ON A RECTANGLE INCLUSION PROBLEM

JANUSZ PAWLIKOWSKI

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Abstract. We show that if every set of reals of size $2^{\aleph_0}$ contains a meager-to-one continuous image of a set that cannot be covered by less than $2^{\aleph_0}$ meager sets, then there exists a null (Lebesgue measure zero) subset of the plane $\mathbb{R} \times \mathbb{R}$ that meets every nonnull rectangle $X \times Y$. The antecedent is satisfied, e.g., if $\omega_2$ Cohen reals are added to a model of the continuum hypothesis.

Martin's Axiom implies that a conull (i.e., with null complement) subset of the Euclidean plane $\mathbb{R} \times \mathbb{R}$ contains a nonnull rectangle $X \times Y$. Fremlin [5], Problem AS (see also [6], 3K), asked if this is true in ZFC.

It is known that there exists a conull subset of $\mathbb{R} \times \mathbb{R}$ which contains no rectangle $X \times Y$ with one side nonnull and the other measurable and nonnull. Namely, let $E = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x + y \in F\}$, where $F = \mathbb{R} \setminus \mathbb{Q}$. Clearly, $E$ is a conull subset of $\mathbb{R} \times \mathbb{R}$. If $X \times Y \subseteq E$, then $X + Y \subseteq F$. But, by a theorem of Steinhaus (see [9]), if $X$ is measurable nonnull and $Y$ is nonnull, then $X + Y$ has nonempty interior; hence $X + Y$ cannot be contained in $F$.

(However, Brodskij and Eggleston (see [4]) showed that a measurable nonnull subset of $\mathbb{R} \times \mathbb{R}$ always contains a rectangle $X \times Y$ with $X$ perfect and $Y$ measurable nonnull.)

Consider the following proposition.

(+) If an $F^\sigma$ subset of $\mathbb{R} \times \mathbb{R}$ contains a nonnull rectangle $X \times Y$, then it contains a measurable nonnull rectangle $A \times B$.

Proposition (+) implies that there exists a conull subset of $\mathbb{R} \times \mathbb{R}$ which contains no nonnull rectangle. Any conull $F^\sigma$ subset of the set $E$ considered above will do.

Proposition (+) has other interesting consequences (see [1]). For instance, it follows from (+) that if $X$ and $Y$ are nonnull subsets of $\mathbb{R}$, then $X + Y$ is nonmeager, hence every meager subgroup of $\mathbb{R}$ is null. If $X + Y$ is covered by an $F^\sigma$ set $F$, then $F^* = \{(x, y) : x + y \in F\}$ is an $F^\sigma$ cover for $X \times Y$. By (+), $F^*$ contains a closed nonnull rectangle $A \times B$. By the theorem of Steinhaus mentioned above, $A + B$ has nonempty interior. Hence, $F$ has nonempty interior.)
Consistency of (+) was shown independently by Friedman and Shelah (see [1] and [2]). The model used was $\omega_2$ Cohen reals over a model of the continuum hypothesis.

In this paper we show that (+) is implied by a combinatorial condition whose consistency has been known for some time.

Denote by $\mathcal{P}(\mathbb{R})$ the family of all subsets of $\mathbb{R}$. Let $\lambda$ be a cardinal number. For $\mathcal{A} \subseteq \mathcal{P}(\mathbb{R})$ and $X \subseteq \mathbb{R}$ write $X \in \mathcal{A}^+$ if $X$ can be covered by less than $\lambda$ sets from $\mathcal{A}$, otherwise write $X \in \mathcal{A}^+$. Write also $X \in \mathcal{A}^+$ if $X \in \mathcal{A}^+_\lambda$.

Let $\mathcal{M}$ be the $\sigma$-ideal of meager subsets of $\mathbb{R}$. Say that a function is meager-to-one if preimages of points are meager.

For $D \subseteq \mathbb{R} \times \mathbb{R}$ and $y \in \mathbb{R}$ let $D^y$ denote the horizontal section of $D$ determined by $y$, i.e., $D^y = \{ x \in \mathbb{R} : (x, y) \in D \}$.

Consider the following condition.

\[ (*) \lambda: \text{Every set of reals of size } \lambda \text{ contains a meager-to-one continuous image of a set from } \mathcal{M}^+ \lambda. \]

Note that we do not weaken $(*)_\lambda$ if we replace 'continuous' by 'Baire measurable'. This is because Baire measurable functions are continuous on comeager sets (see [9]).

It is folklore that $(*)_{2^{\aleph_0}}$ holds if $\omega_2$ Cohen reals are added to a model of the continuum hypothesis. For example, Miller [8], p. 577, showed that in this model every set of reals of size $2^{\aleph_0}$ contains a one-to-one continuous image of a $(2^{\aleph_0}, \aleph_1)$-Lusin set. ($X \subseteq \mathbb{R}$ is a $(\lambda, \kappa)$-Lusin set if $|X| = \lambda$ and $|X \cap S| < \kappa$ for all meager $S$.) This immediately gives $(*)_{2^{\aleph_0}}$.

We prove:

**Theorem.** $(*)_{2^{\aleph_0}} \Rightarrow (+)$.

Theorem is a consequence of the following Lemma.

**Lemma.** Assume $(*)_{2^{\aleph_0}}$. Then, with every nonnull set $X \subseteq \mathbb{R}$ we can associate a family $\{E_n : n < \omega\}$ of closed nonnull subsets of $\mathbb{R}$ such that if $X \subseteq \bigcup_{m < \omega} D_m$, $D_m \subseteq \mathbb{R}$ closed sets, then for some $m$ and $n$, $E_n \subseteq D_m$.

**Proof of the Theorem.** Suppose that $X \times Y \subseteq \mathbb{R} \times \mathbb{R}$ is a nonnull rectangle covered by $\bigcup_{m < \omega} E_n$ where $F_m \subseteq \mathbb{R} \times \mathbb{R}$ is closed. Clearly $X$ and $Y$ are nonnull. Let the $E_n$'s be as in the Lemma. Let $G_{nm} = \{ y : E_n \subseteq (F_m)^y \}$. Then $E_n \times G_{nm} \subseteq F_m$. Each $G_{nm}$ is closed. By the Lemma, the $G_{nm}$'s cover $Y$. Indeed, let $y \in Y$. We have $X \times \{ y \} \subseteq \bigcup_m F_m$. So, $X \subseteq \bigcup_m (F_m)^y$; hence, by the Lemma, some $(F_m)^y$ contains some $E_n$, i.e., $y \in G_{nm}$.

Since $Y$ is nonnull, it follows that some $G_{nm}$ is nonnull. Thus, we can take $E_n \times G_{nm}$ as our measurable nonnull rectangle, which is contained in $F_m$. □

**Proof of the Lemma.** Assume $(*)_{2^{\aleph_0}}$. Fix a nonnull set $X \subseteq \mathbb{R}$.

**Claim.** There is $Y \in \mathcal{M}^+_{2^{\aleph_0}}$ and a meager-to-one continuous function $f : Y \rightarrow X$ such that for each closed nonnull set $W$, $f^{-1}[W] \in \mathcal{M}^+_{2^{\aleph_0}}$.

**Proof.** Let $\{E_\xi : \xi < 2^{\aleph_0}\}$ be an enumeration of all closed null sets. Pick inductively $x_\xi \in X \setminus (\{x_\xi : \xi < \xi\} \cup \bigcup_{\xi < \xi} E_\xi)$ ($\xi < 2^{\aleph_0}$). This is possible because $X$ is not null and $\{x_\xi : \xi < \xi\} \cup \bigcup_{\xi < \xi} E_\xi$ is null. (By $(*)_{2^{\aleph_0}}$, $\mathbb{R} \in \mathcal{M}^+_{2^{\aleph_0}}$; by [7], Thm. 2.1, this implies that a union of less than $2^{\aleph_0}$ closed null sets is...
null.) By the construction, the \( x_\xi \)'s are distinct and for every closed null set \( W, |W \cap \{ x_\xi : \xi < 2^{\omega_0} \}| < 2^{\omega_0}. \)

By (*)2\(\omega_0 \) there is \( Y \in 2^{\omega_0} \) and a meager-to-one continuous function \( f : Y \mapsto \{ x_\xi : \xi < 2^{\omega_0} \}. \) Clearly \( f^{-1}[W \cap X] \in 2^{\omega_0}, \) for each closed null \( W. \)

Let \( \mathcal{U} \) be a countable base for \( \mathbb{R}. \) Let \( \{ U_n : n < \omega \} \) be an enumeration of all \( U \in \mathcal{U} \) with the property that \( f(U \cap Y) \) is not null. Let \( E_n = f[U_n \cap Y] \) \((n < \omega).\)

To see that this works suppose that \( X \subseteq \bigcup_{m<\omega} D_m, D_m \subseteq \mathbb{R} \) closed sets. Then \( f^{-1}[D_m \cap X]'s \) are relatively closed in \( Y \) sets that cover \( Y. \)

Suppose that for each \( m \) and \( U \in \mathcal{U} \) with \( U \cap Y \subseteq f^{-1}[D_m \cap X], \) \( f[U \cap Y] \) is null. Then, by the claim, \( U \cap Y \in 2^{\omega_0} \). It follows that \( Y \) is a union of countably many sets from \( 2^{\omega_0} \) and countably many nowheredense sets. Hence \( Y \subseteq 2^{\omega_0}, \) which is a contradiction.

Thus, for some \( U \in \mathcal{U} \) with \( U \cap Y \subseteq f^{-1}[D_m \cap X], \) \( f[U \cap Y] \) is not null, so it must be one of the \( E_n \)'s. Clearly, \( f[U \cap Y] \subseteq D_m. \)

We shall now generalize the Lemma and the Theorem. Fix cardinals \( \lambda \leq \lambda < 2^{\omega_0}, \) \( c(\kappa) > \omega, \) and an arbitrary family \( \mathcal{F} \subseteq 2^\mathbb{R} \) of closed sets. Note that \( \mathcal{F}^{\omega_1} \) is a \( \sigma \)-ideal.

**Definition.** Let \( X \subseteq \mathbb{R}. \) Say that a family \( \{ F_n : n < \omega \} \) of closed sets is \( \kappa \)-dense (for \( \mathcal{X} \)) if, whenever \( X \) is covered by less than \( \kappa \) closed sets, then some one of these closed sets covers some \( F_n. \) Say \( \sigma \)-dense for \( \kappa\) dense.

**Note.** Let \( \mathcal{F} \subseteq 2^\mathbb{R} \) be a \( \sigma \)-ideal. Every closed set from \( \mathcal{F}^{\omega_1} \) (so also every superset of such a set) has a \( \sigma \)-dense family contained in \( \mathcal{F}^{\omega_1}. \) Indeed, suppose that \( X \in \mathcal{F}^{\omega_1} \) is closed. Let \( \mathcal{U} \) be a countable base for \( \mathbb{R}. \) Let \( X^* = X \setminus \bigcup\{ U \in \mathcal{U} : U \cap X \in \mathcal{F} \}. \) Then \( X^* \) is closed and each \( U \cap X^* \) \((U \in \mathcal{U})\) is either empty or belongs to \( \mathcal{F}^{\omega_1}. \) As a \( \sigma \)-dense family we can just take the collection of those sets \( U \cap X^* \) \((U \in \mathcal{U})\) which belong to \( \mathcal{F}^{\omega_1}. \) If \( X^* \subseteq \bigcup_{m<\omega} D_m, D_m \subseteq \mathbb{R} \) closed, then for some \( m, X^* \cap D_m \) has nonempty interior relatively to \( X^* \) (Baire's category theorem). So, for some \( U \in \mathcal{U}, U \cap X^* \neq \emptyset \) and \( U \cap X^* \subseteq D_m. \) Then \( U \cap X^* \subseteq D_m, \) and, by the definition of \( X^* \), \( U \cap X^* \in \mathcal{F}^{\omega_1}. \)

It also follows that if \( X \) is arbitrary and we can find a family of closed sets \( \{ F_n : n < \omega \} \subseteq \mathcal{F}^{\omega_1} \) such that every \( F_\sigma \) set covering \( X \) contains some \( F_n, \) then \( X \) has a \( \sigma \)-dense family contained in \( \mathcal{F}^{\omega_1}. \)

**Lemma 1.** Let \( \{ F_n : n < \omega \} \) be a \( \kappa \)-dense family for \( X. \) Suppose that \( X \times Y \subseteq \bigcup_{\xi<\mu} D_\xi, \) where \( \mu < \kappa \) and \( D_\xi \subseteq \mathbb{R} \times \mathbb{R} \) \((\xi < \mu)\) are closed. Then there are \( Y_{n\xi} \subseteq Y \) \((n < \omega, \xi < \mu)\) such that \( \bigcup_{n,\xi} Y_{n\xi} = Y \) and for all \( n \) and \( \xi, \) \( F_n \times Y_{n\xi} \subseteq D_\xi. \)

**Proof.** Let \( Y_{n\xi} = \{ y \in Y : F_n \subseteq (D_\xi)^y \}. \) Then \( F_n \times Y_{n\xi} \subseteq D_\xi. \) Also, given \( y \in Y, (D_\xi)^y \) \((\xi < \mu)\) cover \( X. \) So, by the definition of a dense family, some \( F_n \) is contained in some \( (D_\xi)^y, \) i.e., \( y \in Y_{n\xi}. \)

**Corollary.** Let \( \mathcal{F} \subseteq 2^\mathbb{R} \) be arbitrary. Let \( \mu < \kappa, \) and let \( D_\xi \subseteq \mathbb{R} \times \mathbb{R} \) \((\xi < \mu)\) be closed. Suppose that \( X \times Y \subseteq \bigcup_{\xi} D_\xi, \) where \( Y \in \mathcal{F}^{\omega_1} \) and \( X \) has a \( \kappa \)-dense
family contained in $\mathcal{F}_k^+$. Then there exist closed sets $A \in \mathcal{F}_k^+$ and $B \in \mathcal{F}_k^+$ such that $A \times B \subseteq D_\xi$ for some $\xi$.

Proof. Let $\{F_n : n < \omega\} \subseteq \mathcal{F}_k^+$ be $\kappa$-dense for $X$. If in Lemma 1, $Y \in \mathcal{F}_k^+$, then some $Y_{n_\xi} \in \mathcal{F}_k^+$. Set $A = F_n$ and $B = \overline{Y_{n_\xi}}$.

Definition. Let $\tilde{\mathcal{F}}_k$ be the collection of sets $X \subseteq \bigcup \mathcal{F}$ with the property that for any continuous function $f : Y \to X$, $Y \subseteq \mathbb{R}$, there is $W \in \mathcal{F}_k$ with $Y \setminus f^{-1}[W] \in \mathcal{M}_k$. Note that $\tilde{\mathcal{F}}_k$ is a $\sigma$-ideal, which extends $\mathcal{M}_k$. 

Lemma 2. If $X \notin \tilde{\mathcal{F}}_k$, then $X$ has a $\kappa$-dense family contained in $\mathcal{F}_k^+$. 

Proof. Suppose $X \notin \tilde{\mathcal{F}}_k$. If $X \subseteq \bigcup \mathcal{F}$, then $\{x\}$ for any $x \in X \setminus \bigcup \mathcal{F}$ is a $\kappa$-dense family. So, let $X \subseteq \bigcup \mathcal{F}$ and let $f : Y \to X$ ($Y \subseteq \mathbb{R}$) be a continuous function such that $\forall W \in \mathcal{F}_k$ $Y \setminus f^{-1}[W] \in \mathcal{M}_k$. Let $\mathcal{U}$ be a countable base for $\mathbb{R}$. Let $V = \bigcup\{U \in \mathcal{U} : f(U \cap Y) \in \mathcal{F}_k\}$. Then $f[V \cap Y] \in \mathcal{F}_k$, so, by our assumption, $Y \setminus V \in \mathcal{M}_k^+$. 

As a $\kappa$-dense family we take $\{f[U \cap Y] : U \in \mathcal{U} \text{ and } f[U \cap Y] \in \mathcal{F}_k^+\}$. To see that this works let $\mu < \kappa$ and suppose that $X$ is covered by closed sets $D_\xi \subseteq \mathbb{R}$ ($\xi < \mu$). Then the sets $f^{-1}[D_\xi] \cap Y$ cover $Y$ and are relatively closed in $Y$. If $\xi$ is such that for every $U \in \mathcal{U}$ with $U \cap Y \subseteq f^{-1}[D_\xi]$, $f[U \cap Y] \in \mathcal{F}_k$, then $f^{-1}[D_\xi] \cap Y \setminus V$ is nowhere dense. Since $Y \setminus V \in \mathcal{M}_k^+$, this cannot happen to every $\xi$. Thus, we can find $\xi$ and $U$ with $f[U \cap Y] \notin \mathcal{F}_k$. Clearly $f[U \cap Y] \subseteq D_\xi$. \[\Box\]

Definition. Say that a sequence $(F_\xi : \xi < \lambda) \subseteq \mathcal{F}$ is $\lambda$-cofinal in $\mathcal{F}$ if $\forall F \in \mathcal{F}_k$ $\exists \xi < \lambda F \subseteq \bigcup_{\xi < \xi} F_\xi$.

Lemma 3. Assume $(*)_\lambda$. If $\mathcal{F}$ has a $\lambda$-cofinal sequence of length $\lambda$, then $\tilde{\mathcal{F}}_k \subseteq \mathcal{F}_k$. 

Proof. Fix $X \in \mathcal{F}_k^+$, $X \subseteq \bigcup \mathcal{F}$. Let $(F_\xi : \xi < \lambda) \subseteq \mathcal{F}$ be a $\lambda$-cofinal sequence. Pick inductively $x_\xi \in X \setminus (\{x_\xi : \xi < \lambda\} \cup \bigcup_{\xi < \xi} F_\xi)$ ($\xi < \lambda$). This is possible because $X \in \mathcal{F}_k^+$ and $X \subseteq \bigcup \mathcal{F}$. By the construction, $x_\xi$'s are distinct. Also, for every $W \in \mathcal{F}_k$ there is $\xi < \lambda$ with $W \subseteq \bigcup_{\xi < \xi} F_\xi$. Hence $|W \cap \{x_\xi : \xi < \lambda\}| < \lambda$.

By $(*)_\lambda$ there is $Y \in \mathcal{M}_k^+$ and a meager-to-one continuous function $f : Y \to \{x_\xi : \xi < \lambda\}$. Note that for every $W \in \mathcal{F}_k$, $f^{-1}[W \cap X] \in \mathcal{M}_k$. Since $Y \in \mathcal{M}_k^+$ and $\kappa \leq \lambda$, we have that $Y \setminus f^{-1}[W \cap X] \notin \mathcal{M}_k$. Thus $f$ witnesses that $X \notin \tilde{\mathcal{F}}_k$. \[\Box\]

Corollary. Assume $(*)_\lambda$. If $\mathcal{F}$ has a $\lambda$-cofinal sequence of length $\lambda$, then each $X \in \mathcal{F}_k^+$ has a $\kappa$-dense family contained in $\mathcal{F}_k^+$. 

Proof. By Lemmas 2 and 3. \[\Box\]

Combining the corollaries of Lemmas 1 and 3 we get the following.

Proposition 1. Let $\kappa \leq \lambda \leq 2^{\aleph_0}$ be cardinals, $\text{cf}(\kappa) > \omega$. Assume $(*)_\lambda$. Let $\mathcal{F} \subseteq \mathcal{P}(\mathbb{R})$ be arbitrary, and let $\mathcal{F} \subseteq \mathcal{P}(\mathbb{R})$ be a family of closed sets which has a $\kappa$-cofinal sequence of length $\lambda$. Let $\mu < \kappa$ and suppose that $X \times Y \subseteq \bigcup_{\xi} D_\xi$, where $D_\xi \subseteq \mathbb{R} \times \mathbb{R}$ ($\xi < \mu$) are closed and $X \in \mathcal{F}_k^+$ and $Y \in \mathcal{F}_k^+$. 

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Then there are closed sets $A \in \mathcal{F}_k^+$ and $B \in \mathcal{F}_k^+$ such that $A \times B \subseteq D_\xi$ for some $\xi$.

To make it more transparent that Proposition 1 generalizes the Theorem recall the following notation (see [3]). Let $\mathcal{F} \subseteq \mathcal{P}(\mathbb{R})$ be such that $\bigcup \mathcal{F} = \mathbb{R} \notin \mathcal{F}$.

$$
\begin{align*}
\text{cof}(\mathcal{F}) &= \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{F}, \forall B \in \mathcal{F} \exists A \in \mathcal{A} B \subseteq A\}; \\
\text{cov}(\mathcal{F}) &= \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{F}, \bigcup \mathcal{A} = \mathbb{R}\}; \\
\text{add}(\mathcal{F}) &= \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{F}, \bigcup \mathcal{A} \notin \mathcal{F}\}.
\end{align*}
$$

It is folklore that add(\mathcal{F}) \leq \text{cov}(\mathcal{F}) \leq \text{cof}(\mathcal{F})$ and add(\mathcal{F}) \leq \text{cf}(\text{cof}(\mathcal{F})).

Let now $\mathcal{F}$ be the family of all closed null subsets of $\mathbb{R}$, and $\mathcal{N}$ the $\sigma$-ideal of null subsets of $\mathbb{R}$.

**Lemma 4.**

(a) $\text{cov}(\mathcal{M}) \leq \text{cof}(\mathcal{M})$ and $(*)_\lambda \Rightarrow \lambda \leq \text{cov}(\mathcal{M})$ (so $(*)_{\text{cof}(\mathcal{M})} \Rightarrow \text{cov}(\mathcal{M}) = \text{cof}(\mathcal{M})$);

(b) $\mathcal{F}_{\text{cov}(\mathcal{M})} \subseteq \mathcal{N}$;

(c) $\text{cof}(\mathcal{F}) \leq \text{cof}(\mathcal{M})$;

(d) $\text{add}(\mathcal{N})$ is a regular cardinal such that $\mathcal{N}_{\text{add}(\mathcal{N})} \subseteq \mathcal{N}$ and $\aleph_0 < \text{add}(\mathcal{N}) \leq \text{add}(\mathcal{M}) \leq \text{cf}(\text{cof}(\mathcal{M})) \leq \text{cof}(\mathcal{M})$.

**Proof.** (a) is trivial; (d) is well known from Cichoń's diagram (see [3]); (b) is a version of Thm. 2.1 of [7]. We sketch a proof of (c), which is a folklore heir to Thm. 2.1 of [7].

Let $\lambda = \text{cof}(\mathcal{M})$. There exist dominating and diagonalizing families of size $\lambda$, i.e., $\{f_\xi : \xi < \lambda\} \subseteq \omega^\omega$ and $\{g_\eta : \eta < \lambda\} \subseteq \omega^\omega$ such that

$$
\begin{align*}
&\forall e \in \omega^\omega \exists \xi \forall n e(n) < f_\xi(n); \\
&\forall h \in \omega^\omega \exists \eta \forall n \exists m > n h(m) = g_\eta(m)
\end{align*}
$$

(this is because $\omega \leq \text{cof}(\mathcal{M})$ and $\text{non}(\mathcal{M}) \leq \text{cof}(\mathcal{M})$ in Cichoń’s diagram; see [3]). We can assume without loss of generality that $\forall \xi \forall n f_\xi(n) > n$.

Let $\langle I^n_i : i < \omega \rangle$ be an enumeration of all finite unions of closed intervals with rational endpoints with measure $\leq 2^{-n}$. Then the sets

$$
F^\eta_\zeta = \bigcap_n \bigcup \{I^n_m(0) : f^\eta_\zeta(0) \leq m < f^{n+1}_\zeta(0)\}
$$

are closed null and every closed null set is covered by some $F^\eta_\zeta$ (here, $f^\eta_\zeta = f_\xi \circ f_\xi \circ \cdots \circ f_\xi$, $n$ times).

Indeed, suppose that $F$ is closed null. Then there is $h \in \omega^\omega$ such that $F \subseteq \bigcap_n I^n_{h(m)}$. Let $\eta$ be such that $\forall n \exists m > n h(m) = g_\eta(m)$. Define $e \in \omega^\omega$ by $e(n) = \min\{m \geq n : h(m) = g_\eta(m)\}$. Let $\zeta$ be such that $\forall n e(n) < f_\zeta(n)$. Then $\forall n e(f^\eta_\zeta(0)) < f^{n+1}_\zeta(0)$, so $\forall n \exists m \in [f^\eta_\zeta(0), f^{n+1}_\zeta(0)) h(m) = g_\eta(m)$. It follows that $\bigcap_n I^n_{h(m)} \subseteq F^\eta_\zeta$. $\square$

From Proposition 1 we get the following.

**Proposition 2.** Assume $(*)_{\text{cof}(\mathcal{M})}$. If less than $\text{add}(\mathcal{N})$ closed subsets of $\mathbb{R} \times \mathbb{R}$ cover a nonnull rectangle $X \times Y$, then some of them cover a closed nonnull rectangle $A \times B$.
Proof. Use Lemma 4. Let $\lambda = \text{cof}(\mathcal{M})$, $\kappa = \text{add}(\mathcal{N})$. By (d), $\aleph_0 < \text{cf}(\kappa)$ and $\kappa \leq \lambda$. By (a), $\lambda = \text{cov}(\mathcal{M})$. So, by (b), $X \in \mathcal{F}_k^+$; and by $\mathcal{M}_\kappa \subseteq \mathcal{N}$, $Y \in \mathcal{N}_\kappa^+$. Also by (c) and $\kappa \leq \text{cf}(\lambda)$, $\mathcal{F}$ has a $\kappa$-cofinal sequence of length $\lambda$. Now use Proposition 1 for $\mathcal{F} = \mathcal{N}$. $\square$

Note that by Lemma 4(a), $(*)_{2^{\aleph_0}} \Rightarrow \text{cov}(\mathcal{M}) = \text{cof}(\mathcal{M}) = 2^{\aleph_0}$. So, Proposition 2 directly generalizes the Theorem.

We conclude the paper with the following variant of Proposition 2.

Proposition 3. Assume $(*)_{\text{cof}(\mathcal{M})}$. Suppose that a coanalytic set $C \subseteq \mathbb{R} \times \mathbb{R}$, whose all horizontal sections are unions of less than $\text{cf}(\text{cof}(\mathcal{M}))$ closed sets, contains a nonnull rectangle $X \times Y$. Then it contains a closed nonnull rectangle $A \times B$.

Proof. Let $\lambda = \text{cof}(\mathcal{M})$, $\kappa = \text{cf}(\lambda)$. As in the proof of Proposition 2 we get that $X \in \mathcal{F}_k^+$ and that there is a $\kappa$-cofinal sequence for $\mathcal{F}$ of length $\lambda$. So, by the corollary to Lemma 3, $X$ has a $\kappa$-dense family $\{F_n : n < \omega\} \subseteq \mathcal{F}_k^+$.

Let $G_n = \{y : F_n \subseteq C^y\}$. Note that $G_n$'s are analytic. Also, they cover $Y$ ($C^y$ is a union of less than $\kappa$ closed sets, so, by $\kappa$-density, $C^y$ contains some $F_n$). As $Y$ is nonnull, some $G_n$ is nonnull and, hence, being measurable, contains a closed nonnull subset. Since $F_n$ is also closed nonnull and $F_n \times G_n \subseteq C$, we are done. $\square$

Note. Our rectangles had sides parallel to the coordinate axes. What if we consider arbitrary rectangles. The sets $E$ and $F$ discussed at the beginning of the paper have a stronger property than stated. Namely, if $E$ contains a rectangle $R$ with one side nonnull and the other measurable and nonnull, then one of the sides must be parallel to the line $y = -x$. Indeed, otherwise $R$ is obtained by a rotation by an angle $\alpha$, $0 \leq \alpha < \pi/4$, of a rectangle $X \times Y$. So, $R = \{(x^*, y^*) : x \in X, y \in Y\}$, where $x^* = x \cos \alpha - y \sin \alpha$ and $y^* = x \sin \alpha + y \cos \alpha$. From $R \subseteq E$ we get that for all $x \in X$ and $y \in Y$, $x^* + y^* \in F$, i.e., $x(\cos \alpha + \sin \alpha) + y(\cos \alpha - \sin \alpha) \in F$. Thus, $a \cdot X + b \cdot Y \subseteq F$ for some nonzero $a$ and $b$. As before, if one of $X, Y$ is nonnull and the other is measurable and nonnull, we get a contradiction.

Now let $E'$ be a rotation of $E$ by an angle $\beta$, $0 < \beta < \pi$. Then $E \cap E'$ is a connul subset of the Euclidean plane, which contains no rectangle with one side nonnull and the other measurable and nonnull. Thus, from the following weaker version of $(+)$:

$(+')$ if an $F_\sigma$ subset of $\mathbb{R} \times \mathbb{R}$ contains a nonnull rectangle, then it contains a measurable nonnull rectangle;

we can construct a connul subset of the plane which contains no nonnull rectangle.

References


Department of Mathematics, University of Wroclaw, pl. Grunwaldzki 2/4, 50-156 Wroclaw, Poland
E-mail address: pawlikow@math.uni.wroc.pl