MARKOV'S EXPONENT OF COMPACT SETS IN $\mathbb{C}^n$

M. BARAN AND W. PLEŚNIAK

(Communicated by Eric Bedford)

ABSTRACT. We introduce the notion of Markov's exponent of a compact set in $\mathbb{C}^n$ and show that it is invariant under regular analytic maps.

1. Markov's Inequality

Given a compact subset $E$ of the space $\mathbb{C}^n$ and a number $r \geq 1$, consider the following two conditions.

$M(r)$ There exists a constant $M_1 > 0$ such that for each $p \in \mathcal{P}_k$, $k = 1, 2, \ldots$,

$$\|\text{grad} \ p\|_E \leq M_1 k^r \|p\|_E.$$  

$P(r)$ There exist two positive constants $M_2$ and $C_2$ such that for each $p \in \mathcal{P}_k$, $k = 1, 2, \ldots$,

$$|p(x)| \leq M_2 \|p\|_E, \text{ as } \text{dist}(x, E) \leq \frac{C_2}{k^r}.$$  

Here $\mathcal{P}_k$ denotes the space of all polynomials of degree at most $k$. The condition $M(r)$ is a multidimensional version of the well-known inequality, proved by A.A. Markov in 1889 in the case where $E = [-1, 1]$. For the proof of this famous result and its one-dimensional generalizations we refer the reader to [RS]. Some criteria for subsets of $\mathbb{C}^n$ satisfying $M(r)$ have been proved in [PP1], [B1] and [B2]. In particular, it is known that every fat subanalytic set and, more generally, every uniformly polynomially cuspidal set satisfies $M(r)$, for some $r \geq 1$. Markov's inequality has been applied in problems connected with approximation and extension of $C^\infty$ functions (see [PP2] and [PI3]).

Given a compact set $E$ in $\mathbb{C}^n$, an important point is to determine the minimal constant $r$ in $M(r)$. This permits, in particular, the minimization of the loss of regularity in problems connected with the linear extension of classes of $C^\infty$ functions with restricted growth of derivatives (see [PS] and [PI4]). In the next section we call such an $r$ Markov's exponent of $E$ and show that it is invariant under regular holomorphic maps. We close this section by proving the following observation.

Received by the editors February 7, 1994.

1991 Mathematics Subject Classification. Primary 41A17; Secondary 41A10, 26C05.

Research supported, in part, by KBN grant 2 1077 91 01.

©1995 American Mathematical Society

2785
Proposition 1.1. For each \( r \geq 1 \), the properties \( M(r) \) and \( P(r) \) of the set \( E \) are equivalent.

Proof. Assume \( M(r) \). Fix \( a \in E \), \( v \in \mathbb{C}^n \) with \( \|v\| = 1 \), and \( p \in \mathcal{P}_k \), and define \( q(t) = p(a + tv) \). Observe that \( q^{[j]}(t) = D_v^n p(a + tv) \). In particular, we have \( q^{[j]}(0) = D_v^n p(a) \). Hence, if \( t \in \mathbb{C} \) and \( |t| \leq C_2/k' \), we can write

\[
|q(t)| = \left| \sum_{j=0}^{k} \frac{1}{j!} t^j q^{(j)}(0) \right| \leq \sum_{j=0}^{k} \frac{1}{j!} |t|^j |D_v^n p(a)| \leq \sum_{j=0}^{k} \frac{1}{j!} |t|^j |p| |t|^j |p| E \leq M_2 |p||E|.
\]

The inverse implication follows easily from Cauchy's integral formula. The proof is completed.

2. Markov's exponent

Given a compact subset \( E \) of the space \( \mathbb{C}^n \), we define

\[
\mu(E) = \inf \{ r : E \text{ satisfies } M(r) \}
\]

and call this number Markov's exponent of \( E \). If \( E \) is a continuum in \( \mathbb{C} \) containing at least two different points, then by [Po], \( 1 \leq \mu(E) \leq 2 \). For any compact subset \( E \) of \( \mathbb{R}^n \), we have \( \mu(E) \geq 2 \). If \( E \) is a fat convex subset of \( \mathbb{R}^n \), then \( \mu(E) = 2 \). If \( E = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1, 0 \leq y \leq x^p \} \), for \( p \geq 1 \), then by [G] \( \mu(E) = 2p \). In a more general case, if \( E \) is an \( m - U P C \) subset of \( \mathbb{R}^n \) \( (m \geq 1) \) (see [PP1]), then by [B2] \( \mu(E) \leq 2m \). If \( E = \{(x, y) \in \mathbb{R}^n : 0 < x \leq 1, 0 < y \leq e^{-1/x} \} \cup \{(0, 0)\} \), then by [Z] \( \mu(E) = \infty \). The following example is a slight generalization of the above-mentioned results of Goetgheluck and Zerner.

Example 2.1. Let \( \phi \) be a convex, increasing \( \mathcal{C}^1 \) function defined on \( [0, 1] \) such that \( \phi(0) = \phi'(0) = 0 \), \( \phi(1) = 1 \). Define

\[
E = E\phi = \{(x, y) \in \mathbb{R}^2 : |x| \leq 1, |y| \leq \phi(1 - |x|)\}.
\]

Then, for any polynomial \( p \in \mathcal{P}_k \), \( k = 0, 1, \ldots \), we have

\[
(2.1.1) \quad \| \frac{\partial p}{\partial x} \|_E \leq \phi'(1) k^2 \| p \|_E.
\]

Moreover, let

\[
\alpha = \liminf_{t \to 0^+} \frac{\log \phi(t)}{\log t}, \quad \beta = \limsup_{t \to 0^+} \frac{\log \phi(t)}{\log t}.
\]

Then, if \( \beta < \infty \), for any \( \epsilon > 0 \) we have

\[
(2.1.2) \quad \| \frac{\partial p}{\partial y} \|_E \leq \text{const.} \ k^{2(\beta + \epsilon)} \| p \|_E,
\]

whence \( \mu(E) \leq 2\beta \) \( (\mu(E) = 2, \text{ if } \beta = 1) \). If \( \alpha = \infty \), then

\[
(2.1.3) \quad \mu(E) = \infty.
\]

If \( \phi \) satisfies Orlicz's \( \Delta_2 \) condition at 0 (i.e., \( \phi(2u) \leq \text{const.} \phi(u) \)), then \( \beta < \infty \) and \( \phi \) does not satisfy this condition in case \( \alpha = \infty \).

Proof. Let \( e_1 = (1, 0) \) and \( e_2 = (0, 1) \), and let

\[
\rho_i(x, y) = \text{dist}_{e_i}((x, y), \mathbb{R}^2 \setminus E)
\]
be the distance of \((x, y) \in E\) from the boundary of \(E\) in direction of the vector \(e_i\), for \(i = 1, 2\). Then one can easily check that

\[
\rho_1(t(x, y)) \geq \frac{1}{\phi'(1)}(1 - |t|) \tag{2.1.4}
\]

and

\[
\rho_2(t(x, y)) \geq \phi(1 - |t|) \tag{2.1.5}
\]

for \(t \in [-1, 1]\) and \((x, y) \in \partial E\). Now, by (2.1.4), applying a version of Markov’s inequality for star-shaped sets (see [B2, Thm. 3.6]) gives (2.1.1) (cf. [B2]). If \(\beta < \infty\), then, for each \(\epsilon > 0\), \(\phi(t) \geq \text{const.} t^{\beta + \epsilon}\) for \(t \in [0, 1]\). Hence by (2.1.5) and by [B2, Thm. 3.6] we get (2.1.2). Following an idea of Zerner [Z], suppose now that \(\alpha = \infty\). Then, for each \(r > 0\), we have

\[
\frac{\log \phi(t)}{\log t} \geq r, \text{ if } 0 < t \leq \delta = \delta(r); \text{ whence } \phi(t) \leq M t^r \text{ for } 0 \leq t \leq 1, \text{ where } M > 0 \text{ is a constant depending on } r. \text{ Now, if we take } p_k(x, y) = x^k y, \text{ then }
\]

\[
\|p_k\|_E = \sup_{|x| \leq 1} |x|^k \phi(1 - |x|) \leq M \sup_{0 \leq t \leq 1} t^k (1 - t)^r = M \left(\frac{k}{k + r}\right)^r \frac{r}{k + r}
\]

Consequently,

\[
\|\frac{\partial p_k}{\partial y}\|_E \geq M_2(r)(k + 1)^r \|p_k\|_E
\]

which shows that \(E\) cannot have Markov’s property. Let

\[
\alpha_0 = \liminf_{t \to 0^+} \frac{t \phi'(t)}{\phi(t)}, \quad \beta_0 = \limsup_{t \to 0^+} \frac{t \phi'(t)}{\phi(t)}
\]

be the lower and upper Simonenko indices at 0, respectively (see [M]). By Cauchy’s mean value theorem

\[
\alpha_0 \leq \alpha \leq \beta \leq \beta_0,
\]

and the relation between \(\alpha, \beta\) and the \(\Delta_2\) condition follows from [M, Thm. 3.2(b)].

Consider e.g. \(\phi(t) = t^p, p > 1\). Then \(\phi'(1) = p\), \(\alpha = \beta = p\). If \(\phi(t) = e^{2(1-t^{-1})}\), then \(\phi'(1) = 2\) and \(\alpha = \beta = \infty\). If \(\phi(t) = t(1 - \log t)^{-1}\), then \(\phi'(1) = 2\) and \(\alpha = \beta = 1\).

To prove the invariance of Markov’s exponent under analytic mappings we need the following.

**Lemma 2.2.** Let \(E\) be a polynomially convex, compact subset of \(\mathbb{C}^n\) satisfying \(M(r)\). Let \(f\) be a holomorphic mapping defined in a neighbourhood \(U\) of \(E\), with values in \(\mathbb{C}^m\), such that \(f(E)\) is not pluripolar. Then there exist positive constants \(M_2\) and \(C_3\) such that for each polynomial \(p \in \mathcal{P}_k(\mathbb{C}^m)\) and \(k = 1, 2, \ldots\), we have

\[
|(p \circ f)(x)| \leq M_2 \|p \circ f\|_E \text{ as } \text{dist}(x, E) \leq \frac{C_3}{k^r}.
\]
Proof. Without loss of generality we may assume that \( f \) is bounded on \( U \). Since \( E \) is polynomially convex, one can find a compact polynomial polyhedron \( P \) such that \( E \subset \text{int} P \subset P \subset U \). By a uniform version of the Bernstein-Walsh-Siciak theorem (see [Pll]), there exist constants \( A > 0 \) and \( a \in (0,1) \) such that for any polynomial \( p \in \bigcup_{k=0}^{\infty} \mathcal{P}_k(\mathbb{C}^m) \), we have

\[
\text{dist}_P(p \circ f, \mathcal{P}_l(\mathbb{C}^n)) \leq Aa^k\|p \circ f\|_U.
\]

Since \( f(E) \) is not pluripolar, we have \( \|p \circ f\|_U = \|p\|_{f(U)} \leq \|p\|_{f(E)}B_k \), where \( B = \sup\{\Phi_{f(E)}(w) : w \in f(U)\} < \infty \) and where \( \Phi_{f(E)} \) denotes Siciak's extremal function associated with \( f(E) \) ([S1],[S2]). Choose \( k_0 \) such that \( E_k(C_2, r) := \{x \in \mathbb{C}^n : \text{dist}(x, E) < C_2/k^r\} \subset P \) for \( k \geq k_0 \). Let \( s \in \mathbb{N} \) be so large that \( a^sB \leq 1 \), and let \( q_k \) be a best approximation polynomial to \( p \circ f \) of degree \( l = sk \). Then by (2.2.1), since \( E \) satisfies \( P(r) \), we can write

\[
|p \circ f(x)| \leq Aa^skB^k\|p \circ f\|_E + |q(x)| \leq A\|p \circ f\|_E + M_2\|q\|_E \leq (A + 2M_2)\|p \circ f\|_E
\]

as \( \text{dist}(x, E) \leq C_2/(s^kr^r) \), which gives the lemma with \( C_3 = C_2/s^r \) and \( M_3 = A + 2M_2 \).

Corollary 2.3. Under the assumptions of Lemma 2.2, there exists a positive constant \( M_3' \) such that for each \( v \in \mathbb{C}^m \) with \( \|v\| = 1 \), we have

\[
\|D_v(p \circ f)\|_E \leq M_3'k^r\|p \circ f\|_E.
\]

Proof. Fix \( a \in E \) and \( v \in \mathbb{C}^m \) with \( \|v\| = 1 \), and define \( g(t) := (p \circ f)(a + tv) \). By Cauchy's integral formula, for \( \delta = C_3/k^r \), we get

\[
|D_v(p \circ f)(a)| = |g'(0)| \leq \text{sup}\{|g(\zeta) : |\zeta| = \delta\}/\delta \leq (M_2/C_3)k^r\|p \circ f\|_E.
\]

Remark 2.4. The assumption that \( f(E) \) is not pluripolar yields immediately the restrictions that \( m \leq n \) and \( f \) is non-degenerate at least in one of the connected components of \( U \), say \( V \), that meets the set \( E \), which means that \( \text{sup}_{x \in V} \text{rank}_x f = m \). If we knew that the Markov property of \( E \) implies that \( E \) is not pluripolar, we could replace the above assumption on \( f(E) \) by the requirement that \( f \) is non-degenerate on at least one of the connected components of \( U \) that meet \( E \) at a non-pluripolar set. This, however, still seems to be unknown except when \( E \) is a Cantor type subset of \( \mathbb{R} \) (see [Pl2],[BC]).

Lemma 2.2 together with Proposition 1.1 permits us to give a "sharp" version of Proposition 4.1 in [P13] by showing that Markov's exponent of a compact set is invariant under holomorphic injections. More precisely, we have the following

Theorem 2.5. Under the assumptions of Lemma 2.2, suppose that \( m = n \) and \( \text{det} d_x f \neq 0 \) for each \( x \in E \). Then \( f(E) \in M(r) \).

Proof. Choose \( c > 0 \) so that \( |J_c f(x)|^2 \geq c \). By the assumptions and the implicit function theorem there exist positive constants \( L \) and \( L_1 \) such that for each \( x \in E \) and each \( \delta \in (0, c] \), \( f(B(x, L\delta)) \subset B(f(x), L_1\delta) \) (see [T, Chap. I, Prop. 5.1]). Choose \( k_0 \in \mathbb{N} \) so that \( C_3/Lk^r \leq c \) for \( k \geq k_0 \). Then by Lemma 2.2, for a fixed \( b \in f(E) \) and \( a \in f^{-1}\{b\} \), we get

\[
|p(w)| \leq M_2\|p\|_{f(E)} \text{ as } |w - b| \leq L_1C_3/Lk^r,
\]
for any polynomial \( p \in \mathcal{P}_k(\mathbb{C}^m) \) and \( k \geq k_0 \). Since \( f(E) \) is not pluripolar, the last inequality holds for any \( k \in \mathbb{N} \) after a suitable change of the constant \( M_2 \). In view of Proposition 1.1, the proof of the theorem is complete.

**Remark 2.6.** Note that for any compact subset \( E \) of \( \mathbb{C}^n \), \( E \) satisfies \( M(r) \) iff \( \hat{E} \in M(r) \) where \( \hat{E} \) denotes the polynomial hull of \( E \). However, the assumption that \( E = \hat{E} \) cannot be removed. To see why, take \( E \) to be the set \( \{ z \in \mathbb{C} : |z| = 1 \} \cup \{0\} \) and \( f(z) = 1/(z - 1/2) \). Choose a polynomial \( p \) such that \( \|p\|_{f([|z| = 1])} \leq 1 \) and \( |p(-2)| \geq 2 \). Then, if \( f(E) \) satisfied \( M(r) \) for a certain \( r > 0 \), we would have, for \( q_n(z) = (z + 2)p^n(z) \), \( 2^n \leq |q_n(-2)| \leq M(nd + 1)^r \|q\|_{f(E)} \leq 4M(nd + 1)^r \), for \( n = 1, 2, \ldots \) with a positive constant \( M \) independent of \( n \) and where \( d \) denotes the degree of \( p \), a contradiction.

Suppose now \( E \) is a compact subset of the space \( \mathbb{R}^n \). (Here we assume that \( \mathbb{R}^n \) is a subset of \( \mathbb{C}^n \) such that \( \mathbb{C}^n = \mathbb{R}^n + i\mathbb{R}^n \).) Suppose moreover \( E \) is a UPC set (cf. [PP1]). This means that there exist positive constants \( M \) and \( m \), a positive integer \( d \) and a mapping \( h : E \times [0, 1] \to E \) such that for each \( x \in E \), \( h(x, \cdot) \) is a polynomial of degree at most \( d \), \( h(x, 1) = x \) and \( \text{dist}(h(x, t), \mathbb{R}^n \setminus E) \geq M(1- t)^m \) for all \( (x, t) \) in \( E \times [0, 1] \). It was shown in [PP2] that if \( f \) is a \( C^\infty \) mapping defined in \( \mathbb{R}^n \) with \( J_{\mathbb{R}}f(x) \neq 0 \) on \( E \), then \( f(E) \) also is a UPC subset of \( \mathbb{R}^n \), whence a Markov set. If we drop the assumption that \( \det d_x f \neq 0 \) everywhere on \( E \), the above theorem fails to hold, which is seen by the following

**Counter-example 2.7.** Take \( E \) to be the set \( \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1, 0 \leq y \leq x \} \). Then \( E \) satisfies \( M(2) \), since it is convex. Consider now the map \( f(x, y) = (x, y \phi(x)) \) for \( (x, y) \in \mathbb{R}^2 \), where

\[
\phi(x) = \begin{cases} 
eq -1/x & \text{if } x > 0, \\ 0 & \text{otherwise.} \end{cases}
\]

Then \( f \) is a \( C^\infty \) mapping on \( \mathbb{R}^2 \) with \( \det d_{(x,y)}f \neq 0 \) on \( E \setminus \{(0, 0)\} \). Nevertheless, the set \( f(E) = \{(u, v) \in \mathbb{R}^2 : 0 \leq u \leq 1, 0 \leq v \leq u \phi(u) \} \) is known to not preserve Markov's inequality for any order \( r > 0 \).

The situation is much better when \( f \) is a polynomial. We have

**Theorem 2.8.** Let \( E \) be a compact subset of \( \mathbb{R}^n \) that is UPC with parameters \( M > 0, m \geq 1 \) and \( d \in \mathbb{N} \). Let \( f : \mathbb{R}^n \to \mathbb{R}^n \) be a polynomial map of degree \( l \) with \( \det d_x f \neq 0 \) on \( \text{int } E \). Then there exist constants \( M_0 > 0 \) and \( m_0 \geq 1 \) such that

\[
(2.8.1) \quad \| (d_x f)^{-1} \| \leq M_0 (\text{dist}(x, \partial E))^{-m_0}
\]

and for any polynomial \( p \in \mathcal{P}_k \), \( k = 0, 1, \ldots, \)

\[
(2.8.2) \quad \| \text{grad } p \|_{f(E)} \leq C k^{2m(m_0+1)} \| p \|_{f(E)}
\]

with \( C = 2M_0M^{-(m_0+1)}(2d)^{2m(m_0+1)} \).

**Proof.** If \( x \in \text{int } E \), by Cramer's Rule and Hadamard's inequality we can write

\[
\| (d_x f)^{-1} \| \leq \text{const. } | \det d_x f |^{-1}.
\]

Let \( X = \{ x \in \mathbb{R}^n : \det d_x f = 0 \} \). By Lojasiewicz's inequality, there exist constants \( A > 0 \) and \( \alpha \geq 1 \) such that for each \( x \in \text{int } E \) we have

\[
| \det d_x f | \geq A[\text{dist}(x, X)]^\alpha \geq A[\text{dist}(x, \partial E)]^\alpha.
\]
This completes the proof of the first part of the theorem. To prove (2.8.2) fix a polynomial $p \in \mathcal{P}_k$ and choose again a point $x \in \text{int } E$. By [B2, Corollary 3.5] $E$ satisfies $M(2m)$ with the constant $M_1 = \sqrt{2(2d)^{2m}}$, whence by (2.8.1) and Corollary 2.3 we can write

$$
\|\text{grad } p(f(x))\| \leq \| (d_x f)^{-1} \| \|\text{grad } (p \circ f)(x)\|
\leq M_0 [\text{dist}(x, \partial E)]^{-m_0} \sqrt{2} M^{-1}(2dk)^{2m} \|p\|_{f(E)}.
$$

By the assumptions, $E$ admits a parametrization $h : E \times [0, 1) \ni (x, t) \rightarrow h(x, t) \in \text{int } E$ such that for each $x$, $h(x, 1) = x$, $h(x, \cdot)$ is a polynomial of degree at most $d$ and $\text{dist}(h(x, t), \partial E) \geq M(1 - t)^m$. Hence we obtain, for any $v \in \mathbb{R}^n$ with $\|v\| = 1$,

$$
\|D_v p(f(h(x, t^2)))\| \leq L M^{-m_0}(1 - t^2)^{-m_0}
$$

with $L = M_0 \sqrt{2} M^{-1}(2dk)^{2m} \|p\|_{f(E)}$. By a generalization of Schur’s theorem (see [B2, Lemma 2.4]), this implies that

$$
\|D_v p(f(h(x, t^2)))\| \leq L M^{-m_0}(2dk)^{2m_0}
$$

for $t \in [-1, 1]$, whence

$$
\|\text{grad } p(f(h(x, t^2)))\| \leq \sqrt{2} L M^{-m_0}(2dk)^{2m_0}
$$

for $t \in [-1, 1]$. Hence by setting $t^2 = 1$ we get the assertion (2.8.2) of the theorem.

The proof of Theorem 2.8 yields the estimate

$$
\mu(f(E)) \leq \mu(E) + 2mm_0,
$$

which is sharp in the following sense.

**Example 2.9.** Take $E$ to be the set $\{ x = (x_1, x_2) \in \mathbb{R}^2 : x_1, x_2 \geq 0, x_1 + x_2 < 1 \}$. Since $E$ is convex, it satisfies $M(r)$ with $r = 2$. Let $f(x_1, x_2) = (x_1^p, x_2^p)$, $p \in \mathbb{N}$, $p \geq 2$. Then we have, for $x \in \text{int } E$,

$$
\| (d_x f)^{-1} \| \leq \frac{1}{2} (\min\{x_1, x_2\})^{-(p-1)} \leq \frac{1}{2} (\text{dist}(x, \partial E))^{-(p-1)}.
$$

Thus, we have $m = 1$ and $m_0 = p - 1$, and by Theorem 2.8, $\mu(f(E)) \leq 2p$. On the other hand, since $f(E) = \{ x \in \mathbb{R}^2 : x_1, x_2 \geq 0, x_1 + \sqrt{x_2} \leq 1 \}$, by [G], $\mu(f(E)) = 2p$.

**References**


Jagiellonian University of Cracow, Institute of Mathematics, Reymonta 4, 30-059 Kraków, Poland

E-mail address, M. Baran: baran@im.uj.edu.pl
E-mail address, W. Pleśniak: plesniak@im.uj.edu.pl