YOUNGER MATES AND THE JACOBIAN CONJECTURE

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Abstract. Let $F, G \in \mathbb{C}[x, y]$. If the Jacobian determinant of $F$ and $G$ is 1, then $G$ is said to be a Jacobian mate of $F$. If, in addition, $G$ has degree less than that of $F$, then $G$ is said to be a younger mate of $F$. In this paper, a necessary and sufficient condition is given for a polynomial to have a younger mate. This also gives rise to a formula for the younger mate if it exists. Furthermore, a conjecture concerning the existence of a younger mate is shown to be equivalent to the Jacobian conjecture.

Throughout this paper, $F$ and $G$ will be polynomials in $\mathbb{C}[x, y]$ where $\mathbb{C}$ denotes the field of complex numbers. We say that $F$ and $G$ satisfy the Jacobian hypothesis if their Jacobian determinant is one, i.e., $F_xG_y - F_yG_x = 1$. In this case, we also say that $G$ is a Jacobian mate of $F$. Furthermore, if the $x$-degree (resp. $y$-degree, total degree) of $G$ is less than that of $F$, then $G$ is said to be a younger mate of $F$ relative to the $x$-degree (resp. $y$-degree, total degree). For instance, $x + y$ has younger mates $y$ and $-x$ relative to the $x$-degree and the $y$-degree, respectively, but has no younger mate relative to the total degree.

This paper was motivated by the Jacobian conjecture which asserts that if $F$ has a Jacobian mate $G$, then $(F, G)$ is an automorphism pair. In Section 1, it is shown that a younger mate is unique (up to an additive constant) and universal, i.e., if a Jacobian mate $G$ of $F$ exists, then any other mate of $F$ can be expressed as $G$ plus a polynomial in $F$. In Section 2, the problem of existence of a younger mate of $F$ is reduced to the case where $F$ is monic in both variables. In Section 3, a necessary and sufficient condition for the existence of a younger mate and a formula for a younger mate provided one exists are given. Finally, in Section 4, a conjecture concerning the existence of younger mates is formulated and shown to be equivalent to the Jacobian conjecture.
1. Properties of Jacobian mates

Following [12, Theorem 33, p. 472], we shall use \( \langle F, G \rangle \) to denote the Jacobian determinant. Note that \( \langle , \rangle \), as a function from \( \mathbb{C}[x, y] \times \mathbb{C}[x, y] \) to \( \mathbb{C}[x, y] \), is bilinear.

**Theorem 1.** Suppose \( \langle F, G \rangle = 1 \) and \( \langle F, K \rangle = 0 \) for \( F, G, K \) in \( \mathbb{C}[x, y] \). Then \( K \in \mathbb{C}[F] \).

**Proof.** Since \( \langle F, G \rangle = 1 \), \( F \) and \( G \) are algebraically independent over \( \mathbb{C} \). Moreover, any \( \mathbb{C} \)-derivation on \( \mathbb{C}[F, G] \) extends uniquely to a \( \mathbb{C} \)-derivation on \( \mathbb{C}[x, y] \) by [13, Theorem 1, (1), (5), pp. 240–241]. In particular, \( \frac{\partial}{\partial G} : \mathbb{C}[F, G] \to \mathbb{C}[F, G] \) extends uniquely to \( D : \mathbb{C}[x, y] \to \mathbb{C}[x, y] \). Indeed, the derivation \( D \) is given by \( D(H) = \langle F, H \rangle \) for \( H \in \mathbb{C}[x, y] \) by [13, proof of (5) \( \Rightarrow \) (2) of Theorem 1, p. 241]. Then by [2, (1.5) Corollary, p. 74], \( \mathbb{C}[F] = \ker(D) \). Since \( D(K) = 0 \),

\[
K \in \ker(D) = \mathbb{C}[F].
\]

**Corollary 2 (Universality of Jacobian Mate).** Suppose \( \langle F, G \rangle = 1 \) and \( \langle F, H \rangle = 1 \) for \( F, G, H \) in \( \mathbb{C}[x, y] \). Then \( G - H \in \mathbb{C}[F] \).

**Proof.** Apply Theorem 1 with \( K = G - H \). □

**Theorem 3 (Uniqueness of Younger Mate).** If a younger mate relative to the \( x \)-degree (resp. \( y \)-degree, total degree) exists, then it is unique up to an additive constant.

**Proof.** Suppose both \( G \) and \( H \) are younger mates of \( F \) relative to the \( x \)-degree. Then

\[
\langle F, G - H \rangle = \langle F, G \rangle - \langle F, H \rangle = 1 - 1 = 0.
\]

By Corollary 2, \( G - H = \phi(F) \in \mathbb{C}[F] \). Since \( \deg_x(G - H) < \deg_x F \), \( \phi(F) \) is a constant. The proofs for \( y \)-degree and total degree are similar. □

2. Reduction

In this section we shall reduce the problem of determining the existence of a younger mate of \( F \) to the case where \( F \) is monic in both \( x \) and \( y \). Recall that the Newton polygon for \( F(x, y) \) is the convex hull of the origin together with the support of \( F \).

**Lemma 4.** If \( \deg F \geq 2, \deg G \geq 2 \) and \( \langle F, G \rangle = 1 \), then the Newton polygons of \( F \) and \( G \) are similar. □

The first proof of this lemma was given by Oka [10, Lemma (6.1), p. 430]. The second proof was given by Applegate and Onishi [1, 14. Lemma, p. 217]. Because these proofs were incomplete, Nowicki and Nakai [8, Lemma B, p. 305] offered the third proof. However, as pointed out by L. Andrew Campbell in his review [Mathematical Reviews 89h:13007] of the article, this new proof still contained a gap. Moreover, M. Nagata also indicated an error of the new proof to the authors, so Nowicki and Nakai corrected this in [9].

However, we shall need the following stronger version which can be derived from Lemma 4 and its various proofs.
Lemma 5. If \( \deg_x F \geq 1 \), \( \deg_y F \geq 1 \), \( \deg_x G \geq 1 \), \( \deg_y G \geq 1 \), and \( \{F, G\} = 1 \), then the Newton polygons of \( F \) and \( G \) are similar with magnification factor equal to
\[
\frac{\deg_x F}{\deg_x G} = \frac{\deg_y F}{\deg_y G} = \frac{\deg F}{\deg G}. \quad \Box
\]

We have indicated in the beginning that a younger mate of \( F \) relative to the \( x \)-degree (resp. \( y \)-degree) may not be a younger mate relative to the total degree. However, the next proposition shows that all three concepts coincide for a "nondegenerate" \( F \).

Proposition 6. If \( \deg_x F \geq 2 \), \( \deg_y F \geq 2 \) and \( \{F, G\} = 1 \), then the following conditions are equivalent.

1. \( \deg_x G < \deg_x F \).
2. \( \deg_y G < \deg_y F \).
3. \( \deg G < \deg F \).

Proof. We first claim that \( \deg_x G \geq 1 \). Otherwise \( G_x = 0 \), and the Jacobian hypothesis implies that \( F_x \in \mathbb{C} \setminus \{0\} \). Hence \( F = ax + p(y) \) for some nonzero constant \( a \) and some polynomial \( p(y) \in \mathbb{C}[y] \). This contradicts the assumption that \( \deg_x F \geq 2 \). Likewise \( \deg_y G \geq 1 \). Now the result follows from Lemma 5. \( \Box \)

\( F(x, y) \) is said to be monic in \( x \) if either it is a polynomial in \( y \) alone or it is of the form
\[
F(x, y) = x^n + p_{n-1}(y)x^{n-1} + \cdots + p_1(y)x + p_0(y)
\]
where every \( p_i(y) \in \mathbb{C}[y] \) and \( n \geq 1 \). For any polynomial \( F(x, y) \), there is an invertible linear substitution
\[
\begin{cases}
  x = \alpha \bar{x} + \beta \bar{y} \\
  y = \gamma \bar{x} + \delta \bar{y}
\end{cases}
\]
(so that \( \alpha, \beta, \gamma, \delta \in \mathbb{C} \) and \( \alpha \delta - \beta \gamma \neq 0 \)) such that
\[
F(\bar{x}, \bar{y}) = F(\alpha \bar{x} + \beta \bar{y}, \gamma \bar{x} + \delta \bar{y})
\]
is monic in \( \bar{x} \) and \( \bar{y} \) and \( \deg F = \deg_x F = \deg_y F \) [5, (2.7), p. 5]. Note that the above substitution does not change the total degree nor the existence of a Jacobian mate of any polynomial. In the following we shall reserve \( F \) for the result of this transformation on the fixed polynomial \( F \).

Proposition 7. Suppose \( \deg F \geq 2 \). Then the following conditions are equivalent.

1. \( F \) has a younger mate relative to the total degree.
2. \( F \) has a younger mate relative to the \( x \)-degree.

Proof. (1) \( \implies \) (2). Note that this implication does not need the hypothesis that \( \deg F \geq 2 \). Suppose \( G \) is a younger mate of \( F \) relative to the total degree. Let \( G(\bar{x}, \bar{y}) = G(\alpha \bar{x} + \beta \bar{y}, \gamma \bar{x} + \delta \bar{y}) \). Then, by the chain rule for Jacobians, \( \frac{1}{\alpha \delta - \beta \gamma} \frac{\partial^2 F}{\partial \bar{x} \partial \bar{y}} \) is a Jacobian mate for \( F \). Moreover, \( \deg_x G \leq \deg G = \deg G < \deg F = \deg_x F \).
(2) $\implies$ (1). Suppose $H$ is a younger mate of $F$ relative to the $x$-degree. Let $G(x, y) = H(\alpha x + \beta y, \gamma x + \delta y)$ where

\[
\begin{bmatrix}
\alpha & \beta \\
\gamma & \delta \\
\end{bmatrix}^{-1} = \begin{bmatrix}
\bar{\alpha} & \bar{\beta} \\
\bar{\gamma} & \bar{\delta} \\
\end{bmatrix}.
\]

Then, by the chain rule for Jacobians, $(\alpha \delta - \beta \gamma)G$ is a Jacobian mate of $F$. Since $\deg_x F = \deg_y F = \deg F \geq 2$, we may apply Proposition 6 to $F$ and $H$ to obtain $\deg H < \deg F$. Thus $\deg G < \deg F$. \(\square\)

Combining Propositions 6 and 7, we see that if $\deg_x F \geq 2$ and $\deg_y F \geq 2$, then $F$ has a younger mate relative to any degree if and only if $F$ has a younger mate relative to the $x$-degree. In the next section we shall derive a necessary and sufficient condition for the latter.

3. Existence of younger mates

Lemma 8. Suppose that $\nu$ is a ring homomorphism from $R$ to $S$ where $R$ is a commutative ring and $S$ is an algebraically closed field. Let $H$ and $K$ be in $R[x]$. Then $\nu(\text{Res}_x(H, K)) = 0$ if and only if either $\deg_x \nu(H) < \deg_x H$ and $\deg_x \nu(K) < \deg_x K$, or $\nu(H)$ and $\nu(K)$ have a common linear factor in $S[x]$. Here $\nu : R[x] \rightarrow S[x]$ is the natural extension of $\nu$.

Proof. The nondegenerate case is proved in [11, Chapter 5, Section 8, p. 104, lines 2 to 5]. \(\square\)

Proposition 9. Assume that $F$ is monic in $x$ with $x$-degree $\geq 1$. If $F$ has a Jacobian mate, then the resultant of $F_x$ and $F_y$ with respect to $x$ must be a nonzero constant.

Proof. To show that $r(y) = \text{Res}_x(F_x, F_y)$ is a nonzero constant, it suffices to show that $r(y)$ has no roots. Suppose $r(y)$ has a root $y_0$ in $C$. Applying Lemma 8 with $R = C[y]$, $S = C$ and $\nu$ being the evaluation map at $y_0$, we see that $F_x(x, y_0)$ and $F_y(x, y_0)$ have a common linear factor in $C[x]$ since $\deg_x F_x(x, y_0) = \deg_x F(x, y) = n - 1$.

Let the common factor be $x - x_0$. Then $F_x(x_0, y_0) = F_y(x_0, y_0) = 0$. However, setting $x = x_0$ and $y = y_0$ in the Jacobian hypothesis leads to $0 = 1$. \(\square\)

Suppose $F(x, y)$ is of the form (3.1) with $n \geq 2$ and $\deg_y F \geq 1$. Let $k$ be the largest $i$ such that $p_i(y)$ is a nonconstant. Since $\deg_y F \geq 1$, $k$ exists. Therefore

\[
F_x(x, y) = n x^{n-1} + (n-1)p_{n-1}(y) x^{n-2} + \cdots + p_1(y),
\]

\[
F_y(x, y) = p'_k(y) x^k + p'_{k-1}(y) x^{k-1} + \cdots + p'_0(y).
\]

Since $n \neq 0$ and $p'_k(y) \neq 0$, $r(y) = \text{Res}_x(F_x, F_y) \in C[y]$ is the determinant of
the following \((k + n - 1) \times (k + n - 1)\) Sylvester matrix \(S\):

\[
\begin{bmatrix}
  n & (n-1)p_{n-1} & \ldots & \ldots & p_1 \\
  n & (n-1)p_{n-1} & \ldots & \ldots & p_1 \\
  & \ddots & \ddots & \ddots & \ddots \\
  n & (n-1)p_{n-1} & \ldots & \ldots & p_1 \\
p_k' & p_{k-1}' & \ldots & \ldots & p_0' \\
p_k' & p_{k-1}' & \ldots & \ldots & p_0' \\
& \ddots & \ddots & \ddots & \ddots \\
p_k' & p_{k-1}' & \ldots & \ldots & p_0'
\end{bmatrix}
\]

Let \(S^{adj}\) denote the adjoint of \(S\). If the bottom row of \(S^{adj}\) has entries \(a_1(y), a_2(y), \ldots, a_{k+n-1}(y)\), let

\[
\begin{align*}
\hat{A}(x, y) &= a_1(y)x^{k-1} + a_2(y)x^{k-2} + \cdots + a_k(y), \\
\hat{B}(x, y) &= -\left( a_{k+1}(y)x^{n-2} + a_{k+2}(y)x^{n-3} + \cdots + a_{k+n-1}(y) \right).
\end{align*}
\]

**Theorem 10.** Suppose \(F(x, y)\) is monic in \(x\) with \(\deg_x F \geq 2\) and \(\deg_y F \geq 1\). Let \(c = \Res_x(F_x, F_y)\). Then the following two conditions are equivalent.

1. \(F\) has a younger mate relative to the \(x\)-degree.
2. \(c \in \mathbb{C} \setminus \{0\}\) and \(\hat{A}_x = \hat{B}_y\).

If these two conditions are satisfied, then the younger mate \(G\) of \(F\) relative to the \(x\)-degree is given by

\[
G = \frac{1}{c} \left( \int \hat{B} \, dx + \int \hat{A} \, dy - \iint \hat{A}_x \, dy \, dx \right).
\]

**Proof.** Multiplying both sides of the matrix equation

\[
S \times \begin{bmatrix}
  x^{k+n-2} \\
  \vdots \\
  x^{n-1} \\
  x^{n-2} \\
  1
\end{bmatrix} = \begin{bmatrix}
  x^{k-1}F_x \\
  \vdots \\
  F_x \\
  \vdots \\
  F_y
\end{bmatrix}
\]
on the left by the adjoint of $S$, we have

$$
\begin{bmatrix}
c \\
\vdots \\
\vdots \\
c
\end{bmatrix}
\times
\begin{bmatrix}
x^{k+n-2} \\
x^{n-1} \\
x^{n-2} \\
\vdots \\
1
\end{bmatrix}
= S_{\text{adj}} \times
\begin{bmatrix}
x^k F_x \\
x^{k-2} F_x \\
x^{k-3} F_x \\
\vdots \\
x_{n-2} F_y \\
x_{n-1} F_y \\
x_n F_y
\end{bmatrix}
$$

where $c = \det S$. By comparing the bottom rows of both sides, we have

$$
c = a_1(y)x^{k-2}F_x + \cdots + a_k(y)F_x
+ a_{k+1}(y)x^{n-2}F_y + \cdots + a_{k+n-1}(y)F_y
= \tilde{A}(x, y)F_x - \tilde{B}(x, y)F_y.
$$

(1) $\implies$ (2). If $F$ has a younger mate relative to the $x$-degree, then $c = \text{Res}_x(F_x, F_y)$ must be a nonzero constant by Proposition 9. The last equation becomes

$$
1 = A F_x - B F_y
$$

with $B = \frac{1}{c} \tilde{B}$ and $A = \frac{1}{c} \tilde{A}$. Now let $G$ be a younger mate of $F$ relative to the $x$-degree. Consider the matrix

$$
V = \begin{bmatrix}
B & B & A \\
F_x & F_x & F_y \\
G_x & G_x & G_y
\end{bmatrix}.
$$

Since the first two columns are identical, $\det V = 0$. On the other hand, expanding the determinant of $V$ according to the first column and using (4.1) and $[F, G] = 1$, we have $\det V = B[F, G] - F_x(G_y B - G_x A) + G_x(F_y B - F_x A) = B + F_x(G_x A - G_y B) - G_x$. Hence $G_x - B = F_x(G_x A - G_y B)$. Likewise, $G_y - A = F_y(G_x A - G_y B)$. We have

$$
G_x - B = F_x(G_x A - G_y B),
G_y - A = F_y(G_x A - G_y B).
$$

We claim that $G_x - B = 0$. Otherwise (4.2) implies that

$$
\text{deg}_x(G_x - B) = \text{deg}_x F_x + \text{deg}_x(G_x A - G_y B) \geq \text{deg}_x F_x.
$$

On the other hand, $\text{deg}_x G < \text{deg}_x F$. This, together with $\text{deg}_x B \leq n - 2 < n - 1 = \text{deg}_x F_x$, shows that $\text{deg}_x(G_x - B) < \text{deg}_x F_x$, a contradiction. Therefore $G_x - B = 0$. By the first equation of (4.2), we have $G_x A - G_y B = 0$ since $F_x \neq 0$. Hence $G_y - A = 0$ follows from the second equation of (4.2). Thus $G_{yx} - A_x = 0$ and $G_{xy} - B_y = 0$.

(2) $\implies$ (1). Assume that $c$ is a nonzero constant and $\tilde{A}_x = \tilde{B}_y$. Let $B = \frac{1}{c} \tilde{B}$ and $A = \frac{1}{c} \tilde{A}$. Then $A_x = B_y$ and so there exists a polynomial $G(x, y) \in \mathbb{C}[x, y]$ such that $G_x = B$ and $G_y = A$. Then the $x$-degree of $G$ is at most $n - 1$, and (4.1) becomes $1 = [F, G]$. $\Box$
Remark. If \( \deg_x F = 1 \), \( F \) does not have a younger mate relative to the \( x \)-degree.

4. The younger mate conjecture

In this section we formulate the following conjecture and show that it is equivalent to the Jacobian conjecture.

The younger mate conjecture. Suppose that \( F \) has a Jacobian mate and \( \deg F \geq 2 \). Then there exists \( G_1 \) such that \( [F, G_1] = 1 \) and \( \deg G_1 < \deg F \). In other words, if \( F \) has a Jacobian mate, then it has a younger mate relative to the total degree.

We say that \((F, G)\) is an automorphism pair if the \( \mathbb{C}\)-algebra homomorphism from \( \mathbb{C}[x, y] \) to \( \mathbb{C}[x, y] \) determined by \( x \mapsto F \) and \( y \mapsto G \) is an automorphism.

Lemma 11. Suppose \([F, G] = 1 \) and \([F, H] = 1 \). Then \((F, G)\) is an automorphism pair if and only if \((F, H)\) is also an automorphism pair.

Proof. By Corollary 2, \((F, G) = (F, H + \phi(F)) = (x, y + \phi(x)) \circ (F, H)\). Now the result follows from the fact that \((x, y + \phi(x))\) is an automorphism pair.

Lemma 12. If \((F, G)\) is an automorphism pair with \( \deg F \leq \deg G \), then \( \deg F \) divides \( \deg G \).

Proof. For an elementary proof, as well as a historical account, see [7].

Lemma 13 (Dixmier [4, 2.7. Lemme, p. 215]). If \([f, g] = 0\) where \( f, g \in \mathbb{C}[x, y] \) are nonzero and homogeneous, then \( \frac{\deg f}{\deg g} \in \mathbb{C} \setminus \{0\} \).

Corollary 14. Suppose \((F, G)\) is an automorphism pair with \( \deg F \leq \deg G \). Then there exists \( \lambda \in \mathbb{C} \setminus \{0\} \) such that \( \deg G_1 < \deg G \) and \( G_1 = G - \lambda F \deg G / \deg F \).

Proof. The case in which \( \deg G = 1 \) is easy to see, so we only consider the case where \( \deg G \geq 2 \). Let \( F^+ \) and \( G^+ \) be the highest homogeneous part of \( F \) and \( G \) respectively. Since \((F, G)\) is an automorphism pair, \([F, G] \in \mathbb{C} \setminus \{0\}\) by the chain rule for Jacobians. Using the biadditivity of the Jacobian and counting degrees, we obtain \([F^+, G^+] = 0\). By Lemma 13, \((F^+)\deg G / (G^+)\deg F \in \mathbb{C} \setminus \{0\}\). Since \( \deg F \) divides \( \deg G \) by Lemma 12, it follows that \((F^+)\deg G / \deg F / G^+ \in \mathbb{C} \setminus \{0\}\) and so there exists \( \lambda \in \mathbb{C} \setminus \{0\} \) such that \( G^+ = \lambda (F^+)\deg G / \deg F \). Hence the result follows.

Lemma 15. Suppose \([F, G] = 1 \) with \( \deg G = 1 \). Then \((F, G)\) is an automorphism pair.

Proof. After a suitable change of coordinates, we may assume that \( G = y \). Then \([F, G] = 1 \) implies that \( F_x = 1 \), i.e., \( F = x + p(y) \) for some \( p(y) \in \mathbb{C}[y] \).

Theorem 16. The younger mate conjecture is equivalent to the Jacobian conjecture.

Proof. \( \Rightarrow \) : Suppose the younger mate conjecture is true, and suppose \([F, G] = 1 \). Then there exists \( G_1 \) such that \([F, G_1] = 1 \) and \( \deg G_1 < \deg F \). Hence \([G_1, -F] = 1 \) and, by the younger mate conjecture, there exists \( G_2 \) such that
\[ [G_1, G_2] = 1 \text{ and } \deg G_2 < \deg G_1 \text{ provided that } \deg G_1 \geq 2. \text{ Repeating this process, we obtain a decreasing sequence} \\
\deg F > \deg G_1 > \deg G_2 > \cdots \\
such that \\
[ [G_i, G_{i+1}] = 1 \\
for } i \geq 0 \text{ where } F = G_0. \text{ This sequence must stop, say, at } G_s. \text{ Then} \\
\deg G_s \leq 1 \text{ and hence } \deg G_i = 1 \text{ by } [G_{i-1}, G_i] = 1. \text{ By Lemma 11,} (F, G) \\
is an automorphism pair if and only if } (F, G_i) \text{ is. Similarly, as } [G_{i+1}, G_{i+2}] = [G_{i+1}, -G_i] = 1, (G_{i+1}, G_{i+2}) \text{ is an automorphism pair if and only if} \\
(G_{i+1}, -G_i) \text{ is. The latter is equivalent to saying that } (G_i, G_{i+1}) \text{ is an} \\
automorphism pair. \text{ So } (F, G) \text{ is an automorphism pair if and only if } (G_{s-1}, G_s) \text{ is. But} \\
(G_{s-1}, G_s) \text{ is an automorphism pair by Lemma 15.} \\
\leq : \text{ Suppose the Jacobian conjecture is true, and suppose that } [F, G] = 1 \\
with \deg F \geq 2. \text{ We may assume that } \deg F \leq \deg G \text{ for otherwise we are done.} \\
Then the Jacobian conjecture implies that } (F, G) \text{ is an automorphism pair.} \\
By Corollary 14, there exists } G_1 \text{ such that } \deg G_1 < \deg G \text{ and } [F, G_1] = 1. \text{ If} \\
\deg G_1 \geq \deg F, \text{ then using Corollary 14 again, we obtain } G_2 \text{ such that} \\
\deg G_2 < \deg G \text{ and } [F, G_2] = 1. \text{ Repeat this process to create a decreasing} \\
sequence \\
\deg G > \deg G_1 > \deg G_2 > \cdots \\
such that \\
[ F, G_i] = 1 \\
for all } i \geq 1. \text{ The sequence must stop, say, at } G_s. \text{ Then } [F, G_s] = 1 \text{ and} \\
\deg G_s < \deg F. \quad \square \\

Remark. \text{ If the conclusion } \deg G_1 < \deg F \text{ in the younger mate conjecture is} \\
replaced by } \deg G_1 < \deg G, \text{ then it becomes false (although it is easy to see} \\
that the result is stronger than before). \text{ For example, if } F = y + (x + y^2)^2 \text{ and} \\
G = x + y^2, \text{ then such } G_1 \text{ does not exist: As } \deg G_1 < \deg G = 2, \text{ we may write} \\
G_1 = \alpha x + \beta y + s \text{ where } \alpha, \beta, s \in \mathbb{C}; \text{ then } [F, G_1] = -\alpha + 2(x + y^2)(\beta - 2\alpha y) \neq 1. \\

Addendum. \text{ Yosef Stein has also announced some conditions for existence of a} \\
Jacobian mate in his talk entitled “The Jacobian problem in two variables as a} \\
system of ordinary differential equations” in the International Conference on 
Polynomial Automorphisms and Related Topics held in CIRM (Centre International de Rencontres Mathématiques), Luminy, Marseille, France in October 1992. 

References


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