THE MULTIPLICATIVITY OF THE MINIMAL INDEX
OF SIMPLE $C^*$-ALGEBRAS

SATOSHI KAWAKAMI AND YASUO WATATANI

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Abstract. We show the multiplicativity of the minimal index for simple $C^*$-algebras. Although our proof is very short and elementary, it is also valid for subfactors, which was first shown by Kosaki and Longo (1992).

1. INTRODUCTION

The multiplicativity of Jones index [9] for subfactors of type $\text{II}_1$-factors is a basic fact: If $N \subset L \subset M$ are factors of type $\text{II}_1$, then

$$[M : N] = [M : L][L : N].$$

For an inclusion of infinite factors $N \subset M$, Index $E$ was introduced by Kosaki [12], depending on a normal faithful conditional expectation $E$ of $M$ onto $N$. Among such conditional expectations, one can choose a unique conditional expectation $E_0$ such that Index $E_0 \leq$ Index $E$ for any $E : M \rightarrow N$ as in Havet [5], Hiai [6] and Longo [14, 15]. This $E_0$ is called a minimal conditional expectation, and Index $E_0$ is called the minimal index of $N \subset M$, often denoted by $[M : N]_0$.

The multiplicativity of the minimal index was shown by Kosaki-Longo [13] in the case of inclusions obtained by basic constructions, reducing to the result in Pimsner-Popa [19, 20]. The general case for subfactors was proved by Longo [16] in his sector theory, applying the above result [13]. H. Kosaki has also informed us that R. Longo had a direct proof free from sector theory [17]. Popa [21] also gives an alternative proof for type $\text{II}_1$-factors. Yamagami [23], Denizeau and Havet [3] have other approaches. Moreover the first-named author has also considered it in a general situation and shown that it is related to the chain rules of indicial derivatives for von Neumann subalgebras [11]. Owing to [6, 7] and [10, 11], the multiplicativity of the minimal index is known to be closely related to the additivity of the relative entropy of Connes-Størmer [2] and Pimsner-Popa [19]. See [4, 18] also for the basic notions of subfactors.

In this note, we shall show the multiplicativity of the minimal index for simple $C^*$-algebras. Although our proof is very short and elementary, it is also
valid for subfactors. We do not require any knowledge on sectors, entropy, nor the Takesaki duality theorem.

2. INDEX FOR $C^*$-SUBALGEBRAS

We recall some notations and properties on the index for $C^*$-subalgebras from [22]. Let $B$ be a unital $C^*$-algebra and $A$ a $C^*$-subalgebra with the same unit $I$. Let $E$ be a conditional expectation of $B$ onto $A$. Throughout this note, conditional expectations are assumed to be faithful. Then, $E$ is called of index-finite type if there exists a finite set $\{u_1, u_2, \ldots, u_n\} \subset B$, called a basis for $E$, such that

$$x = \sum_{i=1}^{n} u_i E(u_i^* x) \quad \text{for any } x \in B.$$  

(A finite family $\{(u_1, u_1^*), (u_2, u_2^*), \ldots, (u_n, u_n^*)\}$ is a quasi-basis for $E$ in the sense of [22].) When $E$ is of index-finite type, the index of $E$ is defined by

$$\text{Index } E = \sum_{i=1}^{n} u_i u_i^*.$$  

The value $\text{Index } E$ does not depend on the choice of a basis for $E$, and $\text{Index } E$ is in $\text{Center } B$, the center of $B$. See Izumi [8] for interesting examples of simple $C^*$-subalgebras of Cuntz algebras with finite index. When $A \subset B$ is a factor-subfactor pair, $\text{Index } E$ coincides with Kosaki's index [12].

Let $\alpha$ be an action of a finite group $G$ on a $C^*$-algebra $A$ and $B = A \rtimes_{\alpha} G$ the crossed product. Then, there is a canonical conditional expectation $E$ of $B$ onto $A$ such that

$$E(\sum_{g \in G} a_g \lambda_g) = a_e I \quad \text{for } \sum_{g \in G} a_g \lambda_g \in A \rtimes_{\alpha} G = B.$$

In this situation, $\{\lambda_g\}$ is a basis for $E$. If an operator $x$ commutes with $A$, then $\sum_{g \in G} \lambda_g x \lambda_g^*$ obviously commutes with $B$ because $B$ is generated by $A$ and $\{\lambda_g | g \in G\}$. This fact suggests the following lemma:

**Lemma 1.** Let $A \subset B \subset C$ be inclusions of unital $C^*$-algebras with the same unit and $E : B \to A$ a conditional expectation of index-finite type with a basis $\{u_1, u_2, \ldots, u_n\}$ for $E$. Then, for $x \in A' \cap C$, $\sum_{i=1}^{n} u_i x u_i^*$ is in $B' \cap C$.

**Proof.** For $x \in A' \cap C$, $\sum_{i=1}^{n} u_i x u_i^*$ commutes with any $b \in B$. Indeed,

$$b(\sum_{i=1}^{n} u_i x u_i^*) = \sum_{i=1}^{n} (bu_i) x u_i^* = \sum_{i=1}^{n} \sum_{j=1}^{n} u_j E(u_j^* bu_i) x u_i^*$$

$$= \sum_{j=1}^{n} \sum_{i=1}^{n} u_j x E(u_j^* bu_i) u_i^* \quad (\text{since } E(u_j^* bu_i) \in A)$$

$$= \sum_{j=1}^{n} u_j x (\sum_{i=1}^{n} E(u_j^* bu_i) u_i^*) = \sum_{j=1}^{n} u_j x (u_j^* b) = (\sum_{i=1}^{n} u_i x u_i^*) b$$

Therefore, we have $\sum_{i=1}^{n} u_i x u_i^* \in B' \cap C$. \qed
Remark A. (1) In the case $C = B$, Lemma 1 asserts that $\sum_{i=1}^{n} u_i x u_i^* \in \text{Center } B$ for $x \in A' \cap B$, especially,

$$\text{Index } E = \sum_{i=1}^{n} u_i u_i^* \in \text{Center } B$$

(2) In the case $C = B(H)$ for some Hilbert space $H$, Lemma 1 suggests that

$$F(x) = \sum_{i=1}^{n} u_i x u_i^* \quad (x \in A')$$

defines a normal bounded operator valued weight $F$ of $A'$ onto $B'$. Indeed, when $A \subset B$ is a factor-subfactor pair, it is known that $F = E^{-1}$ by [22], [7].

3. Minimal index

We recall that one can minimize indices of conditional expectations of unital $C^*$-algebras with trivial centers, for example unital simple $C^*$-algebras.

**Proposition 2** [22]. Let $A \subset B$ be an inclusion of unital $C^*$-algebras with Center $A = \text{Center } B = \mathbb{C}I$. Assume that there exists a conditional expectation $E : B \to A$ of index-finite type. Then, there exists a unique minimal conditional expectation $E_0 : B \to A$, i.e., $\text{Index } E_0 \leq \text{Index } E$ for any conditional expectation $E : B \to A$. Moreover, $E = E_0$ if and only if

$$\sum_{i=1}^{n} u_i x u_i^* = cE(x) \quad (x \in A' \cap B)$$

for some constant $c > 0$, where $\{u_1, u_2, \ldots, u_n\}$ is a basis for $E$.

**Remark B.** (1) The above constant $c$ is given by $c = \text{Index } E$.

(2) Index $E_0$ is called the minimal index for a pair $A \subset B$ of $C^*$-algebras and is often denoted by $[B : A]_0$.

(3) When $A \subset B$ is a factor-subfactor pair of finite index, every conditional expectation of $B$ onto $A$ is automatically normal and of finite index. Therefore, observing Remark A(2), we see that the above Proposition 2 is exactly the same as Hiai's characterization of minimal index in [6].

Now we are ready to describe the main theorem, which asserts the multiplicativity of the minimal index for unital simple $C^*$-algebras.

**Theorem 3.** Let $A \subset B \subset C$ be inclusions of unital $C^*$-algebras with Center $A = \text{Center } B = \text{Center } C = \mathbb{C}I$. Let $E : B \to A$ and $F : C \to B$ be conditional expectations of index-finite type. Then, $E \circ F$ is minimal if and only if $E$ and $F$ are minimal. Moreover, minimal index is multiplicative, that is,

$$[C : A]_0 = [C : B]_0 \cdot [B : A]_0$$

**Proof.** Suppose that $E \circ F$ is minimal. Then, the fact that

$$\text{Index}(E \circ F) = (\text{Index } E)(\text{Index } F) \quad [22, \text{Proposition 1.7.1}]$$

implies that both $E$ and $F$ need to be minimal.

Conversely, suppose that $E$ and $F$ are minimal. Let $\{u_1, u_2, \ldots, u_n\} \subset B$ be a basis for $E$ and $\{v_1, v_2, \ldots, v_m\} \subset C$ a basis for $F$. Then, $\{v_j u_i \mid i =
Applying a characterization of the minimal conditional expectation in Proposition 2 to $E$ and $F$, we have, for $x \in A' \cap C$,
\[
\sum_{i=1}^{n} \sum_{j=1}^{m} (v_j u_i)x(v_j u_i)^* = \sum_{j=1}^{m} v_j(\sum_{i=1}^{n} u_i x u_i^*)v_j^*
\]
\[
= (\text{Index } F)F(\sum_{i=1}^{n} u_i x u_i^*) \quad (\text{since } \sum_{i=1}^{n} u_i x u_i^* \in B' \cap C \text{ by Lemma 1})
\]
\[
= (\text{Index } F)\sum_{i=1}^{n} u_i F(x) u_i^* \quad (\text{since } u_i \in B \text{ and } F : C \to B)
\]
\[
= (\text{Index } F)(\text{Index } E)E(F(x)) \quad (\text{since } F(x) \in A' \cap B).
\]
Using Proposition 2 again for $E \circ F$, we conclude that $E \circ F$ is minimal. The rest is now clear. \qed

Combining Theorem 3 with Remark B(3), we immediately get the following corollary:

**Corollary 4** [11], [16], [21]. Let $N \subseteq L \subseteq M$ be inclusions of factors with finite index in Kosaki's sense. Then, for normal conditional expectations $E : L \to N$ and $F : M \to L$, $E \circ F$ is minimal if and only if $E$ and $F$ are minimal. Moreover,
\[
[M : N]_0 = [M : L]_0 \cdot [L : N]_0.
\]

**References**


DEPARTMENT OF MATHEMATICS, NARA UNIVERSITY OF EDUCATION, NARA, 630, JAPAN
E-mail address: f61007@sinet.ad.jp

DEPARTMENT OF MATHEMATICS, KYUSHU UNIVERSITY, ROPPONMATSU, FUKUOKA, 810, JAPAN
E-mail address: watatani@math.kyushu-u.ac.jp