbert space $H$, or equivalently, there is a representation $\Phi : A \rightarrow L(H)$ such that $\|\Phi(f)\| = \|f + J\|$ for all $f \in A$, where $\|f + J\|$ is the quotient norm of the coset $f + J$ of $f$ in $A/J$. We say a representation $\Phi$ of $A$ is $Q$-isometric if $\|\Phi(f)\| = \|f + \ker\Phi\|$ for all $f \in A$, and a $Q$-isometric representation $\Phi : A \rightarrow L(K)$ is a $Q$-isometric dilation of a representation $\Phi : A \rightarrow L(H)$ if $H \subseteq K$ and $\Phi(f) = P_H\Phi(f)|_H$ for all $f \in A$. A
$Q$-isometric representation of $A$ is used by Cole, Lewis and Wermer [2] to generalize Pick's conditions of the interpolation problem for the disk algebra to the case of the uniform algebra $A$. The result of Cole stated above shows that any representation $\Phi$ of $A$ has a $Q$-isometric dilation. Indeed, by Cole's result, there exists a $Q$-isometric representation $\Psi$ such that $\ker\Psi = \ker\Phi$. Then the representation $\Phi$ defined by $\Phi(f) = \Phi(f) \oplus \Psi(f)$, $f \in A$, is a $Q$-isometric dilation of $\Phi$. It also follows from our proof of the dilation result (Theorem 1) that if a representation $\Phi: A \to L(H)$ satisfies $\dim(A/\ker\Phi) = 2$, then $\Phi$ has a $Q$-isometric dilation $\Phi: A \to L(K)$ which is minimal in the sense that $K = \bigvee_{f \in A} \Phi(f)H$. In Section 3 it is shown that every representation of the disk algebra has a minimal $Q$-isometric dilation.

2. TWO-DIMENSIONAL REPRESENTATIONS

In this section we prove the following theorem, which extends the result by Paulsen [6].

**Theorem 1.** Let $\Phi: A \to L(H)$ be a representation of $A$. If $\dim(A/\ker\Phi) = 2$, then $\Phi$ has a dilation.

Using Misra's method [5], we first determine representations $\Phi: A \to L(H)$ such that $\dim(A/\ker\Phi) = 2$.

Let $J$ be an ideal of $A$ with $\dim(A/J) = 2$. Then

(1) $J = \{ f \in A : f(x) = f(y) = 0 \}$,

where $x$ and $y$ are two points in the maximal ideal space $M(A)$ of $A$, or

(2) $J = \{ f \in A : f(x) = \delta(f) = 0 \}$,

where $x \in M(A)$ and $\delta$ is a bounded point derivation at $x$, that is, $\delta$ is a bounded linear functional on $A$ such that $\delta(fg) = f(x)\delta(g) + g(x)\delta(f)$ for $f, g \in A$ (see, e.g., [3]).

**Lemma 1.** Let $\Phi: A \to L(H)$ be a homomorphism with $\Phi(1) = I_H$ and assume that $\dim(A/\ker\Phi) = 2$. Then, according as $J = \ker\Phi$ is of the form (1) or (2), $\Phi(f)$ is expressed as

(3) $\Phi(f) = \begin{pmatrix} f(x)I_{H_1} & (f(x) - f(y))C \\ 0 & f(y)I_{H_2} \end{pmatrix}$ on $H = H_1 \oplus H_2$

or

(3') $\Phi(f) = \begin{pmatrix} f(x)I_{H_1} & \delta(f)C \\ 0 & f(x)I_{H_2} \end{pmatrix}$ on $H = H_1 \oplus H_2$

for all $f \in A$, where $C$ is a bounded linear operator from $H_2$ to $H_1$.

**Proof.** Suppose that $J$ is of the form (1). Take functions $f_1$ and $f_2$ in $A$ such that $f_1(x) = f_2(y) = 1$ and $f_1(y) = f_2(x) = 0$. Then $\Phi(f_1)$ is idempotent and so

$\Phi(f_1) = \begin{pmatrix} I & C \\ 0 & 0 \end{pmatrix}$ on $H = \text{ran} \Phi(f_1) \oplus (\text{ran} \Phi(f_1))^\perp$.

Since $\Phi(f_1) + \Phi(f_2) = I$ and $f - f(x)f_1 - f(y)f_2 \in J$ for $f \in A$, we have

$\Phi(f) = \begin{pmatrix} f(x)I & (f(x) - f(y))C \\ 0 & f(y)I \end{pmatrix}$ on $H = \text{ran} \Phi(f_1) \oplus (\text{ran} \Phi(f_1))^\perp$. 

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for all \( f \in A \). For the case where \( J \) is of the form (2), take \( f_0 \in A \) such that \( f_0(x) = 0 \) and \( \delta(f_0) = 1 \), and note that \( \Phi(f_0)^2 = 0 \) and \( f - f(x) - \delta(f)f_0 \in J \) for \( f \in A \).

**Lemma 2** (cf. [5, the proof of Theorem 2.3]). Let \( C : H_2 \to H_1 \) and \( D : K_2 \to K_1 \) be two operators, where \( H_1, H_2, K_1 \) and \( K_2 \) are Hilbert spaces. If \( \|C\| \leq \|D\| \), then

\[
\left\| \begin{pmatrix} aI_{H_1} & C \\ 0 & bI_{H_2} \end{pmatrix} \right\| \leq \left\| \begin{pmatrix} aI_{K_1} & D \\ 0 & bI_{K_2} \end{pmatrix} \right\|
\]

for any scalars \( a \) and \( b \).

**Proof.** If \( a = 0 \) or \( D = 0 \), the inequality is clear. So suppose that \( a \) and \( D \) are nonzero. By considering \( (1 + \varepsilon)D \) (\( \varepsilon > 0 \)) instead of \( D \), we can also assume that \( \|C\| < \|D\| \). Take any unit vector \( \left( \begin{array}{c} x' \\ y' \end{array} \right) \) in \( H = H_1 \oplus H_2 \) (\( y' \neq 0 \)). Since \( \|C\| < \|D\| \), there is \( y' \in K_2 \) such that \( \|Cy\| < \|Dy'\| \) and \( \|y'\| = \|y\| \). Set \( x' = \frac{d}{\|y'\|} \left( \begin{array}{c} y \\ y' \end{array} \right) \). Then \( \|\left( \begin{array}{c} x' \\ y' \end{array} \right)\| = 1 \), and we have

\[
\left\| \begin{pmatrix} aI_{H_1} & C \\ 0 & bI_{H_2} \end{pmatrix} \left( \begin{array}{c} x' \\ y' \end{array} \right) \right\| < \left\| \begin{pmatrix} aI_{K_1} & D \\ 0 & bI_{K_2} \end{pmatrix} \left( \begin{array}{c} x' \\ y' \end{array} \right) \right\|
\]

which implies the required inequality.

Let \( \mu \) be a probability measure on \( X \), and let \( H^2(\mu) \) and \([J]_{\mu}\) denote the closure in \( L^2(\mu) \) of \( A \) and of an ideal \( J \), respectively. For each \( f \in A \), we define an operator \( S^\mu_f \) on \( H = H^2(\mu) \oplus [J]_{\mu} \) by \( S^\mu_f h = P_H(fh) \) for each \( h \in H \). Then the map \( \Phi^\mu : f \mapsto S^\mu_f \) is a representation of \( A \) on \( H \) such that \( \ker \Phi^\mu \supset J \) and has a dilation \( \tilde{\Phi}^\mu : f \mapsto M^\mu_f \), where for \( f \in C(X) \), \( M^\mu_f \) denotes the multiplication operator by \( f \) on \( L^2(\mu) \). B. Cole (see [1]) showed that for each \( f \in A \), there exists a probability measure \( \nu \) such that \( \|S^\nu_f\| = \|f + J\| \).

For \( x, y \in M(A) \) and a bounded point derivation \( \delta \) at \( x \), let

\[\sigma(x, y) = \sup\{|f(y)| : f(x) = 0 \text{ and } \|f\| \leq 1\}\]

and

\[\rho(x, \delta) = \sup\{|\delta(f)| : f(x) = 0 \text{ and } \|f\| \leq 1\}.
\]

**Lemma 3** (cf. [5, Theorem 1.1 and Corollary 1.1]). Let \( \Phi : A \to L(H) \) be a homomorphism with \( \Phi(1) = 1 \) such that \( \dim(A/\ker \Phi) = 2 \), and let \( C \) be as in Lemma 1. Then \( \Phi \) is a representation of \( A \) on \( H \) if and only if, according as \( J = \ker \Phi \) is of the form (1) or (2),

\[
\|C\| \leq \left( \frac{1}{\sigma(x, y)^2 - 1} \right)^{1/2} \text{ or } \rho(x, \delta)^{-1}.
\]

Furthermore, the equality in (4) holds if and only if \( \|\Phi(f)\| = \|f + J\| \) for all \( f \in A \).

**Proof.** By [5, Remark 2], the condition that \( \Phi \) is contractive is equivalent to the condition that \( \|\Phi(f)\| \leq \|f\| \) for all \( f \in J_x = \{f : f(x) = 0\} \). Since \( \dim(J_x/J) = 1 \) by assumption, the latter is equivalent to the condition that \( \|\Phi(f)\| \leq \|f + J\| \) for some \( f \in J_x \setminus J \). In the case where \( J \) is of the form (1), for \( f \in J_x \setminus J \), by (3) we have

\[
\Phi(f)^*\Phi(f) = \begin{pmatrix} 0 & 0 \\ 0 & |f(y)|^2(C^*C + I) \end{pmatrix},
\]
hence
\[ \|\Phi(f)\| = \|\sigma(x, y)f + J\| \]
for \( f \in J \). Hence the first part follows. Also, if \( \|\Phi(f)\| = \|f + J\| \)
for \( f \in J \), then it follows that \( \Phi \) is contractive and the equality in (4) holds. Conversely, assume that the equality in (4) holds. By Cole's result, for each \( f \in A \), there is a probability measure \( \nu \) such that \( \|f + J\| = \|S_f^\nu\| \). Since the map \( \Phi'': g \mapsto S_g^\nu \) is a representation of \( A \) such that \( \ker \Phi'' \supset J \), it follows from the first part and Lemma 2 that \( \|S_f^\nu\| \leq \|\Phi(f)\| \). (Note that if \( \dim(A/\ker \Phi'') = 1 \), then \( S_f^\nu \) is the operator of multiplication by \( f(x) \) or \( f(y) \) on the one-dimensional space and so \( \|S_f^\nu\| \leq \|\Phi(f)\| \).) Therefore \( \|f + J\| = \|\Phi(f)\| \) for all \( f \in A \).

**Corollary 1.** Let \( J \) be an ideal of \( A \) such that \( \dim(A/J) = 2 \). Then there is a probability measure \( \mu \) such that \( \|S_f^\mu\| = \|f + J\| \) for all \( f \in A \).

**Proof.** The ideal \( J \) is of the form (1) or (2). Take an \( f \in A \) such that \( f(x) = 0 \). By Cole’s result, there exists a probability measure \( \mu \) such that \( \|f + J\| = \|S_f^\mu\| \). The map \( \Phi''': g \mapsto S_g^\mu \) is a representation of \( A \) such that \( \ker \Phi''' \supset J \). If \( \ker \Phi'''' = J \), then it follows from Lemma 3 (and its proof) that \( \mu \) is the required measure. On the other hand, if \( \ker \Phi'''' \neq J \), then, since \( S_f^\mu \neq 0 \), the ideal \( J \) is of the form (1) and \( S_f^\mu = f(y) \). It follows that \( \|f + J\| = \|f(y)\| \)
\( (\neq 0) \), hence \( \sigma(x, y) = 1 \), which means \( x \) and \( y \) belong to the different Gleason parts of \( M(A) \). In this case, by Lemma 3 any representation \( \Phi \) of \( A \) such that \( \ker \Phi = J \) satisfies \( \|\Phi(g)\| = \|g + J\| \) for all \( g \in A \). Therefore we have only to take a probability measure \( \mu \) such that \( \dim(H^2(\mu) \oplus [J]_\mu) = 2 \), for example, \( \mu = (\nu_1 + \nu_2)/2 \), where \( \nu_1 \) and \( \nu_2 \) are representing measures of \( x \) and \( y \), respectively.

**Proof of Theorem 1.** Suppose that \( J = \ker \Phi \) is of the form (1). By Lemma 3, \( \Phi(f) \) (\( f \in A \)) is expressed as (3) with \( \|\sigma\| \leq \alpha = (\sigma(x, y)^{-2} - 1)^{1/2} \). If \( \alpha = 0 \), then \( C = 0 \) and clearly \( \Phi \) has a dilation, which is unitarily equivalent to the representation
\[
\begin{pmatrix}
\sum_{1 \leq n \leq d_1} \oplus M_{f_{11}}^\mu & \oplus \\
\sum_{1 \leq n \leq d_2} \oplus M_{f_{22}}^\mu & \oplus
\end{pmatrix}
\]
of \( C(X) \) on the space \( (\sum_{1 \leq n \leq d_1} \oplus L^2(\mu_1)) \oplus (\sum_{1 \leq n \leq d_2} \oplus L^2(\mu_2)) \), where \( \mu_1 \) and \( \mu_2 \) are representing measures of \( x \) and \( y \), respectively, and \( d_i = \dim H_i \) for \( i = 1, 2 \). So assume \( \alpha \neq 0 \). Then we can define an operator
\[
W = \begin{pmatrix}
(I_{H_1} - \alpha^{-2}CC^*)^{1/2} & 0 \\
\alpha^{-1}C^* & 0 \\
0 & I_{H_2}
\end{pmatrix}
: H_1 \oplus H_2 \to H_1 \oplus H_2 \oplus H_2.
\]
Also, define a representation \( \Psi \) of \( A \) on \( K = H_1 \oplus H_2 \oplus H_2 \) by
\[
\Psi(f) = \begin{pmatrix}
f(x)I_{H_1} & 0 & 0 \\
0 & f(x)I_{H_1} & \alpha(f(x) - f(y))I_{H_2} \\
0 & 0 & f(y)I_{H_2}
\end{pmatrix}.
\]
Then the operator $W$ is isometric and satisfies $\Psi(f)^* W = W \Phi(f)^*$ for $f \in A$. Therefore $\text{ran} \, W$ is invariant for the algebra $\{\Psi(f)^*: f \in A\}$ and the representation $\Phi$ is unitarily equivalent to a representation $\Psi_0$ of $A$ on $\text{ran} \, W$ defined by $\Psi_0(f) = P_{\text{ran} \, W} \Psi(f)|_{\text{ran} \, W}$. By Corollary 1 and Lemma 3, there exists a probability measure $\mu$ such that for $f \in A$, the operator $S_f^\mu$ on $H^2(\mu) \oplus [J]_\mu$ is expressed as

$$S_f^\mu = \begin{pmatrix} f(x) & \alpha(f(x) - f(y)) \\ 0 & f(y) \end{pmatrix}$$

(with respect to some orthonormal basis). Also, if $\nu$ is a representing measure of $x$, then $S_f^\nu$ is the multiplication operator by $f(x)$ on the one-dimensional space. Thus $\Psi$ has a dilation, which is unitarily equivalent to the representation

$$f \mapsto \left( \sum_{1 \leq n \leq d_1} \oplus M_f^\nu \right) \oplus \left( \sum_{1 \leq n \leq d_2} \oplus M_f^\mu \right)$$

of $C(X)$ on $(\sum_{1 \leq i \leq d_1} L^2(\nu)) \oplus (\sum_{1 \leq i \leq d_2} L^2(\mu))$. Hence it follows that $\Phi$ has a dilation.

The above argument is also applied to the case where $J$ is of the form (2), if the definition of $\Psi(f)$ is replaced by

$$T(f) = \begin{pmatrix} f(x) |H_1 \rangle \langle 0 | & 0 \\ 0 & f(x) |H_2 \rangle \langle 0 | \end{pmatrix}$$

where $\alpha = \rho(x, \delta)^{-1}$ ($> 0$). Thus the proof is complete.

**Corollary 2.** If $\Phi$ is a representation of $A$ with $\dim(A/\ker \Phi) = 2$, then $\Phi$ has a minimal $Q$-isometric dilation.

**Proof.** Let $\Psi$, $\Psi_0$ and $W$ be as in the proof of Theorem 1. Then the invariant subspace $K_1 = \bigvee_{f \in A} \Psi(f) \text{ran} \, W$ of the algebra $\{\Psi(f): f \in A\}$ generated by $\text{ran} \, W$ includes the space $\{0\} \oplus H_2 \oplus H_2$, hence the representation of $A: f \mapsto \Psi(f)|_{K_1}$ is a minimal $Q$-isometric dilation of $\Psi_0$. Since $\Phi$ is unitarily equivalent to $\Psi_0$, it follows that $\Phi$ has a minimal $Q$-isometric dilation. (Note that if $\alpha = 0$, then $\Phi$ is $Q$-isometric by Lemma 3.)

**3. Representations of the disk algebra**

We consider a minimal $Q$-isometric dilation of a representation of the disk algebra. In the following, $A$ denotes the disk algebra, i.e., $A$ is the algebra of all continuous functions on the unit circle $T$ whose Fourier coefficients vanish on the negative integers. Let $H^p$ (1 $\leq p \leq \infty$) denote the Hardy space on $T$, thus $H^p$ is the closure of $A$ in $L^p = L^p(m)$ or the weak*-closure of $A$ in $L^\infty = L^\infty(m)$ according as $p < \infty$ or $p = \infty$, where $m$ is the Lebesgue measure of $T$.

We use results from the dilation theory of Sz.-Nagy and Foias [8]. Let $T$ be a contraction (i.e., $\|T\| \leq 1$) on a Hilbert space $H$. Then, as is well known, $T$ can be decomposed as $T = U \oplus T_1$ on $H = H_u \oplus H_1$ where $U$ is a unitary operator on $H_u$ and $T_1$ is a completely nonunitary contraction on $H_1$, that is, $T_1$ has no nonzero invariant subspace $M$ such that $T_1|_M$ is unitary (see [8, Chap. I, Theorem 3.2]). For a completely nonunitary contraction $T$ on
$H$, the Sz.-Nagy and Foias functional calculus defines the weak $^\ast$-continuous algebra homomorphism $\Phi_T : f \mapsto f(T)$ from $H^\infty$ to $L(H)$, and $T$ is said to be of class $C_0$ if $\Phi_T$ is not injective (see [8, Chap. III]). If $T$ is of class $C_0$, then $T^n \to 0$ strongly (see [8, Chap. III, Proposition 4.2]), thus $T$ is unitarily equivalent to the (functional model) operator
\[ S(M) = P_{H^2(E) \ominus M} S|H^2(E) \ominus M , \]
where $H^2(E)$ is the $E$-valued Hardy space ($E$ is a Hilbert space), $S$ is the unilateral shift on $H^2(E)$ and $M$ is an invariant subspace of $S$ such that $\bigvee_{n \geq 0} S^n(H^2(E) \ominus M) = H^2(E)$ (see [8, Chap. VI]). Also, since $\ker \Phi_T (\neq \{0\})$ is a weak $^\ast$-closed ideal in $H^\infty$, we have $\ker \Phi_T = qH^\infty$ for an inner function $q$. The following lemma immediately follows from these facts.

**Lemma 4.** If $T$ is a contraction on $H$ of class $C_0$, then there is a contraction $\tilde{T}$ on $\tilde{H}$ (of $H$) of class $C_0$ satisfying the following conditions:

(i) $T^\ast = \tilde{T}^\ast | \tilde{H}$;
(ii) $\|f(\tilde{T})\| = \|f + \ker \Phi_T\|$ for all $f \in H^\infty$;
(iii) $\tilde{H} = \bigvee_{n \geq 0} \tilde{T}^n H$.

**Proof.** We may consider $T$ as the functional model $S(M) = P_{H^2(E) \ominus M} S|H^2(E) \ominus M$. Let $\ker \Phi_T = qH^\infty$, where $q$ is inner. Since $q(S(M)) = 0$, we have $M \supset qH^2(E)$. Define a contraction $\tilde{T}$ on $\tilde{H} = H^2(E) \ominus qH^2(E)$ by $\tilde{T}^\ast = S^\ast | \tilde{H}$. Then clearly (i) holds and the condition $\bigvee_{n \geq 0} S^n(H^2(E) \ominus M) = H^2(E)$ implies (iii). Also, $\tilde{T}$ is unitarily equivalent to the direct sum $\sum_{1 \leq n \leq d} \oplus S(q)$, where $d = \dim E$ and $S(q)$ is an operator on $H^2 \ominus qH^2$ defined by $S(q) h = P_{H^2 \ominus qH^2} (zh)$ ($h \in H^2 \ominus qH^2$). Therefore, for $f \in H^\infty$, we have
\[ \|f(\tilde{T})\| = \left\| f \left( \sum_{1 \leq n \leq d} \oplus S(q) \right) \right\| = \|f(S(q))\| , \]
and so $\|f(\tilde{T})\| = \|f + qH^\infty\|$ (see [7]).

For a closed subset $K$ of $T$ (of measure zero), let $I(K)$ denote the ideal consisting of all functions of $A$ which vanish on $K$. For each $f \in A$, $\|f + I(K)\| = \|f\|_K$, where $\|f\|_K = \sup \{|f(z)| : z \in K\}$ (see the proof of [4, p. 81, Theorem]). Also, for an inner function $q$, let $\text{supp } q$ denote the support of $q$, that is, $\text{supp } q$ is the set of all points on $T$ for which there exists a sequence $\{z_n\}$ from the open unit disc such that $z_n \to z$ and $q(z_n) \to 0$. Thus, if a nonzero function $f$ belongs to $qH^\infty \cap A$, then $f = 0$ on $\text{supp } q$, so it follows that $\text{supp } q$ is of measure zero (see [4, p. 52]) and $qf$ is equal a.e. to a function in $A$. Also, the inner function $q$ is analytic at each point on $T$ which does not belong to $\text{supp } q$. Therefore we have $qH^\infty \cap I(K) = qI(\text{supp } q \cup K)$ for an inner function $q$ and a closed subset $K$. It is known (see [4, p. 85, Theorem]) that $J$ is a nonzero closed ideal of $A$ if and only if $J = qI(K)$ where $K$ is a closed subset of measure zero and $q$ is an inner function such that $\text{supp } q \subset K$.

**Lemma 5.** Let $J$ be a closed ideal of $A$ and $J = qI(K)$, where $K$ is a closed subset of measure zero and $q$ is an inner function with $\text{supp } q \subset K$. Then, for
all \( f \in A \),

\[
\| f + J \| = \max \{ \| f + qH^\infty \|, \| f \|_K \}
\]

\[
= \max \{ \| f + qH^\infty \|, \| f \|_{K \setminus \text{supp } q} \}.
\]

**Proof.** Let \( f \in A \) and take a measure \( \mu \) on \( T \) annihilating \( J = qI(K) \) such that \( \| \mu \| = 1 \) and

\[
\| f + J \| = \int_T f \, d\mu.
\]

Since \( \mu \) annihilates \( J \), the proof of [4, p. 85, Theorem] shows that \( d\mu = \bar{q}h \, dm + d\nu \) where \( h \in zH^1 \) and \( \nu \) is a measure on \( T \) such that \( \text{supp } \nu \subset K \). Therefore we have

\[
\| f + J \| = \int_T f \, d\mu = \int_T f \bar{q}h \, dm + \int_T f \, d\nu
\]

\[
\leq \| f + qH^\infty \| \| h \|_1 + \| f \|_K \| \nu \|
\]

\[
\leq \max \{ \| f + qH^\infty \|, \| f \|_K \} (\| h \|_1 + \| \nu \|)
\]

\[
= \max \{ \| f + qH^\infty \|, \| f \|_K \}.
\]

The converse inequality is obvious, so the first equality is proved. For the proof of the second equality, it suffices to show \( \| f + qH^\infty \| \geq \| f \|_{\text{supp } q} \). Take any \( z \in \text{supp } q \). Then there is a sequence \( \{ z_n \} \) from the open unit disc such that \( z_n \to z \) and \( q(z_n) \to 0 \). Therefore, for all \( h \in H^\infty \), \( \| f + qh \| \geq | f(z_n) + q(z_n)h(z_n) | \to | f(z) | \), so that \( \| f + qh \| \geq \| f \|_{\text{supp } q} \). It follows that \( \| f + qH^\infty \| \geq \| f \|_{\text{supp } q} \).

**Theorem 2.** If \( \Phi \) is a representation of the disk algebra \( A \) on \( H \), then \( \Phi \) has a minimal \( Q \)-isometric dilation.

**Proof.** Let \( T = \Phi(z) \). First suppose that \( T \) is unitary. Then it follows from the spectral theory of unitary operators that \( \ker \Phi = I(\text{supp } T) \), where for a unitary operator \( U \), \( \text{supp } U \) denotes the support of the spectral measure of \( U \). (Note that \( \text{supp } T \) is of measure zero because \( \Phi \neq 0 \), and so \( T \) is singular.) We also have

\[
\| \Phi(f) \| = \| f(T) \| = \| f \|_{\text{supp } T} = \| f + I(\text{supp } T) \|
\]

for \( f \in A \), hence \( \Phi \) is \( Q \)-isometric.

Next suppose that \( T \) is not unitary. The contraction \( T \) is decomposed as \( T = U \oplus T_1 \) on \( H = H_u \oplus H_1 \), where \( U \) is unitary and \( T_1 \) is completely nonunitary. If \( T_1 \) is not of class \( C_0 \), then \( \ker \Phi = \{ 0 \} \). In this case we define \( \tilde{T} : A \to L(H_u \oplus K) \) by \( \tilde{T}(f) = f(U \oplus V) \), where \( V \) is the minimal isometric dilation on \( K \) of the contraction \( T_1 \). Since \( V \) has a unilateral shift summand, \( \tilde{T} \) is isometric. It is easy to show that \( \Phi \) is a minimal \( Q \)-isometric dilation of \( \Phi \). If \( T_1 \) is of class \( C_0 \), then \( \ker \Phi_{T_1} = qH^\infty \) where \( q \) is inner and

\[
\ker \Phi = I_{\text{supp } U} \cap qH^\infty = I(K),
\]

where \( K = \text{supp } U \cup \text{supp } q \), which is of measure zero. By Lemma 4, there exists a contraction \( \tilde{T}_1 \) on \( H_1 \) satisfying the conditions (i), (ii) and (iii) in Lemma 4. Define \( \tilde{\Phi} : A \to L(H_u \oplus \tilde{H}_1) \) by \( \tilde{\Phi}(f) = f(U \oplus \tilde{T}_1) \). Then it easily follows from the conditions (i) and (iii) that \( P_{H} \tilde{\Phi}(f)H = \Phi(f) \) for all \( f \in A \) and \( \forall f \in A \), \( \| \tilde{\Phi}(f) \| H = H_u \oplus \tilde{H}_1 \). For any \( f \in A \), \( \| \tilde{\Phi}(f) \| = \max \{ \| f \|_{\text{supp } U}, \| f \|_{\text{supp } \tilde{T}_1} \} \) and \( \| f(\tilde{T}_1) \| = \| f + qH^\infty \| \) by the condition (ii) of \( \tilde{T}_1 \), so it follows from
Lemma 5 that $||\tilde{\Phi}(f)|| = ||f + \ker \Phi||$. Thus $\tilde{\Phi}$ is a minimal $Q$-isometric dilation of $\Phi$.

We are informed by the referee that the Ph.D. thesis of Che-Chen Chu, *Finite dimensional representation of a function algebra*, submitted to the University of Houston, 1992, contains the following stronger result of Theorem 1: If $\Phi: A \to L(H)$ is a homomorphism and $\dim H = 2$, then the cb-norm of $\Phi$ is equal to the norm of $\Phi$. However our proof of Theorem 1, which directly constructs the dilation, is different from Chu’s proof.

REFERENCES


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