A NOTE ON COHOMOLOGICAL DIMENSION OF APPROXIMATE MOVABLE SPACES

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Abstract. We show that any approximate movable compact metric space \( X \) satisfies the equality \( \dim X = \dim_{\mathbb{Z}} X \) without finite dimensional condition. Thus there is no approximate movable compact metric space \( X \) with \( \dim X = \infty \) and \( \dim_{\mathbb{Z}} X < \infty \). Since ANRs and some generalized ANRs are approximate movable, they satisfy the above equality.

All spaces are compact metric and all polyhedra are finite. Let \( X \) be a space. By \( \dim X \) and \( \dim_{\mathbb{Z}} X \) we denote covering dimension and integral cohomological dimension of \( X \), respectively. It is well known (the fundamental cohomological dimension theorem) that if \( \dim X \) is finite, then \( \dim X = \dim_{\mathbb{Z}} X \) (see P. S. Aleksandrov [1]). Recently, A. N. Dranishnikov [5] constructed a space \( X \) with \( \dim X = \infty \) and \( \dim_{\mathbb{Z}} X = 3 \). So his example means that the equality \( \dim X = \dim_{\mathbb{Z}} X \) does not hold without finite dimensional condition. In this note we investigate this equality for some nice spaces:

Theorem 1. If \( X \) is approximate movable, then \( \dim X = \dim_{\mathbb{Z}} X \) holds.

Corollary 2. There does not exist an approximate movable space \( X \) with \( \dim X = \infty \) and \( \dim_{\mathbb{Z}} X < \infty \).

In [9] the author introduced an approximate shape theory and approximate movability which is an approximate invariant property.

Let \( X \) be a space, and let \( \mathcal{P} = \{P_i, f_{ij}, N\} \) be an inverse sequence of polyhedra \( P_i \) and maps \( f_{ij}: X_j \to X_i, \; i < j \), such that \( X \) is an inverse limit of \( \mathcal{P} \). Lemma (1.6) of [9, II] means the following:

Lemma 3. \( X \) is approximate movable if and only if for each integer \( k \) and each \( \varepsilon > 0 \) there is an integer \( j > k \) with the following property: For each integer \( i \geq k \) there is a map \( r_i: X_j \to X_i \) such that \( f_{ik}r_i \) and \( f_{jk} \) are \( \varepsilon \)-near.

For our proof we need some characterizations of dimension and cohomological dimension. For any integer \( n \) and any triangulation \( K \), \( K^{(n)} \) denotes the \( n \)-th skeleton of \( K \) and \( |K| \) denotes the realization of \( K \). Lemmas 4 and 5 are Theorem 4.1 and Theorem 5.1 of [8].

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Lemma 4. $X$ has $\dim X \leq n$ if and only if for each integer $k$ and each $\varepsilon > 0$ there exist an integer $j > k$, a triangulation $L_k$ of $P_k$, and a map $g_{jk}: P_j \rightarrow |L_k^{(n)}|$ which is $\varepsilon$-close to $f_{jk}$.

Lemma 5 (R. D. Edwards). $X$ has $\dim_{\mathbb{Z}} X \leq n$ if and only if, given an integer $i \geq 1$, for each integer $k$ and each $\varepsilon > 0$ there is a triangulation $L_k$ of $P_k$ and an integer $j > k$ such that for any triangulation $L_j$ of $P_j$ there is a map $g_{jk}: |L_j^{(n)}| \rightarrow |L_k^{(n)}|$ which is $\varepsilon$-close to the restriction of $f_{jk}$.

Proof of Theorem 1. First, we show the inequality $\dim X \leq \dim_{\mathbb{Z}} X$. If $\dim_{\mathbb{Z}} X = \infty$, there is nothing to prove, so we consider the case $\dim_{\mathbb{Z}} X \leq n < \infty$ for some integer $n$. Take any integer $k$ and any $\varepsilon > 0$. Put $\delta = \varepsilon / 3$. Since $X$ is approximate movable, by Lemma 3 there is an integer $j > k$ satisfying

(1) for each $i \geq k$ there is a map $r_i: P_j \rightarrow P_i$ such that $f_{ik}r_i$ and $f_{jk}$ are $\delta$-near.

Since $P_j$ is a finite polyhedron, take a triangulation $L_j$ of $P_j$ and let $s = \dim L_j < \infty$. Since $\dim_{\mathbb{Z}} X \leq n < \infty$, by Lemma 5 there exist a triangulation $L_k$ of $P_k$ and an integer $i > k$ such that

(2) for any triangulation $L_i$ of $P_i$ there is a map $g_{ik}: |L_i^{(n+s)}| \rightarrow |L_k^{(n)}|$ which is $\delta$-close to the restriction of $f_{ik}$.

Since $f_{ik}: X_i \rightarrow X_k$ is uniform, there is an $\eta > 0$ such that if points $x$ and $x'$ in $X_i$ are $\eta$-near, then $f_{ik}(x)$ and $f_{ik}(x')$ are $\delta$-near. Take a triangulation $L_i$ of $P_i$ such that any simplex of $L_i$ has a diameter $< \eta/2$. By the simplicial approximation theorem there are a subdivision $L'_j$ of $L_j$ and a simplicial map $\varphi: L'_j \rightarrow L_i$ which approximates $r_i$, i.e., its realization $|\varphi|$ and $r_i$ are $\eta$-near. By the choice of $\eta$, $f_{ik}|\varphi|$ and $f_{ik}r_i$ are $\delta$-near. Since $\varphi$ is simplicial and $s = \dim L_j = \dim L_j$, $\varphi$ induces a map $h = |\varphi|: P_j = |L'_j| = |L_j^{(s)}| \rightarrow |L_i^{(s)}| \subset |L_i^{(n+s)}|$. Thus

(3) $f_{ik}h$ and $f_{ik}r_i$ are $\delta$-near.

Since $h: P_j \rightarrow |L_i^{(n+s)}|$, by (2)

(4) $g_{ik}h$ and $f_{ik}h$ are $\delta$-near.

By (1), (3) and (4), $f_{jk}$ and $g_{ik}h: P_j \rightarrow |L_k^{(n)}|$ are $\varepsilon$-near. Thus $j$ and the map $g_{ik}h$ satisfies the condition in Lemma 4 for $k$ and $\varepsilon$. Then $\dim X \leq n$. This means the inequality $\dim X \leq \dim_{\mathbb{Z}} X$.

Next, we show the inequality $\dim_{\mathbb{Z}} X \leq \dim X$. If $\dim X = \infty$, there is nothing to prove, so we consider the case $\dim X \leq n < \infty$ for some integer $n$. It is easy to show $\dim_{\mathbb{Z}} X \leq n$ by Lemmas 4 and 5. This means the inequality $\dim_{\mathbb{Z}} X \leq \dim X$. Therefore, we have the required equality.

Corollary 2 follows from Theorem 1 and also means that Dranishnikov's example is not approximate movable.

AANR_N, AANR_C, NE-set and AP are approximate movable (see the table of [9, II, p. 337]). Thus we have

**Corollary 6.** If \( X \) is ANR, ANR_N, ANR_C, NE-set or AP, then \( \dim X = \dim_Z X \) holds.

**REFERENCES**