COMPLETE HYPERSURFACES
WITH CONSTANT MEAN CURVATURE
AND NON-NEGATIVE SECTIONAL CURVATURES

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Abstract. We classify the complete and non-negatively curved hypersurfaces of constant mean curvature in spaces of constant sectional curvature.

1. Introduction

Let \( M^{n+1}(c) \) be an \((n+1)\)-dimensional space of constant sectional curvature \( c \). When \( c < 0 \), \( M^{n+1}(c) = H^{n+1}(c) \); when \( c = 0 \), \( M^{n+1}(c) = R^{n+1} \); when \( c > 0 \), \( M^{n+1}(c) = S^{n+1}(c) \), respectively. Let \( M^n \) be an \( n \)-dimensional hypersurface with constant mean curvature \( H \) in \( M^{n+1}(c) \). Let \( S \) denote the square of the length of the second fundamental form. The main purpose of this paper is to give a characterization of non-negatively curved hypersurfaces of \( M^{n+1}(c) \) by the relationship between \( S \) and \( H \), this can be compared with the results obtained by Nomizu and Smyth [3], Okumura [4], Goldberg [1], Hasanis [2] and Smyth [6].

Theorem. Let \( M^n \) be a complete non-negatively curved hypersurface of \( M^{n+1}(c) \) with constant mean curvature \( H \). Then \( M^n \) is totally umbilical or

\[
\sup S = nc + \frac{n^3 H^2}{2k(n-k)} \pm \frac{n(n-2k)}{2k(n-k)} |H| \sqrt{n^2 H^2 + 4k(n-k)c},
\]

for some \( k = 1, 2, \ldots, n-1 \), when \( c \geq 0 \); or

\[
\sup S = nc + \frac{n^3 H^2}{2(n-1)} - \frac{n(n-2)}{2(n-1)} |H| \sqrt{n^2 H^2 + 4(n-1)c}, \quad \text{when} \; c < 0.
\]

In particular, if \( M^n \) is connected and \( S = \text{constant} \) (when \( c > 0 \), \( S = \text{constant} \) may be replaced by the condition that \( M^n \) is compact), then in the second case we have the following:

1. When \( c > 0 \), \( M^n = S^k(c_1) \times S^{n-k}(c_2) \), for some \( k = 1, 2, \ldots, n-1 \), where \( c_1 > 0 \), \( c_2 > 0 \) and \( 1/c_1 + 1/c_2 = 1/c \).

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(2) When \( c = 0 \), \( M^n = R^k \times S^{n-k}(c_1) \) for some \( k = 1, 2, \ldots, n-1 \), where \( c_1 > 0 \).

(3) When \( c < 0 \), \( M^n = H^1(c_1) \times S^{n-1}(c_2) \), where \( c_1 < 0 \), \( c_2 > 0 \) and \\
\( 1/c_1 + 1/c_2 = 1/c \).

From the theorem, we easily see the following

**Corollary.** Let \( M^n \) be a compact non-negatively curved hypersurface of \( M^{n+1}(c) \) \( (c \leq 0) \) with constant mean curvature \( H \). Then \( M^n \) is totally umbilical and has positive sectional curvature \( H^2 + c \).

**Remark.** The main theorem was partially proved by Nomizu and Smyth in [3] whose results were extended to arbitrary codimension by Smyth [6] and later by Yau in [7] under the condition that \( M^n \) is compact.

### 2. Preliminaries

Let \( M^n \) be a hypersurface of \( M^{n+1}(c) \) and let \( e_1, \ldots, e_n, e_{n+1} \) be a local field of orthonormal frames in \( M^{n+1}(c) \), such that, restricted to \( M^n \), the vector field \( e_{n+1} \) is normal to \( M^n \). Then the second fundamental form \( B \) and the mean curvature \( H \) for \( M^n \) can be written as

\[
(2.1) \quad B = \sum_{i,j} h_{ij} \omega_i \omega_j e_{n+1}, \quad H = \frac{1}{n} \sum_i h_{ii}.
\]

The Gauss equation for \( M^n \) is

\[
(2.2) \quad R = n(n-1)c + n^2 H^2 - S,
\]

where \( S = \text{tr} \ B^2 = \sum_{i,j} h_{ij}^2 \) and \( R \) denotes the scalar curvature of \( M^n \).

We denote by \( \Delta \) the Laplacian relative to the induced metric on \( M^n \). If \( H = \text{constant} \), then ([3])

\[
(2.3) \quad (1/2) \Delta S = |\nabla B|^2 - S^2 + ncS - n^2 c H^2 + nH \text{tr} B^3.
\]

For any point \( p \) in \( M^n \), we can choose a local frame field \( e_1, e_2, \ldots, e_n \) so that the matrix \( (h_{ij}) \) is diagonalized at that point, say, \( h_{ij} = \lambda_i \delta_{ij} \). Then (2.3) can be rewritten as ([3])

\[
(2.4) \quad (1/2) \Delta S = |\nabla B|^2 + \sum_{i<j} (\lambda_i - \lambda_j)^2 K_{ij},
\]

where \( K_{ij} = c + \lambda_i \lambda_j \) is the sectional curvature of the plane section spanned by \( e_i \) and \( e_j \).

**Lemma** (see Omori [5] and Yau [8]). Let \( M^n \) be an \( n \)-dimensional complete Riemannian manifold whose Ricci curvature is bounded from below. Let \( F \) be a \( C^2 \)-function bounded from above on \( M^n \). Then for any \( \varepsilon > 0 \), there exists a point \( p \) in \( M^n \) such that

\[
\sup F - \varepsilon < F(p), \quad |\nabla F|(p) < \varepsilon, \quad \Delta F(p) < \varepsilon.
\]

### 3. Standard models

This section is concerned with some standard models of complete non-negatively curved hypersurfaces with constant mean curvature of the space form
\( \overline{M}^{n+1}(c) \). In particular, we only consider non-totally umbilical cases, and the length of the second fundamental form of such hypersurfaces are calculated.

First, we consider a class of hypersurfaces \( R^k \times S^{n-k}(c) \) of \( R^{n+1} \), where \( k = 1, 2, \ldots, n - 1 \). The number of distinct principal curvatures of each hypersurface is exactly two, say 0 and \( \sqrt{c_1} \), with multiplicities \( k \) and \( n - k \), respectively. The sectional curvatures of a plane spanned by two principal directions are 0 and \( c_1 \), respectively. It is easily seen that \( H \) and \( S \) are constant, and they satisfy, for \( R^k \times S^{n-k}(c) \) in \( R^{n+1} \), \( S = n^2H^2/(n - k) \).

We next consider the case \( c > 0 \). Let

\[
S^k(c_1) = \{ (x_1, \ldots, x_{k+1}) \in R^{k+1}; \, x_1^2 + \cdots + x_{k+1}^2 = 1/c_1 \}, \\
S^{n-k}(c_2) = \{ (y_1, \ldots, y_{n-k+1}) \in R^{n-k+1}; \, y_1^2 + \cdots + y_{n-k+1}^2 = 1/c_2 \}, \\
S^{n+1}(c) = \{ (x_1, \ldots, x_{k+1}, y_1, \ldots, y_{n-k+1}) \in R^{n+2}; \, x_1^2 + \cdots + x_{k+1}^2 + y_1^2 + \cdots + y_{n-k+1}^2 = 1 \},
\]

where \( 1/c_1 + 1/c_2 = 1/c \), \( k = 1, 2, \ldots, n - 1 \). Then \( S^k(c_1) \times S^{n-k}(c_2) \) is a family of hypersurfaces in \( S^{n+1}(c) \). The number of distinct principal curvatures of such hypersurfaces in this family is exactly two, and they are constant. One principal curvature is equal to \( \pm \sqrt{c_1 - c} \) with multiplicity \( k \), and the other is equal to \( \pm \sqrt{c_2 - c} \) with multiplicity \( n - k \). The sectional curvatures of a plane spanned by two principal directions are \( c_1 \), \( c_2 \) and 0, respectively. For \( S^k(c_1) \times S^{n-k}(c_2) \) in \( S^{n+1}(c) \) we can easily show that

\[
S = nc + \frac{n^3H^2}{2k(n - k)} \pm \frac{n(n - 2k)}{2k(n - k)} |H| \sqrt{n^2H^2 + 4k(n - k)c},
\]

where the plus (resp. minus) sign is taken if \( k\sqrt{c_1 - c} \geq (n - k)\sqrt{c_2 - c} \) (resp. \( k\sqrt{c_1 - c} < (n - k)\sqrt{c_2 - c} \)).

Finally, we consider the case \( c < 0 \). Let

\[
H^{n+1}(c) = \{ (x_0, x_1, \ldots, x_{n+1}) \in R^{n+2}; \, -x_0^2 + x_1^2 + \cdots + x_{n+1}^2 = 1/c \}.
\]

Define a family of hypersurfaces \( H^k(c_1) \times S^{n-k}(c_2) \) in \( H^{n+1}(c) \) by

\[
H^k(c_1) \times S^{n-k}(c_2) = \{ (x_0, x_1, \ldots, x_{n+1}) \in R^{n+2}; \, -x_0^2 + x_1^2 + \cdots + x_{k}^2 + x_{k+1}^2 + \cdots + x_{n+1}^2 = 1/c \},
\]

where \( c_1 < 0 \), \( c_2 > 0 \), \( 1/c_1 + 1/c_2 = 1/c \) and \( k = 1, 2, \ldots, n - 1 \). The number of distinct principal curvatures of such hypersurfaces in this family is exactly two and they are constant. One principal curvature is equal to \( \sqrt{c_1 - c} \) with multiplicity \( k \) and the other is equal to \( \sqrt{c_2 - c} \) with multiplicity \( n - k \). The sectional curvatures are 0, \( c_2 \) and \( c_1 \) for \( 1 < k < n \); they are 0 and \( c_2 \) for \( k = 1 \). For \( H^k(c_1) \times S^{n-k}(c_2) \) in \( H^{n+1}(c) \), we can also easily show that

\[
S = nc + \frac{n^3H^2}{2k(n - k)} \pm \frac{n(n - 2k)}{2k(n - k)} |H| \sqrt{n^2H^2 + 4k(n - k)c},
\]
where the plus (resp. minus) sign is taken if $k\sqrt{c_1-c} \geq (n-k)\sqrt{c_2-c}$ (resp. $k\sqrt{c_1-c} < (n-k)\sqrt{c_2-c}$).

4. Proof of Theorem

Since $K_{ij} = c + \lambda_i\lambda_j \geq 0$ for any distinct indices $i$ and $j$, (2.4) means that

$$\frac{1}{2}A_5 = |\nabla B|^2 + \sum_{i<j}(\lambda_i - \lambda_j)^2(c + \lambda_i\lambda_j) \geq 0. \tag{4.1}$$

On the other hand, $K_{ij} \geq 0$ implies that the Ricci curvature of $M^n$ is bounded from below by zero. From (2.2) we have that $S$ is bounded from above by a constant $n(n-1)c + n^2H^2$. Then we can apply the lemma to the function $S$. Then we get $\{p_m\}$ in $M^n$ such that

$$\lim_{m \to \infty} S(p_m) = \sup S, \quad \lim_{m \to \infty} \Delta S(p_m) < 0. \tag{4.2}$$

This implies that (4.1) and (4.2) give rise to

$$(c + \lambda_i\lambda_j)(\lambda_i - \lambda_j)^2(p_m) \to 0 \quad (m \to \infty), \tag{4.3}$$

for any distinct indices $i$ and $j$.

Now, since $S = \sum \lambda_j^2$ is bounded, any principal curvature $\lambda_j$ is bounded and hence so is any sequence $\{\lambda_j(p_m)\}$. Then there exists a subsequence $\{p_{m'}\}$ of $\{p_m\}$ such that

$$\lambda_j(p_{m'}) \to \lambda_j \quad (m' \to \infty) \quad \text{for some } \lambda_j \text{ and any } j. \tag{4.4}$$

In fact, since a sequence $\{\lambda_j(p_m)\}$ is bounded, it converges to some $\lambda_j$ by taking a subsequence $\{p_{m'}\}$ if necessary. For the point sequence $\{p_{m'_1}\}$, a sequence $\{\lambda_2(p_{m'_1})\}$ is also bounded and hence there is a subsequence $\{p_{m'_2}\}$ of $\{p_{m'_1}\}$ such that $\{\lambda_2(p_{m'_2})\}$ converges to some $\lambda_2$ as $m_2$ tends to infinity. Thus we can inductively show that there exists a point sequence $\{p_{m'}\}$ of $\{p_m\}$ such that the property (4.4) holds. By (4.3) and (4.4) we get

$$(c + \lambda_i\lambda_j)(\lambda_i - \lambda_j)^2 = 0, \tag{4.5}$$

for any distinct indices $i$ and $j$. By a simple algebraic calculation it is easily seen that the number of distinct limits in $\{\lambda_i\}$ is at most two.

Case 1. If all limits $\lambda_i$ coincide with each other, because $S - nH^2 = \sum \lambda_i^2 - (1/n)(\sum \lambda_i)^2 = (1/n)\sum_{i<j}(\lambda_i - \lambda_j)^2$, it follows from the above property that $\lim_{m \to \infty}(S - nH^2)(p_{m'}) = 0$. Combining this with (4.2), we get $\sup S = nH^2$. Hence the function $S$ becomes a constant $nH^2$. Accordingly, the hypersurface $M^n$ is totally umbilical.

Case 2. If $\{\lambda_i\}$ has exactly two distinct elements, without loss of generality, we may set

$$\lambda_1 = \cdots = \lambda_k = \lambda, \quad \lambda_{k+1} = \cdots = \lambda_n = \mu, \quad \lambda < \mu,$$

for some $k = 1, 2, \ldots, n-1$. From (4.5) we have

$$\lambda\mu = -c. \tag{4.6}$$
Now, we only consider the case $c < 0$. As for the cases $c > 0$ and $c = 0$, the proof is simpler and similar to each other, we omit them.

Without loss of generality, we may assume that $H \geq 0$. Then two limits $\lambda$ and $\mu$ are positive. In this case we define a negative number $c_1$ and a positive number $c_2$ by $\lambda^2 = c_1 - c$ and $\mu^2 = c_2 - c$, respectively, so these numbers satisfy

$$c < c_1 < 0, \quad c_2 > 0,$$

$$\sqrt{(c_1 - c)(c_2 - c)} = -c, \quad \text{i.e.,} \quad 1/c_1 + 1/c_2 = 1/c.$$

By $H = (1/n) \sum \lambda_i(p_{m'}) = \text{constant}$, we have

$$nH = k\lambda + (n-k)\mu = k\sqrt{c_1 - c} + (n-k)\sqrt{c_2 - c}.$$

At the same time we have

$$\sup S = \lim_{m' \to \infty} \lambda_i^2(p_{m'}) = k\lambda^2 + (n-k)\mu^2$$

$$= k(c_1 - c) + (n-k)(c_2 - c).$$

On the other hand, by using (4.7) and (4.8) we can easily see that

$$nc + \frac{n^3H^2}{2k(n-k)} \pm \frac{n(n-2k)}{2k(n-k)}|H|\sqrt{n^2H^2 + 4k(n-k)c}$$

$$= k(c_1 - c) + (n-k)(c_2 - c),$$

where the plus (resp. minus) sign is taken if $k\sqrt{c_1 - c} \geq (n-k)\sqrt{c_2 - c}$ (resp. $k\sqrt{c_1 - c} < (n-k)\sqrt{c_2 - c}$).

Thus, from (4.9) we have

$$\sup S = nc + \frac{n^3H^2}{2(n-1)} \pm \frac{n(n-2k)}{2(n-1)}|H|\sqrt{n^2H^2 + 4(n-1)c}.$$

When $k > 1$, the sectional curvature $K_{12}(p_{m'}) = (c + \lambda_1\lambda_2)(p_{m'}) \to c + \lambda^2 = c_1 < 0$, so from the assumption we have $k = 1$. Then $k\sqrt{c_1 - c} < (n-k)\sqrt{c_2 - c}$. Therefore we have at last

$$\sup S = nc + \frac{n^3H^2}{2(n-1)} - \frac{n(n-2)}{2(n-1)}|H|\sqrt{n^2H^2 + 4(n-1)c}.$$

This completes the proof of the first part of the theorem.

In particular, if $S = \text{constant}$ and $M^n$ is connected, then (2.4) says that all the principal curvatures are constant and that they satisfy

$$(c + \lambda_i\lambda_j)(\lambda_i - \lambda_j)^2 = 0,$$

for any distinct indices $i$ and $j$. Hence the number of distinct elements in $\{\lambda_i\}$ is at most two.

If $\lambda_1 = \cdots = \lambda_n$, then $M^n$ is totally umbilical.

If $\{\lambda_i\}$ has exactly two distinct elements, without loss of generality we may assume that $\lambda_1 = \cdots = \lambda_k = \lambda$, $\lambda_{k+1} = \cdots = \lambda_n = \mu$, $\lambda < \mu$, for some $k = 1, 2, \ldots, n-1$. From (4.10) we have

$$\lambda\mu = -c.$$

For $c \geq 0$, the theorem has been proved by Nomizu and Smyth [3], so we just consider the case $c < 0$. Without loss of generality we may assume that
Then, similar to the first part, we get \( \lambda = \sqrt{c_1 - c}, \mu = \sqrt{c_2 - c} \) and \( k = 1 \), where \( c_1, c_2 \) satisfy (4.7). Thus the second fundamental form of \( M^n \) in \( H^{n+1}(c) \) is given by

\[
(h_{ij}) = \text{diag}(\sqrt{c_1 - c}, \sqrt{c_2 - c}, \ldots, \sqrt{c_2 - c}).
\]

Then, by using the method similar to that of [3] and combining with Section 3, we can show that \( M^n \) is isometric to \( H^1(c_1) \times S^{n-1}(c_2) \). Q.E.D.

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