ANOTHER GENERALIZATION OF ANDERSON'S THEOREM

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Abstract. In this paper, we prove that if \( A \) and \( B \) are normal operators on a Hilbert space \( H \), then, for every operator \( S \) satisfying \( ASB = S \), \( \|AXB - X + S\| \geq \|A^{-1}\|\|B^{-1}\|\|S\| \) for all operators \( X \in B(H) \), and that if \( A \) and \( B \) are contractions, then, for every operator \( S \) satisfying \( ASB = S \) and \( A^*SB^* = S \), \( \|AXB - X + S\| \geq \|S\| \) for all operators \( X \in B(H) \), where \( B(H) \) denotes the set of all bounded linear operators on \( H \).

1. INTRODUCTION

Let \( B(H) \) denote the set of all bounded linear operators on a Hilbert space \( H \). Anderson ([1]) proved that

**Theorem A.** If \( A \) is a normal operator, then, for every operator \( S \) satisfying \( AS = SA \),

\[
\|S - (AX - XA)\| \geq \|S\|
\]

for all operators \( X \in B(H) \).

Recently, H. Du and W. Xu ([2]) obtained an alternative proof of Anderson's theorem that depends only on the spectral representation of normal operators and proved that

**Theorem D-X.** Let operators \( A \) and \( B \) be in \( B(H) \). If an operator \( C \) satisfies \( AC = CB \), \( A^*C = CB^* \) and \( \|C\| > \|C\|_e \), then

\[
\|C - (AX - XB)\| \geq \|C\|
\]

for all \( X \in B(H) \), where \( \|C\|_e \) denotes the essential norm of \( C \).

Duggal ([3]) proved that if \( A \) and \( B \) are contractions, then \( S \in C_2 \) and \( ASB - S = 0 \) imply \( \|AXB - X + S\|_2 \geq \|AXB - X\|_2 + \|S\|_2 \) for all \( X \in B(H) \), where \( C_2 \) denotes the Hilbert-Schmidt class of \( B(H) \).

In this note, we shall prove the following theorems:
Theorem 1. If $A$ is a normal operator in $B(H)$, then, for every operator $S$ satisfying $ASA = S$,
\[\|AXA - X + S\| \geq \|A\|^{-2}\|S\|,\]
for all $X \in B(H)$.

Theorem 2. If $A$ is a contraction in $B(H)$, then, for every operator $S$ satisfying $ASA^* = S$ and $A^*SA = S$,
\[\|AXA - X + S\| \geq \|S\|,\]
for all $X \in B(H)$.

Remark. In the above two theorems, putting $A = I$, it is easy to see that the estimates are sharp.

2. Proof of the theorems

Proof of Theorem 1. If $\|A\| < 1$, since $ASA = S$ implies $S = 0$, it is nothing to prove. So we assume that $\|A\| \geq 1$. In this case, take any $\alpha$ such that $(1 - \alpha)\|A\| < 1$, so $\alpha > 1 - \|A\|^{-1}$. Denote $\Delta_\alpha = \{\lambda \in \sigma(A) : |\lambda| \leq 1 - \alpha\}$ and by $E(\cdot)$ the spectral measure of $A$ (the spectrum of an operator $T$ is denoted by $\sigma(T)$). Then $E(\Delta_\alpha)H$ reduces $A$, so $A$ and $S$ have the operator matrix forms
\[A = \begin{pmatrix} A_{1\alpha} & 0 \\ 0 & A_{2\alpha} \end{pmatrix}, \quad S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix},\]
with respect to the space decomposition $H = E(\Delta_\alpha)H \oplus E(\Delta_\alpha')H$, respectively, where $\Delta_\alpha' = \sigma(A) \setminus \Delta_\alpha$. It is easy to see that $\|A_{1\alpha}\| \leq 1 - \alpha < 1$, $\|A_{2\alpha}\| = \|A\|$ and $\sigma(A_{2\alpha}) \subset \Delta_\alpha'$, so $A_{2\alpha}$ is invertible on $E(\Delta_\alpha')H$ and $\|A_{2\alpha}^{-1}\| \leq \frac{1}{1-\alpha}$. By the hypothesis $ASA = S$, we obtain
\[\begin{pmatrix} A_{1\alpha} & 0 \\ 0 & A_{2\alpha} \end{pmatrix} \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} A_{1\alpha} & 0 \\ 0 & A_{2\alpha} \end{pmatrix} = \begin{pmatrix} A_{1\alpha}S_{11}A_{1\alpha} & A_{1\alpha}S_{12}A_{2\alpha} \\ A_{2\alpha}S_{21}A_{1\alpha} & A_{2\alpha}S_{22}A_{2\alpha} \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix},\]
so
(1) $A_{1\alpha}S_{11}A_{1\alpha} = S_{11},$
(2) $A_{1\alpha}S_{12}A_{2\alpha} = S_{12},$
(3) $A_{2\alpha}S_{21}A_{1\alpha} = S_{21},$
(4) $A_{2\alpha}S_{22}A_{2\alpha} = S_{22}.$

From (1), (2), (3) and every positive integer $n$, we get
(5) $A_{1\alpha}^nS_{11}A_{1\alpha}^n = S_{11},$
(6) $A_{1\alpha}^nS_{12}A_{2\alpha}^n = S_{12},$
(7) $A_{2\alpha}^nS_{21}A_{1\alpha}^n = S_{21},$
respectively. Therefore,
(8) $\|A_{1\alpha}\|^2\|S_{11}\| \geq \|S_{11}\|,$
(9) $\|A_{1\alpha}\|^n\|S_{12}\|\|A_{2\alpha}\|^n \geq \|S_{12}\|,$
(10) $\|A_{2\alpha}\|^n\|S_{21}\|\|A_{1\alpha}\|^n \geq \|S_{21}\|.$
But, by the choice of \( \alpha \), \( \|A_{1\alpha}\|^n \to 0 \) and \( (\|A_{1\alpha}\|^n\|A_{2\alpha}\|^n) \leq ((1 - \alpha)\|A\|)^n \to 0 \) (as \( n \to \infty \)), hence \( S_{11} = 0 \), \( S_{12} = 0 \) and \( S_{21} = 0 \). It shows that

\[
S = \begin{pmatrix} 0 & 0 \\ 0 & S_{22} \end{pmatrix}
\]

and \( \|S_{22}\| = \|S\| \). Letting \( X_\alpha = E(\Delta'_{\alpha})XE(\Delta'_{\alpha}) \), we now get

\[
\|AXA - X + S\| \geq \|E(\Delta'_{\alpha})(AXA - X + S)E(\Delta'_{\alpha})\|
\]

\[
= \|A_{2\alpha}X_\alpha A_{2\alpha} - X_\alpha + S_{22}\|
\]

\[
= \|A_{2\alpha}(X_\alpha A_{2\alpha} - A_{2\alpha}^{-1}X_\alpha + A_{2\alpha}^{-1}S_{22})\|
\]

\[
\geq \frac{1}{\|A_{2\alpha}\|} \|X_\alpha A_{2\alpha} - A_{2\alpha}^{-1}X_\alpha + A_{2\alpha}^{-1}S_{22}\|
\]

\[
\geq (1 - \alpha)\|X_\alpha A_{1\alpha} - A_{2\alpha}^{-1}X_\alpha + A_{2\alpha}^{-1}S_{22}\|
\]

Note that by (4) \( A_{2\alpha}^{-1}S_{22}A_{2\alpha} - A_{2\alpha}^{-1}A_{2\alpha}^{-1}S_{22} = A_{2\alpha}^{-2}(A_{2\alpha}S_{22}A_{2\alpha} - S_{22}) = 0 \), so moreover by Anderson’s Theorem

\[
\|X_\alpha A_{2\alpha} - A_{2\alpha}^{-1}X_\alpha + A_{2\alpha}^{-1}S_{22}\| \geq \|A_{2\alpha}^{-1}S_{22}\|
\]

\[
\geq \frac{1}{\|A_{2\alpha}\|} \|S_{22}\| = \|A\|^{-1}\|S\|.
\]

That is, \( \|AXA - X + S\| \geq (1 - \alpha)\|A\|^{-1}\|S\| \). But, by the choice of \( \alpha \), we may choose \( \alpha \) such that \( (1 - \alpha)\|A\| \) is sufficiently close to 1, so

\[
\|AXA - X + S\| \geq \|A\|^{-2}\|S\|.
\]

We have finished the proof.

**Corollary 2.1.** If \( A \) and \( B \) are normal, then for every operator \( S \) satisfying \( ASB = S \) and all operators \( X \in B(H) \),

\[
\|AXB - X + S\| \geq \|A\|^{-1}\|B\|^{-1}\|S\|.
\]

**Proof.** Suppose that

\[
\tilde{A} = \begin{pmatrix} \sqrt{\frac{\|B\|}{\|A\|}} A & 0 \\ 0 & \sqrt{\frac{\|A\|}{\|B\|}} B \end{pmatrix}, \quad \tilde{X} = \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad \tilde{S} = \begin{pmatrix} 0 & S \\ 0 & 0 \end{pmatrix}.
\]

Then

\[
\tilde{A} \tilde{S} \tilde{A} = \begin{pmatrix} \sqrt{\frac{\|B\|}{\|A\|}} A & 0 \\ 0 & \sqrt{\frac{\|A\|}{\|B\|}} B \end{pmatrix} \begin{pmatrix} 0 & S \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{\frac{\|B\|}{\|A\|}} A & 0 \\ 0 & \sqrt{\frac{\|A\|}{\|B\|}} B \end{pmatrix} = \begin{pmatrix} 0 & ASB \\ 0 & 0 \end{pmatrix} = \tilde{S},
\]

\( \|\tilde{S}\| = \|S\| \), and \( \|\tilde{A}\| = \sqrt{\|A\|\|B\|} \). By Theorem 1,

\[
\|AXB - X + S\| = \|\tilde{A} \tilde{X} \tilde{A} - \tilde{X} + \tilde{S}\|
\]

\[
\geq ||\tilde{A}||^{-2}\|\tilde{S}\| = ||A||^{-1}\|B\|^{-1}\|S\|.
\]

Now we turn to the proof of Theorem 2.
Proof of Theorem 2. If \( \|A\| < 1 \), it is clear that \( S = 0 \), so there is nothing to do. Now we suppose \( \|A\| = 1 \). For convenience, we will divide the proof into several steps.

(1) Let \( A = UP \) be the polar decomposition of \( A \), we shall show that \( PS = S = SP \). From the hypotheses \( ASA^* = S \) and \( A^*SA = S \), we get \( A^*ASA^*A = S \); so

\[
P^2SP^2 = S.
\]

Hence for all positive integers \( n \),

\[
P^{2n}SP^{2n} = S.
\]

If the spectral measure of \( P \) is denoted by \( E(\cdot) \), let \( \Delta_\epsilon = \{ \lambda \in \sigma(P) : \lambda < 1 - \epsilon \} \) for arbitrary \( \epsilon > 0 \); multiplying both sides of the above equation on the left by \( E(\Delta_\epsilon) \), we get

\[
(E(\Delta_\epsilon)P(E(\Delta_\epsilon)))^{2n}SP^{2n} = E(\Delta_\epsilon)S.
\]

Since \( \|E(\Delta_\epsilon)P(E(\Delta_\epsilon))\| < 1 - \epsilon \), it follows that \( \lim_{n \to \infty} (E(\Delta_\epsilon)P(E(\Delta_\epsilon)))^{2n} = 0 \), so

\[
E(\Delta_\epsilon)S = 0.
\]

Similarly, \( SE(\Delta_\epsilon) = 0 \). We shall show that \( PS = S \) and \( SP = S \). In fact, it is clear that the spectrum \( \sigma(P) \) is included in \([0, 1]\), so we can suppose that \( P = \int_0^1 \lambda dE_\lambda \) is the spectral representation of \( P \). Then for any \( 1 > \epsilon > 0 \), since \( \int_0^{1-\epsilon} \lambda dE_\lambda S = 0 \), we obtain

\[
\|PS - S\| = \left\| \int_0^1 (\lambda - 1) dE_\lambda S \right\| = \left\| \int_0^{1-\epsilon} (\lambda - 1) dE_\lambda S + \int_{1-\epsilon}^1 (\lambda - 1) dE_\lambda S \right\| \leq \epsilon \|S\|.
\]

Because \( \epsilon \) is arbitrary, \( PS = S \). Similarly, \( SP = S \).

(2) We will prove that \( SU = US \) and \( SU^* = U^*S \).

Denote the range of an operator \( T \) by \( R(T) \), note that \( A^*SA = S \) implies that \( R(S) \subset R(A^*) \) and \( U^*U \) is the projection on \( R(A^*)^\perp \), where the \( R(A^*)^\perp \) means the closure of \( R(A^*) \). So, since \( ASA^* = UPSU^* = USU^* \) from (1) and multiplying both sides of \( ASA^* = S \) on the left by \( U^* \), it follows that \( U^*S = SU^* \). Similarly, we have \( US = SU \).

(3) Suppose that \( S = VQ \) is the polar decomposition of \( S \), where \( Q = (S^*S)^{-\frac{1}{2}} \), and \( \|S\| \) is an isolated point of \( \sigma(Q) \). We shall show that in this case the theorem holds.

Since \( S^*SA = AS^*A^*SA = AS^*S \), so \( QA = AQ \). Let \( Q = \int_0^{\|S\|} \lambda dF_\lambda \) be the spectral representation of \( Q \); by the hypothesis that \( \|S\| \) is an isolated point of \( \sigma(Q) \), then \( F(\|S\|)A = AF(\|S\|) \). From (1), we get \( A^*S^*SA = S^*S \), that is, \( A^*Q^2A = Q^2 \). Multiplying both sides of the above on the left and the right by \( F(\|S\|) \), respectively, and defining \( A_1 = F(\|S\|)AF(\|S\|) \), then

\[
A_1^*\|S\|^2A_1 = \|S\|^2.
\]

We therefore obtain \( A_1^*A_1 = I \) on the space \( F(\|S\|)H \), where \( I \) denotes the identity on \( F(\|S\|)H \). Similarly, \( A_1A_1^* = I \). These show that \( A_1 \) is a unitary
operator on the space $F(||S||)H$. In this case,

$$\|S\| \|AXA^* - X + S\|$$

$$\geq \|S^*AXA^* - S^*X + S^*S\|$$

$$= \|AS^*XA^* - S^*X + S^*S\|$$

$$\geq \|F(||S||)(AS^*XA^* - S^*X + S^*S)F(||S||)||$$

$$= \|A_1F(||S||)S^*XF(||S||)A_1^* - F(||S||)S^*XF(||S||) + ||S||^2F(||S||)||. $$

Let $X_1 = F(||S||)S^*XF(||S||)$ and note that $A_1$ is a unitary operator on $F(||S||)H$; then

$$\|AXA^* - X + S\| \geq \|S\|. $$

(4) The general case. As in case (3), let $S = VQ$ be the polar decomposition of $S$. For any $\varepsilon > 0$, define $Q_\varepsilon = \int_0^{1-\varepsilon} \lambda dF - \lambda + F([||S|| - \varepsilon, ||S||])$ and $S_\varepsilon = VQ_\varepsilon$; since $QA = AQ$, we get $F([||S|| - \varepsilon, ||S||])A = AF([||S|| - \varepsilon, ||S||])$ and

$$\int_0^{||S||-\varepsilon} \lambda dF_A = F([0, ||S|| - \varepsilon]) \int_0^{||S||} dF_A = F([0, ||S|| - \varepsilon])AQ$$

$$= AF([0, ||S|| - \varepsilon])Q = A \int_0^{||S||-\varepsilon} \lambda dF - \lambda.$$

So $Q_\varepsilon A = AQ_\varepsilon$.

Next, from $ASA^* = S$ follows $AVA^* = VQ$, hence $(AVA^* - V)Q = 0$. It is clear that $R(Q_\varepsilon) = R(Q)$; therefore $(AVA^* - V)Q_\varepsilon = 0$, that is, $AS_\varepsilon A^* = S_\varepsilon$.

Similarly, $A^*S_\varepsilon A = S_\varepsilon$. Clearly, $||S|| = ||S_\varepsilon||$. By (3), we get

$$\|AXA^* - X + S\| = \|AXA^* - X + S_\varepsilon - S_\varepsilon + S\|$$

$$\geq \|AXA^* - X + S_\varepsilon\| - ||S_\varepsilon - S||$$

$$= ||S_\varepsilon|| - \varepsilon = ||S|| - \varepsilon.$$ 

Finally, since $\varepsilon$ is arbitrary,

$$\|AXA^* - X + S\| \geq ||S||.$$ 

The proof is completed.

**Corollary 2.2.** Let $A$ and $B$ be operators in $B(H)$. If $\|A\| \|B\| \leq 1$, then, for every operator $S$ satisfying $ASB = S$ and $A^*SB^* = S$,

$$\|AXB - X + S\| \geq ||S||,$$

for all $X \in B(H)$.

**Proof.** Define

$$\tilde{A} = \begin{pmatrix} \sqrt{||B||/A} & 0 \\ 0 & \sqrt{||A||/B^*} \end{pmatrix}, \quad \tilde{X} = \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad \tilde{S} = \begin{pmatrix} 0 & S \\ 0 & 0 \end{pmatrix}.$$
Then $\tilde{A}\tilde{S}\tilde{A}^* = \tilde{S}$ and $\tilde{A}^*\tilde{S}\tilde{A} = \tilde{S}$; by Theorem 2

$$\|AXB - X + S\| = \|\tilde{A}\tilde{X}\tilde{A}^* - \tilde{X} + \tilde{S}\| \geq \|	ilde{S}\| = \|S\|.$$ 

REFERENCES


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