ANOTHER GENERALIZATION OF ANDERSON'S THEOREM

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ABSTRACT. In this paper, we prove that if $A$ and $B$ are normal operators on a Hilbert space $H$, then, for every operator $S$ satisfying $AS = SA$, $\|AXB - X + S\| \geq \|A\|^{-1}\|B\|^{-1}\|S\|$ for all operators $X \in B(H)$, and that if $A$ and $B$ are contractions, then, for every operator $S$ satisfying $AS = S$ and $A^*SB^* = S$, $\|AXB - X + S\| \geq \|S\|$ for all operators $X \in B(H)$, where $B(H)$ denotes the set of all bounded linear operators on $H$.

1. INTRODUCTION

Let $B(H)$ denote the set of all bounded linear operators on a Hilbert space $H$. Anderson ([1]) proved that

Theorem A. If $A$ is a normal operator, then, for every operator $S$ satisfying $AS = SA$,

$$\|S - (AX -XA)\| \geq \|S\|$$

for all operators $X \in B(H)$.

Recently, H. Du and W. Xu ([2]) obtained an alternative proof of Anderson's theorem that depends only on the spectral representation of normal operators and proved that

Theorem D-X. Let operators $A$ and $B$ be in $B(H)$. If an operator $C$ satisfies $AC = CB$, $A^*C = CB^*$ and $\|C\| > \|C\|_e$, then

$$\|C - (AX - XB)\| \geq \|C\|,$$

for all $X \in B(H)$, where $\|C\|_e$ denotes the essential norm of $C$.

Duggal ([3]) proved that if $A$ and $B$ are contractions, then $S \in C_2$ and $ASB - S = 0$ imply $\|AXB - X + S\|_2^2 = \|AXB - X\|_2^2 + \|S\|_2^2$ for all $X \in B(H)$, where $C_2$ denotes the Hilbert-Schmidt class of $B(H)$.

In this note, we shall prove the following theorems:
Theorem 1. If $A$ is a normal operator in $B(H)$, then, for every operator $S$ satisfying $ASA = S$,  
$$\|AXA - X + S\| \geq \|A\|^{-2}\|S\|,$$
for all $X \in B(H)$.

Theorem 2. If $A$ is a contraction in $B(H)$, then, for every operator $S$ satisfying $ASA^* = S$ and $A^*SA = S$,  
$$\|AXA - X + S\| \geq \|S\|,$$
for all $X \in B(H)$.

Remark. In the above two theorems, putting $A = I$, it is easy to see that the estimates are sharp.

2. Proof of the theorems

Proof of Theorem 1. If $\|A\| < 1$, since $ASA = S$ implies $S = 0$, it is nothing to prove. So we assume that $\|A\| \geq 1$. In this case, take any $\alpha$ such that $(1 - \alpha)\|A\| < 1$, so $\alpha > 1 - \|A\|^{-1}$. Denote $\Delta_\alpha = \{\lambda \in \sigma(A) : |\lambda| \leq 1 - \alpha\}$ and by $E(\cdot)$ the spectral measure of $A$ (the spectrum of an operator $T$ is denoted by $\sigma(T)$). Then $E(\Delta_\alpha)H$ reduces $A$, so $A$ and $S$ have the operator matrix forms  
$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \quad S = \begin{pmatrix} S_11 & S_{12} \\ S_{21} & S_{22} \end{pmatrix},$$
with respect to the space decomposition $H = E(\Delta_\alpha)H \oplus E(\Delta_\alpha')H$, respectively, where $\Delta_\alpha' = \sigma(A) \backslash \Delta_\alpha$. It is easy to see that $\|A_1\| \leq 1 - \alpha < 1$, $\|A_2\| = \|A\|$ and $\sigma(A_2) \subset \Delta_\alpha'$, so $A_2$ is invertible on $E(\Delta_\alpha')H$ and $\|A_2^{-1}\| \leq \frac{1}{1+\alpha}$. By the hypothesis $ASA = S$, we obtain  
$$\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \begin{pmatrix} S_11 & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$
so  
(1) $A_1S_{11}A_1 = S_{11}$,  
(2) $A_{11}S_{12}A_2 = S_{12}$,  
(3) $A_{21}A_{11} = S_{21}$,  
(4) $A_{22}A_{22} = S_{22}$.

From (1), (2), (3) and every positive integer $n$, we get  
(5) $A_{11}^nS_{11}A_1^n = S_{11}$,  
(6) $A_{11}^nS_{12}A_2^n = S_{12}$,  
(7) $A_{21}^nS_{21}A_1^n = S_{21}$, 
respectively. Therefore,  
(8) $\|A_1\|^{2n}\|S_{11}\| \geq \|S_{11}\|$,  
(9) $\|A_1\|^{n}\|S_{12}\|^{n}\|A_2\|^{n} \geq \|S_{12}\|$,  
(10) $\|A_2\|^{n}\|S_{21}\|^{n}\|A_1\|^{n} \geq \|S_{21}\|$.
But, by the choice of $\alpha$, $\|A_{1\alpha}\|^n \to 0$ and $(\|A_{1\alpha}\|^n\|A_{2\alpha}\|^n) \leq ((1 - \alpha)\|A\|)^n \to 0$ (as $n \to \infty$), hence $S_{11} = 0$, $S_{12} = 0$ and $S_{21} = 0$. It shows that

$$S = \begin{pmatrix} 0 & 0 \\ 0 & S_{22} \end{pmatrix}$$

and $\|S_{22}\| = \|S\|$. Letting $X_\alpha = E(A'_\alpha)XE(A'_\alpha)$, we now get

$$\|AXA - X + S\| \geq \|E(A'_\alpha)(AXA - X + S)E(A'_\alpha)\|$$

$$= \|A_{2\alpha}X_\alpha A_{2\alpha} - X_\alpha + S_{22}\|$$

$$= \|A_{2\alpha}(X_\alpha A_{2\alpha} - A_{2\alpha}^{-1}X_\alpha A_{2\alpha}^{-1}S_{22})\|$$

$$\geq \frac{1}{\|A_{2\alpha}\|}\|X_\alpha A_{2\alpha} - A_{2\alpha}^{-1}X_\alpha A_{2\alpha}^{-1}S_{22}\|$$

$$\geq (1 - \alpha)\|X_\alpha A_{1\alpha} - A_{2\alpha}^{-1}X_\alpha A_{2\alpha}^{-1}S_{22}\|.$$ 

Note that by (4) $A_{2\alpha}^{-1}S_{22}A_{2\alpha} - A_{2\alpha}^{-1}A_{2\alpha}S_{22} = A_{2\alpha}^{-2}(A_{2\alpha}S_{22}A_{2\alpha} - S_{22}) = 0$, so moreover by Anderson's Theorem

$$\|X_\alpha A_{2\alpha} - A_{2\alpha}^{-1}X_\alpha A_{2\alpha}^{-1}S_{22}\| \geq \|A_{2\alpha}^{-1}S_{22}\| \geq \|A\|^{-1}\|S\|.$$ 

That is, $\|AXA - X + S\| \geq (1 - \alpha)\|A\|^{-1}\|S\|$. But, by the choice of $\alpha$, we may choose $\alpha$ such that $(1 - \alpha)\|A\|$ is sufficiently close to 1, so

$$\|AXA - X + S\| \geq \|A\|^{-2}\|S\|.$$ 

We have finished the proof.

**Corollary 2.1.** If $A$ and $B$ are normal, then for every operator $S$ satisfying $ASB = S$ and all operators $X \in B(H)$,

$$\|AXB - X + S\| \geq \|A\|^{-1}\|B\|^{-1}\|S\|.$$ 

**Proof.** Suppose that

$$\tilde{A} = \begin{pmatrix} \sqrt{\frac{\|B\|}{\|A\|}} A & 0 \\ 0 & \sqrt{\frac{\|A\|}{\|B\|}} B \end{pmatrix}, \quad \tilde{X} = \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}, \quad \tilde{S} = \begin{pmatrix} 0 & S \\ 0 & 0 \end{pmatrix}.$$ 

Then

$$\tilde{A}S\tilde{A} = \begin{pmatrix} \sqrt{\frac{\|B\|}{\|A\|}} A & 0 \\ 0 & \sqrt{\frac{\|A\|}{\|B\|}} B \end{pmatrix} \begin{pmatrix} 0 & S \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{\frac{\|B\|}{\|A\|}} A & 0 \\ 0 & \sqrt{\frac{\|A\|}{\|B\|}} B \end{pmatrix}$$

$$= \begin{pmatrix} 0 & ASB \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & S \\ 0 & 0 \end{pmatrix} = \tilde{S},$$

$\|\tilde{S}\| = \|S\|$, and $\|\tilde{A}\| = \sqrt{\|A\|\|B\|}$. By Theorem 1,

$$\|AXB - X + S\| = \|\tilde{A}\tilde{X}\tilde{A} - \tilde{X} + \tilde{S}\|$$

$$\geq \|\tilde{A}\|^{-2}\|\tilde{S}\| = \|A\|^{-1}\|B\|^{-1}\|S\|.$$ 

Now we turn to the proof of Theorem 2.
Proof of Theorem 2. If \( \| A \| < 1 \), it is clear that \( S = 0 \), so there is nothing to do. Now we suppose \( \| A \| = 1 \). For convenience, we will divide the proof into several steps.

(1) Let \( A = UP \) be the polar decomposition of \( A \), we shall show that \( PS = S = SP \). From the hypotheses \( ASA^* = S \) and \( A^*SA = S \), we get \( A^*ASA^*A = S \); so

\[
P^2SP^2 = S.
\]

Hence for all positive integers \( n \),

\[
P^{2n}SP^{2n} = S.
\]

If the spectral measure of \( P \) is denoted by \( E(\cdot) \), let \( \Delta_\varepsilon = \{ \lambda \in \sigma(P): \lambda < 1 - \varepsilon \} \) for arbitrary \( \varepsilon > 0 \); multiplying both sides of the above equation on the left by \( E(\Delta_\varepsilon) \), we get

\[
(E(\Delta_\varepsilon)P E(\Delta_\varepsilon))^{2n}SP^{2n} = E(\Delta_\varepsilon)S.
\]

Since \( \| E(\Delta_\varepsilon)P E(\Delta_\varepsilon) \| \leq 1 - \varepsilon \), it follows that \( \lim_{n \to \infty} (E(\Delta_\varepsilon)P E(\Delta_\varepsilon))^{2n} = 0 \), so

\[
E(\Delta_\varepsilon)S = 0.
\]

Similarly, \( SE(\Delta_\varepsilon) = 0 \). We shall show that \( PS = S \) and \( SP = S \). In fact, it is clear that the spectrum \( \sigma(P) \) is included in \([0, 1]\), so we can suppose that \( P = \int_0^1 \lambda dE_\lambda \) is the spectral representation of \( P \). Then for any \( 1 > \varepsilon > 0 \), since \( \int_0^{1-\varepsilon} \lambda dE_\lambda S = 0 \), we obtain

\[
\| PS - S \| = \left\| \int_0^1 (\lambda - 1) dE_\lambda S \right\|
\]

\[
= \left\| \int_0^{1-\varepsilon} (\lambda - 1) dE_\lambda S + \int_{1-\varepsilon}^1 (\lambda - 1) dE_\lambda S \right\| \leq \varepsilon \| S \|.
\]

Because \( \varepsilon \) is arbitrary, \( PS = S \). Similarly, \( SP = S \).

(2) We will prove that \( SU = US \) and \( SU^* = U^*S \).

Denote the range of an operator \( T \) by \( R(T) \), note that \( A^*SA = S \) implies that \( R(S) \subset R(A^*) \) and \( U^*U \) is the projection on \( R(A^*)^{-} \), where the \( R(A^*)^{-} \) means the closure of \( R(A^*) \). So, since \( ASA^* = USPU^* = USU^* \) from (1) and multiplying both sides of \( ASA^* = S \) on the left by \( U^* \), it follows that \( U^*S = SU^* \). Similarly, we have \( US = SU \).

(3) Suppose that \( S = VQ \) is the polar decomposition of \( S \), where \( Q = (S^*)^{-1} \), and \( \| S \| \) is an isolated point of \( \sigma(Q) \). We shall show that in this case the theorem holds.

Since \( S^*SA = AS^*A^*SA = AS^*S \), so \( QA = AQ \). Let \( Q = \int_0^{\| S \|} \lambda dF_\lambda \) be the spectral representation of \( Q \); by the hypothesis that \( \| S \| \) is an isolated point of \( \sigma(Q) \), then \( F(\| S \|)A = AF(\| S \|) \). From (1), we get \( A^*S^*SA = S^*S \), that is, \( A^*Q^2A = Q^2 \). Multiplying both sides of the above on the left and the right by \( F(\| S \|) \), respectively, and defining \( A_1 = F(\| S \|)AF(\| S \|) \), then

\[
A_1^*\| S \|^2A_1 = \| S \|^2.
\]

We therefore obtain \( A_1^*A_1 = I \) on the space \( F(\| S \|)H \), where \( I \) denotes the identity on \( F(\| S \|)H \). Similarly, \( A_1A_1^* = I \). These show that \( A_1 \) is a unitary
operator on the space $F(||S||)H$. In this case,

$$\|S\| \|AXA^* - X + S\|
\geq \|S^*AXA^* - S^*X + S^*S\|
= \|AS^*XA^* - S^*X + S^*S\|
\geq \|F(||S||)(AS^*XA^* - S^*X + S^*S)F(||S||)||
= \|A_1F(||S||)S^*XF(||S||)A_1^* - F(||S||)S^*XF(||S||) + ||S||^2F(||S||)||.
$$

Let $X_1 = F(||S||)S^*XF(||S||)$ and note that $A_1$ is a unitary operator on $F(||S||)H$; then

$$\|A_1X_1 - X_1A_1 + ||S||^2A_1\| \geq \|S\|^2||A_1|| = \|S\|^2,$$

so

$$\|AXA^* - X + S\| \geq \|S\|.$$

(4) The general case. As in case (3), let $S = VQ$ be the polar decomposition of $S$. For any $\epsilon > 0$, define $Q_\epsilon = \int_0^{1-\epsilon} \lambda dF_\lambda - \lambda + F([\|S\| - \epsilon, ||S||])$ and $S_\epsilon = VQ_\epsilon$; since $QA = AQ$, we get $F([\|S\| - \epsilon, ||S||])A = AF([\|S\| - \epsilon, ||S||])$ and

$$\int_0^{\|S\| - \epsilon} \lambda dF_\lambda A = F([0, \|S\| - \epsilon]) \int_0^{\|S\| - \epsilon} dF_\lambda A = F([0, \|S\| - \epsilon])AQ
= AF([0, \|S\| - \epsilon])Q = A \int_0^{\|S\| - \epsilon} \lambda dF_\lambda.$$

So $Q_\epsilon A = AQ_\epsilon$.

Next, from $ASA^* = S$ follows $AVA^* = VQ$, hence $(AVA^* - V)Q = 0$. It is clear that $R(Q_\epsilon) = R(Q)$; therefore $(AVA^* - V)Q_\epsilon = 0$, that is, $ASA^* = S_\epsilon$.

Similarly, $A^*S_\epsilon A = S_\epsilon$. Clearly, $\|S\| = \|S_\epsilon\|$. By (3), we get

$$\|AXA^* - X + S\| = \|AXA^* - X + S_\epsilon - S_\epsilon + S\|
\geq \|AXA^* - X + S_\epsilon\| - \|S_\epsilon - S\|
= \|S\| - \epsilon = ||S|| - \epsilon.$$

Finally, since $\epsilon$ is arbitrary,

$$\|AXA^* - X + S\| \geq ||S||.$$

The proof is completed.

**Corollary 2.2.** Let $A$ and $B$ be operators in $B(H)$. If $\|A\| \|B\| \leq 1$, then, for every operator $S$ satisfying $ASB = S$ and $A^*SB^* = S$,

$$\|AXB - X + S\| \geq ||S||,$$

for all $X \in B(H)$.

**Proof.** Define

$$\tilde{A} = \left( \begin{array}{cc} \sqrt{\frac{\|B\|}{\|A\|}} A & 0 \\ 0 & \sqrt{\frac{\|A\|}{\|B\|}} B^* \end{array} \right), \quad \tilde{X} = \left( \begin{array}{cc} 0 & X \\ 0 & 0 \end{array} \right), \quad \tilde{S} = \left( \begin{array}{cc} 0 & S \\ 0 & 0 \end{array} \right).$$
Then $\tilde{A}\tilde{S}\tilde{A}^* = \tilde{S}$ and $\tilde{A}^*\tilde{S}\tilde{A} = \tilde{S}$; by Theorem 2
\[
\|AXB - X + S\| = \|\tilde{A}\tilde{X}\tilde{A}^* - \tilde{X} + \tilde{S}\| \\
\geq \|\tilde{S}\| = \|S\|.
\]

References


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