

REPRESENTING ABSTRACT MEASURES BY LOEB MEASURES: A GENERALIZATION OF THE STANDARD PART MAP

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ABSTRACT. Loeb measures have been utilized to represent Radon and τ -smooth measures on topological spaces via the standard part map. The purpose of this paper is to show how to extend these results to a nontopological setting.

1. INTRODUCTION

This article considers only finite measures. In his papers [L1] and [L2], Peter Loeb discovered what now are known as Loeb measures, and utilized them to represent standard measures via the standard part map. Since then, this technique has turned out to be very useful, specially in applications to probability theory. The question that immediately arises is, which measures can be represented by Loeb measures? Robert Anderson proved that if X is a Hausdorff space and μ a Borel measure, then $L(*\mu) \circ st^{-1}(\cdot) = \mu$ if and only if μ is Radon, and also that $*\mu$ can be replaced by an internal discrete measure with hyperfinite support (see [A]). Landers and Rogge found a way to represent more general measures, namely those that are regular and τ -smooth, by using the outer measure generated by a Loeb measure, instead of the Loeb measure itself (cf. [LR]).

The purpose of this article is to extend the results obtained by Anderson, and Landers and Rogge, to abstract measure spaces. We shall work with a κ -saturated nonstandard model, where κ is larger than the cardinality of the power set of any standard set being considered. Let (Y, \mathcal{T}) be a topological Hausdorff space. For each $y \in *Y$, the monad of y , $m(y)$, is defined to be the set $\cap\{*\mathcal{O} : y \in \mathcal{O} \in \mathcal{T}\}$. The standard part map associates to each subset A of Y , the set $st^{-1}(A) = \cup\{m(y) : y \in A\} \subset *Y$. Given an algebra \mathcal{A} of subsets of a set X , we shall follow a similar strategy: Select a suitable collection $V \subset \mathcal{A}$, closed under finite unions and finite intersections, and containing \emptyset and X ; such a collection is said to be a $[\emptyset, X, \cup f, \cap f]$ -paving. Then, define the monads by using the paving V instead of a topology. More precisely, for each $x \in *X$, let $m(V, x)$ be the set $\cap\{*\mathcal{O} : x \in *\mathcal{O} \text{ and } \mathcal{O} \in V\}$, and

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for each $B \subset X$, let $st^{-1}(V, B)$ be the set $\cup\{m(V, x) : x \in B\}$. Since V is closed under finite unions and finite intersections, saturation arguments will still apply.

For the purposes of this paper, it is enough to give the expression $st^{-1}(V, \cdot)$ the meaning stated above. We shall have no need to define a partial map $st : {}^*X \rightarrow X$; such a definition requires either some type of Hausdorff condition on V , or resorting to arbitrary choices, since a point in *X may belong to more than one monad. At this point one faces the following difficulty: Given two disjoint sets $A, B \subset X$, it may happen that $st^{-1}(V, A) \cap st^{-1}(V, B) \neq \emptyset$ (this is never the case if X is a Hausdorff space, since then different points have disjoint monads). However, we will show that if a finitely additive measure μ on \mathcal{A} is outer regular with respect to V , then there exists a sufficiently large class \mathcal{C} of subsets of X , such that $A, B \in \mathcal{C}$ and $A \cap B = \emptyset$ imply $L({}^*\mu)[st^{-1}(V, A) \cap st^{-1}(V, B)] = 0$. This will allow us to obtain the results mentioned above.

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2. NOTATION, DEFINITIONS, AND MAIN RESULTS

A paving V is just a collection of subsets of some given set, and cV , the complement paving of V , consists of all complements of sets in V . We assume that V is a paving on X , closed under finite unions and finite intersections, and also that V contains X and \emptyset ; thus V is a $[\emptyset, X, \cap, \cup]$ -paving. Notice that cV is also a $[\emptyset, X, \cap, \cup]$ -paving. The symbol $T(V)$ denotes the topology generated by V . When we say that a set is compact, we mean compact with respect to $T(V)$. Likewise, a set is Borel if it belongs to the smallest σ -algebra generated by $T(V)$. Given a measure space (X, \mathcal{A}, μ) , where \mathcal{A} is an algebra and μ a finite, finitely additive measure on \mathcal{A} , we always assume that $V \subset \mathcal{A}$.

2.1. Definitions. Let V be a paving on a standard set X . For each x in *X , the V monad of x , $m(V, x)$, is the set $m(V, x) = \cap\{O : x \in {}^*O \text{ and } O \in V\}$. The set of near standard points of *X with respect to V , $ns(V, {}^*X)$, is the union of all V monads of standard points in *X ; and for each subset B of X , the standard part inverse with respect to V of the set B is given by $st^{-1}(V, B) := \cup\{m(V, x) : x \in B\}$.

If we consider V as a base for the topology $T(V)$, it follows from the usual topological case that if O is open with respect to $T(V)$, then $st^{-1}(V, O) \subset {}^*O$; if C is closed, then $ns(V, {}^*X) \cap {}^*C \subset st^{-1}(V, C)$; and if K is compact, then ${}^*K \subset st^{-1}(V, K)$. One can easily check that the Hausdorff condition is not needed in the proofs of these facts.

Let λ be a finite, internal, finitely additive measure on $({}^*X, {}^*\mathcal{A})$. In [LR], Landers and Rogge define, for each $A \subset {}^*X$,

$$L(\lambda)(A) := \inf\{L(\lambda)(B) : A \subset B \in {}^*\mathcal{A}\}.$$

A Borel measure μ is τ -smooth if for every collection \mathcal{G} of open sets, directed by inclusion, we have $\mu(\cup\mathcal{G}) = \sup\{\mu(G) : G \in \mathcal{G}\}$. Next we give the corresponding notion in the nontopological case.

2.2. Definitions. Let λ be a finite, finitely additive measure on an algebra \mathcal{A} . If a collection of sets \mathcal{G} is directed by inclusion (i.e., \mathcal{G} is directed upwards) and $\cup \mathcal{G} = O$, we write $\mathcal{G} \uparrow O$. The measure λ is said to be τ -smooth with respect to V if for every subcollection \mathcal{G} of V such that $\mathcal{G} \uparrow \cup \mathcal{G} \in V$, we have $\lambda(\cup \mathcal{G}) = \sup\{\lambda(G) : G \in \mathcal{G}\}$. We say that λ is *outer regular* with respect to V if for every $\varepsilon > 0$ and every $A \in \mathcal{A}$ there exists a set O in V such that $A \subset O$ and $\lambda(O \setminus A) < \varepsilon$. *Inner regularity* with respect to a given paving is defined in an analogous fashion.

2.3. Remark. If λ is outer regular with respect to V , then λ is inner regular with respect to cV . To see why this is true, remember that λ is finite, and just take complements. In the presence of outer regularity (with respect to V), saying that λ is τ -smooth with respect to V is equivalent to the seemingly weaker condition

(1) For every collection \mathcal{G} of V sets with $\mathcal{G} \uparrow X$, we have $\lambda(X) = \sup\{\lambda(G) : G \in \mathcal{G}\}$.

This equivalence can be proven by an argument similar to the one showing that every closed subset of a compact set is compact. Assume (1) and let ε be positive. If \mathcal{G} is directed by inclusion and $\cup \mathcal{G} \in V$, then select a V set O with $\neg \cup \mathcal{G} \subset O$ and $\lambda(O \cap (\cup \mathcal{G})) < \varepsilon$. Close $\mathcal{G} \cup \{O\}$ under finite unions. By (1) there exists a finite subcollection G_1, \dots, G_n of $\mathcal{G} \cup \{O\}$ with $\lambda(\cup_1^n G_i) + \varepsilon > \lambda(X)$. Thus there exists a finite subcollection Λ of \mathcal{G} with $\lambda(\cup \Lambda) + 2\varepsilon > \lambda(\cup \mathcal{G})$, and since \mathcal{G} is directed by inclusion we have $\lambda(\cup \mathcal{G}) = \sup\{\lambda(G) : G \in \mathcal{G}\}$.

2.4. Lemma. Let $\{A_\alpha\}$ be a collection of subsets of X . Then

$$st^{-1}(V, \cup_\alpha A_\alpha) = \cup_\alpha st^{-1}(V, A_\alpha).$$

Proof. $st^{-1}(V, \cup_\alpha A_\alpha) = \cup\{m(V, x) : x \in \cup_\alpha A_\alpha\} = \cup_\alpha \cup\{m(V, x) : x \in A_\alpha\} = \cup_\alpha st^{-1}(V, A_\alpha)$. \square

2.5. Lemma. Let the finite, finitely additive measure μ on (X, \mathcal{A}) be τ -smooth with respect to $V \subset \mathcal{A}$. Then for all $O \in V$, $\mu(O) = \bar{L}(*\mu)(st^{-1}(V, O))$.

Proof. Since $st^{-1}(V, O) \subset *O$, it is enough to show that

$$\mu(O) \leq \bar{L}(*\mu)(st^{-1}(V, O)).$$

Fix a positive standard ε , and select an internal measurable set E containing $st^{-1}(V, O)$, with $*\mu(E) < \bar{L}(*\mu)(st^{-1}(V, O)) + \varepsilon$. Then

$$\mu(O) = *\mu(*O \cap E) + *\mu(*O \cap \neg E) < \bar{L}(*\mu)(st^{-1}(V, O)) + \varepsilon + *\mu(*O \cap \neg E).$$

We conclude the proof by showing that $*\mu(*O \cap \neg E) < \varepsilon$. For each $x \in O$, $m(V, x) \subset E$, so by κ -saturation there exists a V set O_x with $x \in O_x \subset O$ and $*O_x \subset E$. By τ -smoothness we can select from $\{O_x : x \in O\}$ a finite subcollection O_1, \dots, O_n with $\mu(\cup_1^n O_i) + \varepsilon > \mu(O)$. Therefore

$$*\mu(*O \cap \neg E) < *\mu(*O) - *\mu(\cup_1^n *O_i) < \varepsilon. \quad \square$$

Landers and Rogge have shown (Theorem 1 of [LR]) that if the collection $\mathcal{D} \subset *\mathcal{A}$ has cardinality smaller than the saturation of the nonstandard model, then $\cup \mathcal{D}$ is $L(*\mu)$ measurable; if in addition $\mathcal{D} \uparrow \cup \mathcal{D}$, we have $L(*\mu)(\cup \mathcal{D}) = \sup_{D \in \mathcal{D}} L(*\mu)(D)$. The proof of this last statement will be given below in the special case where the sets in \mathcal{D} are of the form $*D$ for $D \in V$. We remark that the proof in the general case is the same.

2.6. **Lemma.** Let μ be a finite, finitely additive measure on (X, \mathcal{A}) . If G is open and \mathcal{O} is any collection of V sets such that $\mathcal{O} \uparrow G$, then

$$\bar{L}(*\mu)(st^{-1}(V, G)) \leq \sup_{O \in \mathcal{O}} \mu(O).$$

If in addition μ is τ -smooth with respect to V , then

$$\bar{L}(*\mu)(st^{-1}(V, G)) = L(*\mu)(\cup_{O \in \mathcal{O}} *O) = \sup_{O \in \mathcal{O}} \mu(O).$$

Proof. By κ -saturation there is a set $Q \in *V$ such that $*O \subset Q$ for all $O \in \mathcal{O}$, and ${}^*\mu(Q) = \sup_{O \in \mathcal{O}} \mu(O)$. Then $st^{-1}(V, G) = \cup_{O \in \mathcal{O}} st^{-1}(V, O) \subset \cup_{O \in \mathcal{O}} *O \subset Q$ and therefore $\bar{L}(*\mu)(st^{-1}(V, G)) \leq L(*\mu)(\cup_{O \in \mathcal{O}} *O) \leq {}^*\mu(Q) = \sup_{O \in \mathcal{O}} \mu(O)$. Next, assume μ is τ -smooth with respect to V . Fix $\varepsilon > 0$ and select $O' \in \mathcal{O}$ with $\mu(O') > \sup_{O \in \mathcal{O}} \mu(O) - \varepsilon$. By Lemma 2.5, $\mu(O') = \bar{L}(*\mu)(st^{-1}(V, O')) \leq \bar{L}(*\mu)(st^{-1}(V, G))$, and since ε is arbitrary, we have $\bar{L}(*\mu)(st^{-1}(V, G)) = \sup_{O \in \mathcal{O}} \mu(O)$. \square

Let \mathcal{B}' denote the σ -algebra generated by the collection $\mathcal{A} \cup T(V)$.

2.7. **Theorem.** Let the finite, finitely additive measure μ on (X, \mathcal{A}) be outer regular with respect to $T(V) \cap \mathcal{A}$. Then, if A and B are disjoint subsets of X with $A, B \in \mathcal{B}'$, we have

$$L(*\mu)[st^{-1}(V, A) \cap st^{-1}(V, B)] = 0.$$

Proof. It is enough to show that the collection \mathcal{E} of sets $A \subset X$ for which $L(*\mu)[st^{-1}(V, A) \cap st^{-1}(V, \neg A)] = 0$ is a σ -algebra containing \mathcal{A} and $T(V)$. Obviously \mathcal{E} is closed under complementation. Let $\{A_n\}$ be a sequence of sets in \mathcal{E} . We need to show that $\cup_n A_n \in \mathcal{E}$, i.e., that

$$L(*\mu)[st^{-1}(V, \cup_n A_n) \cap st^{-1}(V, \neg \cup_n A_n)] = 0.$$

But by Lemma 2.4

$$\begin{aligned} &st^{-1}(V, \cup_n A_n) \cap st^{-1}(V, \cap_n \neg A_n) \\ &= [\cup_n st^{-1}(V, A_n)] \cap st^{-1}(V, \cap_n \neg A_n) \\ &\subset \cup_n [st^{-1}(V, A_n) \cap st^{-1}(V, \neg A_n)], \end{aligned}$$

so

$$\begin{aligned} &L(*\mu)[st^{-1}(V, \cup_n A_n) \cap st^{-1}(V, \neg \cup_n A_n)] \\ &\leq \sum_n L(*\mu)[st^{-1}(V, A_n) \cap st^{-1}(V, \neg A_n)] = 0. \end{aligned}$$

It is easily seen that \mathcal{E} contains \mathcal{A} : If $A \in \mathcal{A}$, then for any $\varepsilon > 0$ there are sets $O_1, O_2 \in T(V) \cap \mathcal{A}$ with $A \subset O_1, \neg A \subset O_2$ and $\mu(O_1 \cap O_2) < \varepsilon$. Since $st^{-1}(V, A) \cap st^{-1}(V, \neg A) \subset *O_1 \cap *O_2$ and ε is arbitrary, we have

$$L(*\mu)[st^{-1}(V, A) \cap st^{-1}(V, \neg A)] = 0.$$

Next we show that all the $T(V)$ sets are in \mathcal{E} . Fix $G \in T(V)$ and $\varepsilon > 0$. The paving V is a base for $T(V)$, closed under finite unions, so there exists a collection \mathcal{O} of V sets with $\mathcal{O} \uparrow G$. Select $O' \in \mathcal{O}$ with $\sup_{O \in \mathcal{O}} \mu(O) < \mu(O') + \varepsilon/2$. By the outer regularity of μ with respect to $T(V) \cap \mathcal{A}$, we can find a set $O'' \in T(V) \cap \mathcal{A}$ such that $\neg O' \subset O''$ and $\mu(O' \cap O'') < \varepsilon/2$. Then

$$\sup_{O \in \mathcal{O}} \mu(O \cap O'') \leq \sup_{O \in \mathcal{O}} \mu(O \setminus O') + \mu(O' \cap O'') < \varepsilon.$$

Since $\neg G \subset \neg O'$, it follows that

$$\begin{aligned} &st^{-1}(V, G) \cap st^{-1}(V, \neg G) \\ &\subset [st^{-1}(V, (G \setminus O') \cup O')] \cap st^{-1}(V, \neg O') \\ &= [st^{-1}(V, G \setminus O') \cup st^{-1}(V, O')] \cap st^{-1}(V, \neg O') \quad \text{by Lemma 2.4} \\ &\subset [st^{-1}(V, G \setminus O')] \cup [st^{-1}(V, O') \cap st^{-1}(V, \neg O')]. \end{aligned}$$

Now $O' \in \mathcal{A}$ implies $L(*\mu)[st^{-1}(V, O') \cap st^{-1}(V, \neg O')] = 0$, and thus it is enough to show that $\bar{L}(*\mu)[st^{-1}(V, G \setminus O')] < \varepsilon$. But this is clear, because

$$\bar{L}(*\mu)[st^{-1}(V, G \setminus O')] \leq \bar{L}(*\mu)[st^{-1}(V, G \cap O'')] \leq \sup_{O \in \mathcal{O}} \mu(O \cap O'')$$

by Lemma 2.6, since $\{O \cap O'' : O \in \mathcal{O}\} \uparrow G \cap O''$. The set O'' was chosen so that $\sup_{O \in \mathcal{O}} \mu(O \cap O'') < \varepsilon$; hence, the proof is complete. \square

2.8. Theorem. *Let the finite, finitely additive measure μ on (X, \mathcal{A}) be outer regular and τ -smooth with respect to V . Then, for all $A \in \mathcal{A}$,*

$$\bar{L}(*\mu)[st^{-1}(V, A)] = \mu(A),$$

and furthermore, $\bar{L}(*\mu)[st^{-1}(V, \cdot)]$ is a τ -smooth and outer regular measure (with respect to $T(V)$), defined on a complete σ -algebra which contains \mathcal{B}' .

Proof. First we show that $\bar{L}(*\mu)[st^{-1}(V, \cdot)] \equiv \mu$ on \mathcal{A} . By the outer regularity of μ , we have $\bar{L}(*\mu)[st^{-1}(V, A)] \leq \mu(A)$ for every $A \in \mathcal{A}$, so

$$\begin{aligned} \bar{L}(*\mu)[ns(V, *X)] &\leq \bar{L}(*\mu)[st^{-1}(V, A)] + \bar{L}(*\mu)[st^{-1}(V, \neg A)] \\ &\leq \mu(A) + \mu(\neg A) = \mu(X) = \bar{L}(*\mu)[ns(V, *X)] \end{aligned}$$

by Lemma 2.5. Thus $\bar{L}(*\mu)[st^{-1}(V, A)] = \mu(A)$.

Next we prove that $\bar{L}(*\mu)[st^{-1}(V, \cdot)]$ is a countably additive measure on \mathcal{B}' , by showing that for each set $B \in \mathcal{B}'$ there is an $L(*\mu)$ measurable set D with $D \cap ns(V, *X) = st^{-1}(V, B)$. Clearly it is enough to do this for the sets in $\mathcal{A} \cup T(V)$, since they generate \mathcal{B}' . This part of the proof does not require the τ -smoothness of μ with respect to V . Fix $G \in T(V)$, and select $\mathcal{O} \subset V$ with $\mathcal{O} \uparrow G$. For each $n \in \mathbb{N}$ choose $O_n \in \mathcal{O}$ and $C_n \in cV$ with $C_n \subset O_n$, $\mu(O_n) + 1/2n > \sup_{O \in \mathcal{O}} \mu(O)$ and $\mu(C_n) + 1/2n > \mu(O_n)$. Then $\mu(C_n) + 1/n > \sup_{O \in \mathcal{O}} \mu(O)$ and $(ns(V, *X) \cap [\cup_n *C_n]) \subset st^{-1}(V, G) \subset \cup_{O \in \mathcal{O}} *O$. Let D be the set $[(\cup_n *C_n) \setminus ns(V, *X)] \cup st^{-1}(V, G)$. Clearly $D \cap ns(V, *X) = st^{-1}(V, G)$ and $(\cup_n *C_n) \subset D \subset (\cup_{O \in \mathcal{O}} *O)$. Since the Loeb measure is complete and $L(*\mu)(\cup_n *C_n) = L(*\mu)(\cup_{O \in \mathcal{O}} *O)$, it follows that D is $L(*\mu)$ measurable. We can utilize a similar reasoning with each set $A \in \mathcal{A}$: For every $n \in \mathbb{N}$, select $O_n \in V$ and $C_n \in cV$ with $C_n \subset A \subset O_n$ and $\mu(O_n \setminus C_n) < 1/n$, and argue as before. Thus $\bar{L}(*\mu)[st^{-1}(V, \cdot)]$ is a countably additive measure on \mathcal{B}' . We remark that if $ns(V, *X)$ is $L(*\mu)$ measurable, then the preceding argument shows that $st^{-1}(V, B)$ is $L(*\mu)$ measurable for all $B \in \mathcal{B}'$, provided that μ is outer regular with respect to V . Therefore we see that the equivalence of the measurability of $ns(V, *X)$ and $st^{-1}(V, \cdot)$ is preserved in the nontopological case. This result is of independent interest, so we will present it as a separate theorem.

In Theorem 2.7 we showed that the class \mathcal{E} of sets $A \subset X$ for which $L(*\mu)[st^{-1}(V, A) \cap st^{-1}(V, \neg A)] = 0$ is a σ -algebra and it contains \mathcal{B}' . Clearly, \mathcal{E} also contains the completion of \mathcal{B}' under $L(*\mu)[st^{-1}(V, \cdot)]$: If $D \subset X$ has outer measure zero, then $L(*\mu)[st^{-1}(V, D) \cap st^{-1}(V, \neg D)] = L(*\mu)[st^{-1}(V, D)] = 0$. Hence $\bar{L}(*\mu)[st^{-1}(V, \cdot)]$ is a complete measure on a σ -algebra containing \mathcal{B}' .

To prove that $\bar{L}(*\mu)[st^{-1}(V, \cdot)]$ is outer regular with respect to $T(V)$, we note, as Landers and Rogge do in the topological case (Lemma 7 of [LR]), that for every $E \subset X$,

$$\bar{L}(*\mu)[st^{-1}(V, E)] = \inf\{\bar{L}(*\mu)[st^{-1}(V, O)] : E \subset O \in T(V)\}.$$

It is enough to show that

$$\bar{L}(*\mu)[st^{-1}(V, E)] \geq \inf\{\bar{L}(*\mu)[st^{-1}(V, O)] : E \subset O \in T(V)\}.$$

Select any $A \in *\mathcal{A}$ with $st^{-1}(V, E) \subset A$. We will produce a set G in $T(V)$ such that $st^{-1}(V, E) \subset st^{-1}(V, G) \subset A$. For every $x \in E$ there exists, by κ -saturation, a V set O_x with $x \in O_x$ and $*O_x \subset A$. Let G be the set $\cup_{x \in E} O_x$. Then $E \subset G \in T(V)$, and thus $st^{-1}(V, E) \subset st^{-1}(V, G) \subset \cup_{x \in E} *O_x \subset A$.

Finally, we show that $\bar{L}(*\mu)[st^{-1}(V, \cdot)]$ is τ -smooth with respect to $T(V)$. Let \mathcal{O} be a subcollection of $T(V)$, directed by inclusion. To see why

$$\bar{L}(*\mu)[st^{-1}(V, \cup \mathcal{O})] = \sup_{O \in \mathcal{O}} \bar{L}(*\mu)[st^{-1}(V, O)],$$

choose, for each $O \in \mathcal{O}$, a collection \mathcal{V}_O of V sets with $O = \cup \mathcal{V}_O$. Let \mathcal{V} be the closure under finite unions of the collection $\cup\{\mathcal{V}_O : O \in \mathcal{O}\}$. Then $\mathcal{V} \subset V$ and $\mathcal{V} \uparrow \cup \mathcal{O}$. By Lemmas 2.5 and 2.6, $\bar{L}(*\mu)[st^{-1}(V, \cup \mathcal{O})] = \sup_{D \in \mathcal{V}} \mu(D) = \sup_{D \in \mathcal{V}} \bar{L}(*\mu)[st^{-1}(V, D)]$. But if $D \in \mathcal{V}$, then D is contained in a finite union of sets from \mathcal{O} . Since \mathcal{O} is directed by inclusion, there is a set in \mathcal{O} that contains D . Therefore

$$\begin{aligned} \sup_{D \in \mathcal{V}} \bar{L}(*\mu)[st^{-1}(V, D)] &\leq \sup_{O \in \mathcal{O}} \bar{L}(*\mu)[st^{-1}(V, O)] \\ &\leq \bar{L}(*\mu)[st^{-1}(V, \cup \mathcal{O})] = \sup_{D \in \mathcal{V}} \bar{L}(*\mu)[st^{-1}(V, D)]. \quad \square \end{aligned}$$

2.9. Theorem. *Let the finite, finitely additive measure μ on (X, \mathcal{A}) be outer regular with respect to V . Then, $st^{-1}(V, A)$ is $L(*\mu)$ measurable for all $A \in \mathcal{B}'$ if and only if $ns(V, *X)$ is $L(*\mu)$ measurable.*

This is a corollary to the proof of Theorem 2.8, as we indicated there.

2.10. Example. The following set-up is often encountered in probability theory as a model of an infinite sequence of coin tosses with a fair coin. Let $(X, \mathcal{A}_n, P_n)_{n \in \mathbb{N}}$ be the filtration obtained by choosing X to be the interval $[0, 1)$, \mathcal{A}_n the algebra generated by the intervals $[\frac{i}{2^n}, \frac{i+1}{2^n})$, $i = 0, 1, \dots, 2^n - 1$, and P_n the probability measure that assigns to each interval in \mathcal{A}_n its length. We will use Theorem 2.8 to find a measure P on $[0, 1)$ such that for all $n \in \mathbb{N}$, $P \upharpoonright \mathcal{A}_n = P_n$. Set $V = \cup_n \mathcal{A}_n$. Then V is an algebra. Define a measure λ on V by $\lambda(A) := P_n(A)$, where n is the smallest integer for which $A \in P_n$. Note that $\sigma(V) = Bo(X)$, the Borel sets of X , since the dyadic rational numbers are dense in $[0, 1)$, whence every open interval belongs to $\sigma(V)$. Since V is the entire algebra and every nonempty set in V is just a

finite union of intervals, the measure λ is trivially outer regular and τ -smooth with respect to V . Thus $P = \bar{L}(*\lambda) \circ st^{-1}(V, \cdot)$ extends λ to the Borel sets of X . P could also be obtained by using the Carathéodory extension theorem, after showing that for any decreasing sequence $\{A_n\}$ of sets in V with $\cap A_n = \emptyset$, $\lim \lambda A_n = 0$. But Theorem 2.8 gives us the additional information that P is τ -smooth with respect to $T(V)$.

Because $\bar{L}(*\lambda) \circ st^{-1}(V, [a, b]) = b - a$ for each dyadic interval $[a, b]$, it follows that $P = \bar{L}(*\lambda) \circ st^{-1}(V, \cdot)$ is Lebesgue measure. With a little more effort, it is possible to show that $ns(V, *[0, 1])$ is $L(*\lambda)$ measurable, so we can use $L(*\lambda) \circ st^{-1}(V, \cdot)$ to extend λ , instead of $\bar{L}(*\lambda) \circ st^{-1}(V, \cdot)$.

Fix $H \in *\mathbb{N} \setminus \mathbb{N}$, and let ν be the internal, discrete probability measure whose support is $\{0, 1/H, 2/H, \dots, (H - 1)/H\}$ and which satisfies $\nu(\{n/H\}) = 1/H$ for $n = 0, 1, \dots, H - 1$. It is easy to see that $\bar{L}(\nu) \circ st^{-1}(V, \cdot)$ (or $L(\nu) \circ st^{-1}(V, \cdot)$) also extends λ . In the following section we present a general result of this kind, for outer regular, τ -smooth measures.

The next corollary is a special case of Theorem 2.8. A family of sets \mathcal{K} is compact if for every collection $\mathcal{F} \subset \mathcal{K}$ with $\cap \mathcal{F} = \emptyset$, there is a finite subcollection $\Lambda \subset \mathcal{F}$ such that $\cap \Lambda = \emptyset$; and a measure μ is compact if it is inner regular with respect to a compact family \mathcal{K} of measurable sets. Let μ be a finite, finitely additive measure on an algebra \mathcal{A} , such that μ is compact with respect to \mathcal{K} . Let V be the closure, under finite unions and finite intersections, of the collection $c\mathcal{K} \cup \{\emptyset, X\}$. Clearly, every covering of X by sets from $c\mathcal{K} \cup \{\emptyset, X\}$ has a finite subcover. Since $c\mathcal{K} \cup \{\emptyset, X\}$ is a subbase for $T(V)$, it follows from Alexander's theorem that X is $T(V)$ compact, and thus, so is every set in \mathcal{K} . Therefore, the closure of \mathcal{K} under arbitrary intersections and finite unions, denoted here by $\hat{\mathcal{K}}$, is also a compact family. By a different technique, David Ross has proven a result basically identical to the next corollary, in the special case where μ is countably additive and \mathcal{A} a σ -algebra (see [R]). We remark that Ross' methods depend in an essential manner on the existence of a compact family of measurable sets, and therefore cannot go beyond the compact case.

2.11. Corollary. *Let the finite, finitely additive measure μ on (X, \mathcal{A}) be inner regular with respect to a compact family of sets \mathcal{K} . Let V be the closure, under finite unions and finite intersections, of the collection $c\mathcal{K} \cup \{\emptyset, X\}$. Then, for all $A \in \mathcal{A}$, $L(*\mu)[st^{-1}(V, A)] = \mu(A)$. Furthermore, $L(*\mu)[st^{-1}(V, \cdot)]$ is a countably additive measure, inner regular with respect to the compact family $\hat{\mathcal{K}}$, and defined on a complete σ -algebra which contains \mathcal{B}' .*

Proof. The measure μ is τ -smooth with respect to V , since for any upward directed collection $\mathcal{O} \subset V$ with $\cup \mathcal{O} \in \mathcal{A}$ and any positive standard ε , there is a set $K \in \mathcal{K}$ such that $K \subset \cup \mathcal{O}$ and $\mu(\cup \mathcal{O} \setminus K) < \varepsilon$. Now \mathcal{O} is upward directed and covers K , so there exists a set $O \in \mathcal{O}$ which contains K . Therefore $\mu(\cup \mathcal{O}) = \sup_{O \in \mathcal{O}} \mu(O)$. Clearly μ is outer regular with respect to V , so by Theorem 2.8, $\bar{L}(*\mu)[st^{-1}(V, \cdot)]$ extends μ to a σ -algebra containing \mathcal{B}' and is outer regular with respect to $T(V)$, hence inner regular with respect to $\hat{\mathcal{K}}$. Finally, $ns(V, *X) = *X$ is $L(*\mu)$ measurable, and hence so is $st^{-1}(V, B)$ for every set $B \in \mathcal{B}'$. Therefore $L(*\mu)[st^{-1}(V, \cdot)] \equiv \bar{L}(*\mu)[st^{-1}(V, \cdot)]$ on the completion of \mathcal{B}' , finishing the proof. \square

Theorem 2.8 entails an interesting result in abstract measure theory, namely, that every finite, finitely additive measure that is τ -smooth and outer regular with respect to some $[\emptyset, X, \cap f, \cup f]$ -paving V has a τ -smooth extension to a σ -algebra containing all unions of sets from V . The standard proof of this result is due to Topsøe (see [T]).

3. REPRESENTATION OF MEASURES BY DISCRETE INTERNAL MEASURES WITH HYPERFINITE SUPPORT

In the applications of Loeb measures, it is often useful to represent a standard measure μ not by $^*\mu$, but by an internal measure ν with a simpler structure. Typically, ν is a discrete measure with hyperfinite support. Next we show that such a representation is also possible in the nontopological case. A measure μ is continuous if $\mu(\{x\}) = 0$ for all $x \in X$; μ is discrete if there is a finite or countable set $D \subset X$ with $\mu(X \setminus D) = 0$ and $\mu(\{x\}) \neq 0$ for all $x \in D$. Every measure μ can be decomposed into a continuous and a discrete part, i.e., $\mu = \mu_c + \mu_d$, where μ_c is continuous and μ_d is discrete. An internal measure is said to be a *hyperfinite counting measure* if its support is hyperfinite and it assigns the same mass to each point in the support.

3.1. Theorem. *Let the finite, finitely additive measure μ on (X, \mathcal{A}) be outer regular and τ -smooth with respect to V . Then there exists an internal discrete measure ν with hyperfinite support such that for all $A \in \mathcal{A}$,*

$$\bar{L}(\nu)[st^{-1}(V, A)] = \mu(A),$$

and furthermore, $\bar{L}(\nu)[st^{-1}(V, \cdot)]$ is a τ -smooth and outer regular measure (with respect to $T(V)$) defined on a complete σ -algebra which contains \mathcal{B}' . If μ is continuous, then ν can be taken to be a hyperfinite counting measure.

Proof. The result is obvious if μ is discrete; ν can be obtained simply by truncating $^*\mu$. By the decomposition result mentioned before, we may assume that μ is continuous. Ward Henson has proven (Theorem 1 of [He]) that there is a hyperfinite counting measure ν on *X , with $\mu(A) = \circ\nu(^*A)$ for all $A \in \mathcal{A}$. First we need to show, as we did with $^*\mu$, that the collection \mathcal{E} of sets $A \subset X$ for which $L(\nu)[st^{-1}(V, A) \cap st^{-1}(V, \neg A)] = 0$ is a σ -algebra containing \mathcal{A} and $T(V)$. The proof that \mathcal{E} is closed under complementation and countable unions is the same as before. To show that \mathcal{E} contains \mathcal{A} and $T(V)$, one can argue as in Theorem 2.7, and then use the equality $\mu(A) = \circ\nu(^*A)$ for all $A \in \mathcal{A}$.

The set function $\bar{L}(\nu)[st^{-1}(V, \cdot)]$ is outer regular, since for every $A \subset X$ and every internal set E containing $st^{-1}(V, A)$ there is a $T(V)$ set G with $st^{-1}(V, A) \subset st^{-1}(V, G) \subset E$. Thus the result will follow from Theorem 2.8 if we show that $\bar{L}(\nu)[st^{-1}(V, G)] = \bar{L}(^*\mu)[st^{-1}(V, G)]$ for all $G \in T(V)$. The set $ns(V, ^*X)$ has full $L(\nu)$ outer measure, for, if E is an internal set contained in $^*X \setminus ns(V, ^*X)$ and $x \in X$, we can select a V set O_x with $x \in O_x$ and $^*O_x \subset \neg E$. Fix $\varepsilon > 0$ and (by the τ -smoothness of μ) find a finite subcollection O_1, \dots, O_n of $\{O_x : x \in X\}$ such that $\mu(\cup_1^n O_i) > \mu(X) - \varepsilon$. Then $\bar{L}(\nu)(E) \leq \circ\nu(^*(\neg \cup_1^n O_i)) = \mu(\neg \cup_1^n O_i) < \varepsilon$, and therefore $ns(V, ^*X)$ has full $L(\nu)$ outer measure. Fix $G \in T(V)$ and $\mathcal{O} \subset V$ with $\mathcal{O} \uparrow G$. Select $O' \in \mathcal{O}$ such that $\sup_{O \in \mathcal{O}} \mu(O) < \mu(O') + \varepsilon/2$, and $C \in cV$ with $C \subset O'$ and

$\mu(O' \setminus C) < \varepsilon/2$. Now

$$\begin{aligned} \bar{L}(\nu)[st^{-1}(V, G)] &\geq \bar{L}(\nu)[st^{-1}(V, O')] \geq \bar{L}(\nu)[*C \cap ns(V, *X)] \\ &= L(\nu)(*C) \quad (\text{since } ns(V, *X) \text{ has full outer measure}) \\ &= \mu(C) > \mu(O') - \varepsilon/2 > \bar{L}(*\mu)[st^{-1}(V, G)] - \varepsilon. \end{aligned}$$

Therefore $\bar{L}(\nu)[st^{-1}(V, G)] \geq \bar{L}(*\mu)[st^{-1}(V, G)]$. The reverse inequality is easy to prove:

$$\begin{aligned} \bar{L}(\nu)[st^{-1}(V, G)] &\leq L(\nu)(\cup_{O \in \mathcal{O}} *O) \\ &= \sup_{O \in \mathcal{O}} L(\nu)(*O) \quad \text{by Theorem 1 of [LR]} \\ &= \sup_{O \in \mathcal{O}} \mu(O) = \bar{L}(*\mu)[st^{-1}(V, G)] \quad \text{by Lemma 2.6. } \square \end{aligned}$$

The analogous result for compact measures can be obtained as a corollary from this theorem.

4. CONCLUDING REMARKS

(1) Suppose we start with an internal measure ψ on the algebra $*\mathcal{A}$, instead of a standard measure on \mathcal{A} . It is still possible to get a standard measure $\bar{L}(\psi)[st^{-1}(V, \cdot)]$ on a σ -algebra containing \mathcal{A} , provided we can find a paving $V \subset \mathcal{A}$ with the following “outer regularity” property: For each standard $\varepsilon > 0$ and each $A \in \mathcal{A}$, there is a set $O \in V$ such that $A \subset O$ and $\psi(*O \setminus *A) < \varepsilon$. It is easy to modify the proof of Theorem 2.7 to cover this case.

(2) In the proof of Theorem 3.1, we used a result of Henson on the existence of hyperfinite counting measures representing standard, continuous, finitely additive measures. Henson utilized hyperfinite partitions (a technique originally due to Loeb, see [L3]) to generalize the work of Bernstein and Wattenberg on the representation of Lebesgue measure (cf. [BW]). These results (which belong to the period before the discovery of Loeb measures, when the representation of measures was done through transfer methods and the map $*$) tell us that, given a standard measure μ on an algebra \mathcal{A} , there is an internal, hyperfinite, discrete measure ϕ such that $\phi(*A) = \mu(A)$ for all $A \in \mathcal{A}$. In the context of Loeb measures far less is needed. Consider, for instance, Lebesgue measure λ on $[0,1]$ with the usual topology. It is well known (cf. [A]) that the hyperfinite counting measure ν of Example 2.10 ($\nu(\{n/H\}) = 1/H$, where H is an infinite integer and $n = 0, 1, \dots, H-1$) can be used to represent Lebesgue measure via $L(\nu) \circ st^{-1}(\cdot)$, despite the fact that $\nu(*\mathbb{Q} \cap * [0, 1]) = 1 \neq \lambda(\mathbb{Q} \cap [0, 1]) = 0$. Let μ be a finitely additive measure on (X, \mathcal{A}) , such that μ is outer regular and τ -smooth with respect to V . To carry out the proof of Theorem 3.1 it is enough to find an internal, discrete measure ϕ with hyperfinite support, such that $\phi(*O) = \mu(O)$ for all $O \in V$. The preceding discussion shows that to require $\phi(*A) = \mu(A)$ for all $A \in \mathcal{A}$ is unnecessarily restrictive when working with Loeb measures, for it may exclude the most natural hyperfinite discrete measures from consideration. In the specific case of Lebesgue measure λ and the hyperfinite counting measure ν we have $\lambda \equiv L(\nu) \circ st^{-1}(\cdot)$, since λ is Radon and $\lambda(O) = \nu(*O)$ for all open sets O , even though this equality does not hold for all Lebesgue measurable sets.

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