A NOTE ON COHOMOLOGICAL DIMENSION OF APPROXIMATE MOVABLE SPACES

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Abstract. We show that any approximate movable compact metric space $X$ satisfies the equality $\dim X = \dim_{\mathbb{Z}} X$ without finite dimensional condition. Thus there is no approximate movable compact metric space $X$ with $\dim X = \infty$ and $\dim_{\mathbb{Z}} X < \infty$. Since ANRs and some generalized ANRs are approximate movable, they satisfy the above equality.

All spaces are compact metric and all polyhedra are finite. Let $X$ be a space. By $\dim X$ and $\dim_{\mathbb{Z}} X$ we denote covering dimension and integral cohomological dimension of $X$, respectively. It is well known (the fundamental cohomological dimension theorem) that if $\dim X$ is finite, then $\dim X = \dim_{\mathbb{Z}} X$ (see P. S. Aleksandrov [1]). Recently, A. N. Dranishnikov [5] constructed a space $X$ with $\dim X = \infty$ and $\dim_{\mathbb{Z}} X = 3$. So his example means that the equality $\dim X = \dim_{\mathbb{Z}} X$ does not hold without finite dimensional condition. In this note we investigate this equality for some nice spaces:

Theorem 1. If $X$ is approximate movable, then $\dim X = \dim_{\mathbb{Z}} X$ holds.

Corollary 2. There does not exist an approximate movable space $X$ with $\dim X = \infty$ and $\dim_{\mathbb{Z}} X < \infty$.

In [9] the author introduced an approximate shape theory and approximate movability which is an approximate invariant property.

Let $X$ be a space, and let $\mathcal{P} = \{P_i, f_{ij}, N\}$ be an inverse sequence of polyhedra $P_i$ and maps $f_{ij}: X_j \to X_i$, $i < j$, such that $X$ is an inverse limit of $\mathcal{P}$. Lemma (1.6) of [9, II] means the following:

Lemma 3. $X$ is approximate movable if and only if for each integer $k$ and each $\varepsilon > 0$ there is an integer $j > k$ with the following property: For each integer $i \geq k$ there is a map $r_i: X_j \to X_i$ such that $f_{ik}r_i$ and $f_{jk}$ are $\varepsilon$-near.

For our proof we need some characterizations of dimension and cohomological dimension. For any integer $n$ and any triangulation $K$, $K^{(n)}$ denotes the $n$-th skeleton of $K$ and $|K|$ denotes the realization of $K$. Lemmas 4 and 5 are Theorem 4.1 and Theorem 5.1 of [8].

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Lemma 4. $X$ has $\text{dim} X \leq n$ if and only if for each integer $k$ and each $\varepsilon > 0$ there exist an integer $j > k$, a triangulation $L_k$ of $P_k$, and a map $g_{jk}: P_j \to |L_k(n)|$ which is $\varepsilon$-close to $f_{jk}$.

Lemma 5 (R. D. Edwards). $X$ has $\text{dim}_Z X \leq n$ if and only if, given an integer $i \geq 1$, for each integer $k$ and each $\varepsilon > 0$ there is a triangulation $L_k$ of $P_k$ and an integer $j > k$ such that for any triangulation $L_j$ of $P_j$ there is a map $g_{jk}: |L_j(n+i)| \to |L_k(n)|$ which is $\varepsilon$-close to the restriction of $f_{jk}$.

Proof of Theorem 1. First, we show the inequality $\text{dim} X \leq \text{dim}_Z X$. If $\text{dim}_Z X = \infty$, there is nothing to prove, so we consider the case $\text{dim}_Z X \leq n < \infty$ for some integer $n$. Take any integer $k$ and any $\varepsilon > 0$. Put $\delta = \varepsilon/3$. Since $X$ is approximate movable, by Lemma 3 there is an integer $j \geq k$ satisfying

1. for each $i \geq k$ there is a map $r_i: P_j \to P_i$ such that $f_{ik}r_i$ and $f_{jk}$ are $\varepsilon$-near.

Since $P_j$ is a finite polyhedron, take a triangulation $L_j$ of $P_j$ and let $s = \text{dim} L_j < \infty$. Since $\text{dim}_Z X \leq n < \infty$, by Lemma 5 there exist a triangulation $L_k$ of $P_k$ and an integer $i > k$ such that

2. for any triangulation $L_i$ of $P_i$ there is a map $g_{ik}: |L_i(n+s)| \to |L_k(n)|$ which is $\delta$-close to the restriction of $f_{ik}$.

Since $f_{ik}: X_i \to X_k$ is uniform, there is an $\eta > 0$ such that if points $x$ and $x'$ in $X_i$ are $\eta$-near, then $f_{ik}(x)$ and $f_{ik}(x')$ are $\delta$-near. Take a triangulation $L_i$ of $P_i$ such that any simplex of $L_i$ has a diameter < $\eta/2$. By the simplicial approximation theorem there are a subdivision $L'_j$ of $L_j$ and a simplicial map $\varphi: L'_j \to L_i$ which approximates $r_i$, i.e., its realization $|\varphi|$ and $r_i$ are $\eta$-near. By the choice of $\eta$, $f_{ik}|\varphi|$ and $f_{ik}r_i$ are $\delta$-near. Since $\varphi$ is simplicial and $s = \text{dim} L_j = \text{dim} L'_j$, $\varphi$ induces a map $h = |\varphi|: P_j = |L'_j| = |L_j(s^i)| \to |L_i(s)| \subset |L_i(n+s)|$. Thus

3. $f_{ik}h$ and $f_{ik}r_i$ are $\delta$-near.

Since $h: P_j \to |L_i(n+s)|$, by (2)

4. $g_{ik}h$ and $f_{ik}h$ are $\delta$-near.

By (1), (3) and (4), $f_{jk}$ and $g_{ik}h: P_j \to |L_k(n)|$ are $\varepsilon$-near. Thus $j$ and the map $g_{ik}h$ satisfies the condition in Lemma 4 for $k$ and $\varepsilon$. Then dim $X \leq n$. This means the inequality dim $X \leq \text{dim}_Z X$.

Next, we show the inequality $\text{dim}_Z X \leq \text{dim} X$. If $\text{dim} X = \infty$, there is nothing to prove, so we consider the case $\text{dim} X \leq n < \infty$ for some integer $n$. It is easy to show $\text{dim}_Z X \leq n$ by Lemmas 4 and 5. This means the inequality $\text{dim}_Z X \leq \text{dim} X$. Therefore, we have the required equality.

Corollary 2 follows from Theorem 1 and also means that Dranishnikov’s example is not approximate movable.

AANR$_N$, AANR$_C$, NE-set and AP are approximate movable (see the table of [9, II, p. 337]). Thus we have

**Corollary 6.** If $X$ is ANR, AANR$_N$, AANR$_C$, NE-set or AP, then $\dim X = \dim_{\mathbb{Z}} X$ holds.

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