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ON A RECTANGLE INCLUSION PROBLEM

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ABSTRACT. We show that if every set of reals of size $2^{\aleph_0}$ contains a meager-to-one continuous image of a set that cannot be covered by less than $2^{\aleph_0}$ meager sets, then there exists a null (Lebesgue measure zero) subset of the plane $\mathbb{R} \times \mathbb{R}$ that meets every nonnull rectangle $X \times Y$. The antecedent is satisfied, e.g., if ω_2 Cohen reals are added to a model of the continuum hypothesis.

Martin's Axiom implies that a conull (i.e., with null complement) subset of the Euclidean plane $\mathbb{R} \times \mathbb{R}$ contains a nonnull rectangle $X \times Y$. Fremlin [5], Problem AS (see also [6], 3K), asked if this is true in ZFC.

It is known that there exists a conull subset of $\mathbb{R} \times \mathbb{R}$ which contains no rectangle $X \times Y$ with one side nonnull and the other measurable and nonnull. Namely, let $E = \{\langle x, y\rangle \in \mathbb{R} \times \mathbb{R} : x + y \in F\}$, where $F = \mathbb{R}\backslash\mathbb{Q}$. Clearly, E is a conull subset of $\mathbb{R} \times \mathbb{R}$. If $X \times Y \subseteq E$, then $X + Y \subseteq F$. But, by a theorem of Steinhaus (see [9]), if X is measurable nonnull and Y is nonnull, then $X + Y$ has nonempty interior; hence $X + Y$ cannot be contained in F. (However, Brodskij and Eggleston (see [4]) showed that a measurable nonnull subset of $\mathbb{R} \times \mathbb{R}$ always contains a rectangle $X \times Y$ with X perfect and Y measurable nonnull.)

Consider the following proposition.

(+) If an $\mathbf{F}_\sigma$ subset of $\mathbb{R} \times \mathbb{R}$ contains a nonnull rectangle $X \times Y$, then it contains a measurable nonnull rectangle $A \times B$.

Proposition (+) implies that there exists a conull subset of $\mathbb{R} \times \mathbb{R}$ which contains no nonnull rectangle. Any conull F_σ subset of the set E considered above will do.

Proposition (+) has other interesting consequences (see [1]). For instance, it follows from (+) that if X and Y are nonnull subsets of $\mathbb{R}$, then $X + Y$ is nonmeager, hence every meager subgroup of $\mathbb{R}$ is null. (If $X + Y$ is covered by an $\mathbf{F}_\sigma$ set F, then $F^* = \{(x, y) : x + y \in F\}$ is an $\mathbf{F}_\sigma$ cover for $X \times Y$. By (+), F^* contains a closed nonnull rectangle $A \times B$. By the theorem of Steinhaus mentioned above, $A + B$ has nonempty interior. Hence, F has nonempty interior.)

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Consistency of $(+)$ was shown independently by Friedman and Shelah (see [1] and [2]). The model used was ω_2 Cohen reals over a model of the continuum hypothesis.

In this paper we show that $(+)$ is implied by a combinatorial condition whose consistency has been known for some time.

Denote by $\mathscr{P}(\mathbb{R})$ the family of all subsets of $\mathbb{R}$. Let λ be a cardinal number. For $\mathscr{A} \subseteq \mathscr{P}(\mathbb{R})$ and $X \subseteq \mathbb{R}$ write $X \in \mathscr{A}_\lambda$ if X can be covered by less than λ sets from $\mathscr{A}$, otherwise write $X \in \mathscr{A}_\lambda^+$. Write also $X \in \mathscr{A}^+$ if $X \in \mathscr{A}_2^+$.

Let $\mathscr{M}$ be the σ-ideal of meager subsets of $\mathbb{R}$. Say that a function is meager-to-one if preimages of points are meager.

For $D \subseteq \mathbb{R} \times \mathbb{R}$ and $y \in \mathbb{R}$ let D^y denote the horizontal section of D determined by y, i.e., $D^y = \{x \in \mathbb{R} : \langle x, y\rangle \in D\}$.

Consider the following condition.

$(*)_\lambda$ Every set of reals of size λ contains a meager-to-one continuous image of a set from $\mathscr{M}_\lambda^+$.

Note that we do not weaken $(*)_\lambda$ if we replace 'continuous' by 'Baire measurable'. This is because Baire measurable functions are continuous on comeager sets (see [9]).

It is folklore that $(*)_{2^{\aleph_0}}$ holds if ω_2 Cohen reals are added to a model of the continuum hypothesis. For example, Miller [8], p. 577, showed that in this model every set of reals of size $2^{\aleph_0}$ contains a one-to-one continuous image of a $(2^{\aleph_0}, \aleph_1)$-Lusin set. ($X \subseteq \mathbb{R}$ is a (λ, κ)-Lusin set if $|X| = \lambda$ and $|X \cap S| < \kappa$ for all meager S.) This immediately gives $(*)_{2^{\aleph_0}}$.

We prove:

Theorem. $(*)_{2^{\aleph_0}} \Rightarrow (+)$.

The Theorem is a consequence of the following Lemma.

Lemma. *Assume $(*)_{2^{\aleph_0}}$. Then, with every nonnull set $X \subseteq \mathbb{R}$ we can associate a family $\{E_n : n < \omega\}$ of closed nonnull subsets of $\mathbb{R}$ such that if $X \subseteq \bigcup_{m<\omega} D_m$, $D_m \subseteq \mathbb{R}$ closed sets, then for some m and n, $E_n \subseteq D_m$.*

Proof of the Theorem. Suppose that $X \times Y \subseteq \mathbb{R} \times \mathbb{R}$ is a nonnull rectangle covered by $\bigcup_{m<\omega} F_m$, $F_m \subseteq \mathbb{R} \times \mathbb{R}$ closed. Clearly X and Y are nonnull. Let the E_n's be as in the Lemma. Let $G_{nm} = \{y : E_n \subseteq (F_m)^y\}$. Then $E_n \times G_{nm} \subseteq F_m$. Each G_{nm} is closed. By the Lemma, the G_{nm}'s cover Y. Indeed, let $y \in Y$. We have $X \times \{y\} \subseteq \bigcup_m F_m$. So, $X \subseteq \bigcup_m (F_m)^y$; hence, by the Lemma, some $(F_m)^y$ contains some E_n, i.e. $y \in G_{nm}$.

Since Y is nonnull, it follows that some G_{nm} is nonnull. Thus, we can take $E_n \times G_{nm}$ as our measurable nonnull rectangle, which is contained in F_m. □

Proof of the Lemma. Assume $(*)_{2^{\aleph_0}}$. Fix a nonnull set $X \subseteq \mathbb{R}$.

Claim. *There is $Y \in \mathscr{M}_{2^{\aleph_0}}^+$ and a meager-to-one continuous function $f : Y \mapsto X$ such that for each closed null set W, $f^{-1}[W] \in \mathscr{M}_{2^{\aleph_0}}$.*

Proof. Let $\{E_\xi : \xi < 2^{\aleph_0}\}$ be an enumeration of all closed null sets. Pick inductively $x_\xi \in X \setminus (\{x_\zeta : \zeta < \xi\} \cup \bigcup_{\zeta<\xi} E_\zeta)$ $(\xi < 2^{\aleph_0})$. This is possible because X is not null and $\{x_\zeta : \zeta < \xi\} \cup \bigcup_{\zeta<\xi} E_\zeta$ is null. (By $(*)_{2^{\aleph_0}}$, $\mathbb{R} \in \mathscr{M}_{2^{\aleph_0}}^+$; by [7], Thm. 2.1, this implies that a union of less than $2^{\aleph_0}$ closed null sets is

null.) By the construction, the x_ξ's are distinct and for every closed null set W, $|W \cap \{x_\xi : \xi < 2^{\aleph_0}\}| < 2^{\aleph_0}$.

By $(*)_{2^{\aleph_0}}$ there is $Y \in \mathscr{M}^+_{2^{\aleph_0}}$ and a meager-to-one continuous function $f : Y \mapsto \{x_\xi : \xi < 2^{\aleph_0}\}$. Clearly $f^{-1}[W \cap X] \in \mathscr{M}_{2^{\aleph_0}}$, for each closed null W. □

Let $\mathscr{U}$ be a countable base for $\mathbb{R}$. Let $\{U_n : n < \omega\}$ be an enumeration of all $U \in \mathscr{U}$ with the property that $\overline{f[U \cap Y]}$ is not null. Let $E_n = \overline{f[U_n \cap Y]}$ ($n < \omega$).

To see that this works suppose that $X \subseteq \bigcup_{m<\omega} D_m$, $D_m \subseteq \mathbb{R}$ closed sets. Then $f^{-1}[D_m \cap X]$'s are relatively closed in Y sets that cover Y.

Suppose that for each m and $U \in \mathscr{U}$ with $U \cap Y \subseteq f^{-1}[D_m \cap X]$, $\overline{f[U \cap Y]}$ is null. Then, by the claim, $U \cap Y \in \mathscr{M}_{2^{\aleph_0}}$. It follows that Y is a union of countably many sets from $\mathscr{M}_{2^{\aleph_0}}$ and countably many nowheredense sets. Hence $Y \in \mathscr{M}_{2^{\aleph_0}}$, which is a contradiction.

Thus, for some $U \in \mathscr{U}$ with $U \cap Y \subseteq f^{-1}[D_m \cap X]$, $\overline{f[U \cap Y]}$ is not null, so it must be one of the E_n's. Clearly, $\overline{f[U \cap Y]} \subseteq D_m$. □

We shall now generalize the Lemma and the Theorem. Fix cardinals $\kappa \leq \lambda \leq 2^{\aleph_0}$, $\mathrm{cf}(\kappa) > \omega$, and an arbitrary family $\mathscr{F} \subseteq \mathscr{P}(\mathbb{R})$ of closed sets. Note that $\mathscr{F}_\kappa{}^+$ is a σ-ideal.

Definition. Let $X \subseteq \mathbb{R}$. Say that a family $\{F_n : n < \omega\}$ of closed sets is κ-dense (for X) if, whenever X is covered by less than κ closed sets, then some one of these closed sets covers some F_n. Say σ-dense for $\aleph_1$-dense.

Note. Let $\mathscr{I} \subseteq \mathscr{P}(\mathbb{R})$ be a σ-ideal. Every closed set from $\mathscr{I}^+$ (so also every superset of such a set) has a σ-dense family contained in $\mathscr{I}^+$. Indeed, suppose that $X \in \mathscr{I}^+$ is closed. Let $\mathscr{U}$ be a countable base for $\mathbb{R}$. Let $X^* = X \setminus \bigcup\{U \in \mathscr{U} : U \cap X \in \mathscr{I}\}$. Then X^* is closed and each $U \cap X^*$ ($U \in \mathscr{U}$) is either empty or belongs to $\mathscr{I}^+$. As a σ-dense family we can just take the collection of those sets $\overline{U \cap X^*}$ ($U \in \mathscr{U}$) which belong to $\mathscr{I}^+$. If $X^* \subseteq \bigcup_{m<\omega} D_m$, $D_m \subseteq \mathbb{R}$ closed, then for some m, $X^* \cap D_m$ has nonempty interior relatively to X^* (Baire's category theorem). So, for some $U \in \mathscr{U}$, $U \cap X^* \neq \varnothing$ and $U \cap X^* \subseteq D_m$. Then $\overline{U \cap X^*} \subseteq D_m$, and, by the definition of X^*, $U \cap X^* \in \mathscr{I}^+$.

It also follows that if X is arbitrary and we can find a family of closed sets $\{F_n : n < \omega\} \subseteq \mathscr{I}^+$ such that every $\mathbf{F}_\sigma$ set covering X contains some F_n, then X has a σ-dense family contained in $\mathscr{I}^+$.

Lemma 1. *Let $\{F_n : n < \omega\}$ be a κ-dense family for X. Suppose that $X \times Y \subseteq \bigcup_{\xi<\mu} D_\xi$, where $\mu < \kappa$ and $D_\xi \subseteq \mathbb{R} \times \mathbb{R}$ ($\xi < \mu$) are closed. Then there are $Y_{n\xi} \subseteq Y$ ($n < \omega, \xi < \mu$) such that $\bigcup_{n,\xi} Y_{n\xi} = Y$ and for all n and ξ, $F_n \times Y_{n\xi} \subseteq D_\xi$.*

Proof. Let $Y_{n\xi} = \{y \in Y : F_n \subseteq (D_\xi)^y\}$. Then $F_n \times Y_{n\xi} \subseteq D_\xi$. Also, given $y \in Y$, $(D_\xi)^y$ ($\xi < \mu$) cover X. So, by the definition of a dense family, some F_n is contained in some $(D_\xi)^y$, i.e., $y \in Y_{n\xi}$. □

Corollary. *Let $\mathscr{I} \subseteq \mathscr{P}(\mathbb{R})$ be arbitrary. Let $\mu < \kappa$, and let $D_\xi \subseteq \mathbb{R} \times \mathbb{R}$ ($\xi < \mu$) be closed. Suppose that $X \times Y \subseteq \bigcup_\xi D_\xi$, where $Y \in \mathscr{I}_\kappa{}^+$ and X has a κ-dense*

family contained in $\mathscr{F}_\kappa^+$. *Then there exist closed sets* $A \in \mathscr{F}_\kappa^+$ *and* $B \in \mathscr{I}_\kappa^+$ *such that* $A \times B \subseteq D_\xi$ *for some* ξ.

Proof. Let $\{F_n : n < \omega\} \subseteq \mathscr{F}_\kappa^+$ be κ-dense for X. If in Lemma 1, $Y \in \mathscr{I}_\kappa^+$, then some $Y_{n\xi} \in \mathscr{I}_\kappa^+$. Set $A = F_n$ and $B = \overline{Y_{n\xi}}$. □

Definition. Let $\widetilde{\mathscr{F}_\kappa}$ be the collection of sets $X \subseteq \bigcup\mathscr{F}$ with the property that for any continuous function $f : Y \to X$, $Y \subseteq \mathbb{R}$, there is $W \in \mathscr{F}_\kappa$ with $Y \setminus f^{-1}[W] \in \mathscr{M}_\kappa$. Note that $\widetilde{\mathscr{F}_\kappa}$ is a σ-ideal, which extends $\mathscr{F}_\kappa$.

Lemma 2. *If* $X \notin \widetilde{\mathscr{F}_\kappa}$, *then* X *has a* κ*-dense family contained in* $\mathscr{F}_\kappa^+$.

Proof. Suppose $X \notin \widetilde{\mathscr{F}_\kappa}$. If $X \not\subseteq \bigcup\mathscr{F}$, then $\{\{x\}\}$ for any $x \in X \setminus \bigcup\mathscr{F}$ is a κ-dense family. So, let $X \subseteq \bigcup\mathscr{F}$ and let $f : Y \to X$ ($Y \subseteq \mathbb{R}$) be a continuous function such that $\forall\, W \in \mathscr{F}_\kappa\ Y \setminus f^{-1}[W] \in \mathscr{M}_\kappa^+$. Let $\mathscr{U}$ be a countable base for $\mathbb{R}$. Let $V = \bigcup\{U \in \mathscr{U} : f[U \cap Y] \in \mathscr{F}_\kappa\}$. Then $f[V \cap Y] \in \mathscr{F}_\kappa$, so, by our assumption, $Y \setminus V \in \mathscr{M}_\kappa^+$.

As a κ-dense family we take $\{\overline{f[U \cap Y]} : U \in \mathscr{U}$ and $f[U \cap Y] \in \mathscr{F}_\kappa^+\}$. To see that this works let $\mu < \kappa$ and suppose that X is covered by closed sets $D_\xi \subseteq \mathbb{R}$ ($\xi < \mu$). Then the sets $f^{-1}[D_\xi] \cap Y$ cover Y and are relatively closed in Y. If ξ is such that for every $U \in \mathscr{U}$ with $U \cap Y \subseteq f^{-1}[D_\xi]$, $f[U \cap Y] \in \mathscr{F}_\kappa$, then $f^{-1}[D_\xi] \cap Y \setminus V$ is nowhere dense. Since $Y \setminus V \in \mathscr{M}_\kappa^+$, this cannot happen to every ξ. Thus, we can find ξ and U with $f[U \cap Y] \notin \mathscr{F}_\kappa$. Clearly $\overline{f[U \cap Y]} \subseteq D_\xi$. □

Definition. Say that a sequence $\langle F_\xi : \xi < \lambda\rangle \subseteq \mathscr{F}$ is κ-cofinal in $\mathscr{F}$ if $\forall F \in \mathscr{F}_\kappa\ \exists\, \xi < \lambda\ F \subseteq \bigcup_{\zeta<\xi} F_\zeta$.

Lemma 3. *Assume* $(*)_\lambda$. *If* $\mathscr{F}$ *has a* κ*-cofinal sequence of length* λ, *then* $\widetilde{\mathscr{F}_\kappa} \subseteq \mathscr{F}_\lambda$.

Proof. Fix $X \in \mathscr{F}_\lambda^+$, $X \subseteq \bigcup\mathscr{F}$. Let $\langle F_\xi : \xi < \lambda\rangle \subseteq \mathscr{F}$ be a κ-cofinal sequence. Pick inductively $x_\xi \in X \setminus (\{x_\zeta : \zeta < \xi\} \cup \bigcup_{\zeta<\xi} F_\zeta)$ ($\xi < \lambda$). This is possible because $X \in \mathscr{F}_\lambda^+$ and $X \subseteq \bigcup\mathscr{F}$. By the construction, x_ξ's are distinct. Also, for every $W \in \mathscr{F}_\kappa$ there is $\xi < \lambda$ with $W \subseteq \bigcup_{\zeta<\xi} F_\zeta$. Hence $|W \cap \{x_\zeta : \zeta < \lambda\}| < \lambda$.

By $(*)_\lambda$ there is $Y \in \mathscr{M}_\lambda^+$ and a meager-to-one continuous function $f : Y \mapsto \{x_\xi : \xi < \lambda\}$. Note that for every $W \in \mathscr{F}_\kappa$, $f^{-1}[W \cap X] \in \mathscr{M}_\lambda$. Since $Y \in \mathscr{M}_\lambda^+$ and $\kappa \le \lambda$, we have that $Y \setminus f^{-1}[W \cap X] \notin \mathscr{M}_\kappa$. Thus f witnesses that $X \notin \widetilde{\mathscr{F}_\kappa}$. □

Corollary. *Assume* $(*)_\lambda$. *If* $\mathscr{F}$ *has a* κ*-cofinal sequence of length* λ, *then each* $X \in \mathscr{F}_\lambda^+$ *has a* κ*-dense family contained in* $\mathscr{F}_\kappa^+$.

Proof. By Lemmas 2 and 3. □

Combining the corollaries of Lemmas 1 and 3 we get the following.

Proposition 1. *Let* $\kappa \le \lambda \le 2^{\aleph_0}$ *be cardinals,* $\mathrm{cf}(\kappa) > \omega$. *Assume* $(*)_\lambda$. *Let* $\mathscr{I} \subseteq \mathscr{P}(\mathbb{R})$ *be arbitrary, and let* $\mathscr{F} \subseteq \mathscr{P}(\mathbb{R})$ *be a family of closed sets which has a* κ*-cofinal sequence of length* λ. *Let* $\mu < \kappa$ *and suppose that* $X \times Y \subseteq \bigcup_\xi D_\xi$, *where* $D_\xi \subseteq \mathbb{R} \times \mathbb{R}$ ($\xi < \mu$) *are closed and* $X \in \mathscr{F}_\lambda^+$ *and* $Y \in \mathscr{I}_\kappa^+$.

Then there are closed sets $A \in \mathscr{F}_\kappa^+$ *and* $B \in \mathscr{I}_\kappa^+$ *such that* $A \times B \subseteq D_\xi$ *for some* ξ.

To make it more transparent that Proposition 1 generalizes the Theorem recall the following notation (see [3]). Let $\mathscr{J} \subseteq \mathscr{P}(\mathbb{R})$ be such that $\bigcup \mathscr{J} = \mathbb{R} \notin \mathscr{J}$.

$$\operatorname{cof}(\mathscr{J}) = \min\{|\mathscr{A}| : \mathscr{A} \subseteq \mathscr{J},\ \forall B \in \mathscr{J}\ \exists A \in \mathscr{A}\ B \subseteq A\};$$
$$\operatorname{cov}(\mathscr{J}) = \min\{|\mathscr{A}| : \mathscr{A} \subseteq \mathscr{J},\ \bigcup \mathscr{A} = \mathbb{R}\};$$
$$\operatorname{add}(\mathscr{J}) = \min\{|\mathscr{A}| : \mathscr{A} \subseteq \mathscr{J},\ \bigcup \mathscr{A} \notin \mathscr{J}\}.$$

It is folklore that $\operatorname{add}(\mathscr{J}) \leq \operatorname{cov}(\mathscr{J}) \leq \operatorname{cof}(\mathscr{J})$ and $\operatorname{add}(\mathscr{J}) \leq \operatorname{cf}(\operatorname{cof}(\mathscr{J}))$.

Let now $\mathscr{F}$ be the family of all closed null subsets of $\mathbb{R}$, and $\mathscr{N}$ the σ-ideal of null subsets of $\mathbb{R}$.

Lemma 4.

(a) $\operatorname{cov}(\mathscr{M}) \leq \operatorname{cof}(\mathscr{M})$ *and* $(*)_\lambda \Rightarrow \lambda \leq \operatorname{cov}(\mathscr{M})$ (*so* $(*)_{\operatorname{cof}(\mathscr{M})} \Rightarrow \operatorname{cov}(\mathscr{M}) = \operatorname{cof}(\mathscr{M})$);

(b) $\mathscr{F}_{\operatorname{cov}(\mathscr{M})} \subseteq \mathscr{N}$;

(c) $\operatorname{cof}(\mathscr{F}) \leq \operatorname{cof}(\mathscr{M})$;

(d) $\operatorname{add}(\mathscr{N})$ *is a regular cardinal such that* $\mathscr{N}_{\operatorname{add}(\mathscr{N})} \subseteq \mathscr{N}$ *and* $\aleph_0 < \operatorname{add}(\mathscr{N}) \leq \operatorname{add}(\mathscr{M}) \leq \operatorname{cf}(\operatorname{cof}(\mathscr{M})) \leq \operatorname{cof}(\mathscr{M})$.

Proof. (a) is trivial; (d) is well known from Cichoń's diagram (see [3]); (b) is a version of Thm. 2.1 of [7]. We sketch a proof of (c), which is a folklore heir to Thm. 2.1 of [7].

Let $\lambda = \operatorname{cof}(\mathscr{M})$. There exist dominating and diagonalizing families of size λ, i.e., $\{f_\xi : \xi < \lambda\} \subseteq \omega^\omega$ and $\{g_\eta : \eta < \lambda\} \subseteq \omega^\omega$ such that

$$\forall e \in \omega^\omega\ \exists \xi\ \forall n\ e(n) < f_\xi(n);$$
$$\forall h \in \omega^\omega\ \exists \eta\ \forall n\ \exists m > n\ h(m) = g_\eta(m))$$

(this is because $\mathfrak{d} \leq \operatorname{cof}(\mathscr{M})$ and $\operatorname{non}(\mathscr{M}) \leq \operatorname{cof}(\mathscr{M})$ in Cichoń's diagram; see [3]). We can assume without loss of generality that $\forall \xi\ \forall n\ f_\xi(n) > n$.

Let $\langle I_i^n : i < \omega \rangle$ be an enumeration of all finite unions of closed intervals with rational endpoints with measure $\leq 2^{-n}$. Then the sets

$$F_{\eta\xi} = \bigcap_n \bigcup \{I^m_{g_\eta(n)} : f_\xi^n(0) \leq m < f_\xi^{n+1}(0)\}$$

are closed null and every closed null set is covered by some $F_{\eta\xi}$ (here, $f_\xi^n = f_\xi \circ f_\xi \circ \cdots \circ f_\xi$, n times).

Indeed, suppose that F is closed null. Then there is $h \in \omega^\omega$ such that $F \subseteq \bigcap_m I^m_{h(m)}$. Let η be such that $\forall n\ \exists m > n\ h(m) = g_\eta(m)$. Define $e \in \omega^\omega$ by $e(n) = \min\{m \geq n : h(m) = g_\eta(m)\}$. Let ξ be such that $\forall n\ e(n) < f_\xi(n)$. Then $\forall n\ e(f_\xi^n(0)) < f_\xi^{n+1}(0)$, so $\forall n\ \exists m \in [f_\xi^n(0), f_\xi^{n+1}(0))\ h(m) = g_\eta(m)$. It follows that $\bigcap_m I^m_{h(m)} \subseteq F_{\eta\xi}$. $\square$

From Proposition 1 we get the following.

Proposition 2. *Assume* $(*)_{\operatorname{cof}(\mathscr{M})}$. *If less than* $\operatorname{add}(\mathscr{N})$ *closed subsets of* $\mathbb{R} \times \mathbb{R}$ *cover a nonnull rectangle* $X \times Y$, *then some of them cover a closed nonnull rectangle* $A \times B$.

Proof. Use Lemma 4. Let $\lambda = \mathrm{cof}(\mathscr{M})$, $\kappa = \mathrm{add}(\mathscr{N})$. By (d), $\aleph_0 < \mathrm{cf}(\kappa)$ and $\kappa \leq \lambda$. By (a), $\lambda = \mathrm{cov}(\mathscr{M})$. So, by (b), $X \in \mathscr{F}_\lambda^+$; and by $\mathscr{N}_\kappa \subseteq \mathscr{N}$, $Y \in \mathscr{N}_\kappa^+$. Also by (c) and $\kappa \leq \mathrm{cf}(\lambda)$, $\mathscr{F}$ has a κ-cofinal sequence of length λ. Now use Proposition 1 for $\mathscr{J} = \mathscr{N}$. □

Note that by Lemma 4(a), $(*)_{2^{\aleph_0}} \Rightarrow \mathrm{cov}(\mathscr{M}) = \mathrm{cof}(\mathscr{M}) = 2^{\aleph_0}$. So, Proposition 2 directly generalizes the Theorem.

We conclude the paper with the following variant of Proposition 2.

Proposition 3. *Assume* $(*)_{\mathrm{cof}(\mathscr{M})}$. *Suppose that a coanalytic set* $C \subseteq \mathbb{R}\times\mathbb{R}$, *whose all horizontal sections are unions of less than* $\mathrm{cf}(\mathrm{cof}(\mathscr{M}))$ *closed sets, contains a nonnull rectangle* $X \times Y$. *Then it contains a closed nonnull rectangle* $A \times B$.

Proof. Let $\lambda = \mathrm{cof}(\mathscr{M})$, $\kappa = \mathrm{cf}(\lambda)$. As in the proof of Proposition 2 we get that $X \in \mathscr{F}_\lambda^+$ and that there is a κ-cofinal sequence for $\mathscr{F}$ of length λ. So, by the corollary to Lemma 3, X has a κ-dense family $\{F_n : n < \omega\} \subseteq \mathscr{F}_\kappa^+$.

Let $G_n = \{y : F_n \subseteq C^y\}$. Note that G_n's are analytic. Also, they cover Y (C^y is a union of less than κ closed sets, so, by κ-density, C^y contains some F_n). As Y is nonnull, some G_n is nonnull and, hence, being measurable, contains a closed nonnull subset. Since F_n is also closed nonnull and $F_n \times G_n \subseteq C$, we are done. □

Note. Our rectangles had sides parallel to the coordinate axes. What if we consider arbitrary rectangles. The sets E and F discussed at the beginning of the paper have a stronger property than stated. Namely, if E contains a rectangle R with one side nonnull and the other measurable and nonnull, then one of the sides must be parallel to the line $y = -x$. Indeed, otherwise R is obtained by a rotation by an angle α, $0 \leq \alpha < \pi/4$, of a rectangle $X \times Y$. So, $R = \{\langle x^*, y^*\rangle : x \in X, y \in Y\}$, where $x^* = x\cos\alpha - y\sin\alpha$ and $y^* = x\sin\alpha + y\cos\alpha$. From $R \subseteq E$ we get that for all $x \in X$ and $y \in Y$, $x^* + y^* \in F$, i.e., $x(\cos\alpha + \sin\alpha) + y(\cos\alpha - \sin\alpha) \in F$. Thus, $a \cdot X + b \cdot Y \subseteq F$ for some nonzero a and b. As before, if one of X, Y is nonnull and the other is measurable and nonnull, we get a contradiction.

Now let E' be a rotation of E by an angle β, $0 < \beta < \pi$. Then $E \cap E'$ is a conull subset of the Euclidean plane, which contains no rectangle with one side nonnull and the other measurable and nonnull. Thus, from the following weaker version of (+):

(+') if an $\mathbf{F}_\sigma$ subset of $\mathbb{R}\times\mathbb{R}$ contains a nonnull rectangle, then it contains a measurable nonnull rectangle;

we can construct a conull subset of the plane which contains no nonnull rectangle.

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