THE BOREL CLASSES
OF MAHLER'S A, S, T, AND U NUMBERS

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Abstract. In this article we examine the A, S, T, and U sets of Mahler's classification from a descriptive set theoretic point of view. We calculate the possible locations of these sets in the Borel hierarchy. A turns out to be $\Sigma^0_2$-complete, while U provides a rare example of a natural $\Sigma^0_2$-complete set. We produce an upperbound of $\Sigma^0_2$ for S and show that T is $\Pi^0_3$ but not $\Sigma^0_2$. Our main result is based on a deep theorem of Schmidt that allows us to guarantee the existence of the T numbers.

Introduction

Mahler [6] divided complex numbers into classes A, S, T, and U according to their properties of approximation by algebraic numbers. Some studies were done on the structural properties of these sets. For example, Kasch and Volkmann [3] verified that the T numbers have Hausdorff dimension zero. Also in harmonic analysis, W. Morgan, C. E. M. Pearce, and A. D. Pollington [7] have shown that the set of T and U numbers supports a measure whose Fourier transform vanishes at infinity. In the present paper we study the A, S, T, and U sets from the point of view of Descriptive Set Theory. Among the few sets whose exact Borel class is known, a large percentage turn out to be $\Pi^0_3$-complete. For example, the collection of reals that are normal or simply normal to base $n$ [4]; $C^\infty(T)$, the class of infinitely differentiable functions (viewed as a $2\pi$-periodic function on $\mathbb{R}$); and $UC_X$, the class of convergent sequences in a separable Banach space $X$ are $\Pi^0_3$-complete [2]. Apparently, there are few known natural $\Sigma^0_3$-complete sets. Of course, the complement of a $\Pi^0_3$-complete set is $\Sigma^0_3$-complete. But the complement of a natural set need not be natural! Tom Linton [5] has shown that the family of $H$-sets, a class of thin sets from harmonic analysis, is $\Sigma^0_3$-complete, and this is the only $\Sigma^0_3$-complete natural set we know of (whose complement is not also natural). A. Kechris proposed to find out what the Borel classes of the A, S, T, and U sets are. It turns out that A is rather simple, being $\Sigma^0_2$-complete. On the other hand, T is $\Pi^0_3$-hard, while U is $\Sigma^0_3$-complete. Our main results are based on a theorem of W. M.
The exact Borel classes of the $S$ and $T$ sets are unknown to us.

**Definitions and background**

For spaces $X$ and $Y$, $XY$ denotes the set of all functions $f$ from $Y$ to $X$, with the usual product topology, $X$ and $Y$ being endowed with their usual topologies ($2 = \{0, 1\}$ and $\mathbb{N} = \{1, 2, 3, \ldots\}$ being discrete). For sets $U$ and $V$, if $S$ is a function from $X^{n+1} \times Y^{n+1}$ to $U^{n+1} \times V^{n+1}$ and $n \in \mathbb{N}$, then $S|_{n}$ is the function from $X^{n+1} \times Y^{n+1}$ to $U^{n} \times V^{n}$ such that if $S((x_1, \ldots, x_{n+1}), (y_1, \ldots, y_{n+1})) = ((u_1, \ldots, u_n), (v_1, \ldots, v_n))$, then $S|_{n}((x_1, \ldots, x_{n+1}), (y_1, \ldots, y_{n+1})) = ((u_1, \ldots, u_n), (v_1, \ldots, v_n))$. $\mathbb{P} = \{x \in \mathbb{R} : x > 1\}$ and $\mathbb{A}$ denotes the class of all nonzero real algebraic numbers in $\mathbb{C}$. We use the standard terminology of Addison to describe the Borel hierarchy. Thus the multiplicative sets of level $n$ are denoted by $\Pi_{n}^{0}$, while the additive class of level $n$ is denoted by $\Sigma_{n}^{0}$. In particular, $\Sigma_{0}^{0}$ = Open, $\Pi_{1}^{0}$ = Closed, $\Sigma_{2}^{0}$ = $F_{\sigma}$, $\Pi_{2}^{0}$ = $G_{\delta}$. In addition, the countable union of $\Pi_{n}^{0}$ sets is $\Sigma_{n+1}^{0}$; the countable intersection of $\Pi_{n}^{0}$ sets is a $\Sigma_{n+1}^{0}$ set; the complement of a $\Pi_{n}^{0}$ set is $\Sigma_{n}^{0}$; the $\Sigma_{n}^{0}$ sets are closed under finite intersection and countable union; while the $\Pi_{n}^{0}$ sets are closed under finite union and countable intersection. If the context demands it, we use $\Pi_{n}^{0}(X)$ to denote the $\Pi_{n}^{0}$ subsets of a space $X$.

Let $\Gamma = \Sigma_{0}^{0}$ or $\Pi_{n}^{0}$. We call a set $C \subseteq X$ (a Polish space) $\Gamma$-hard if for any $B \in \Gamma(2^{\mathbb{N}})$, there is a continuous function $f$ from $2^{\mathbb{N}}$ to $X$, such that $B = f^{-1}(C)$. If, moreover, $C \in \Gamma(X)$, we call $C$ $\Gamma$-complete. It is well known (see [2]) that a $\Pi_{n}^{0}$-complete set in an uncountable Polish space is $\Pi_{n}^{0}$ but not $\Sigma_{n}^{0}$, and if $A$ is $\Pi_{n}^{0}$-hard, then $A$ is not $\Sigma_{n}^{0}$. As well, in uncountable Polish spaces every $\Pi_{n}^{0}$ set and every $\Sigma_{n}^{0}$ set is $\Pi_{n+1}^{0}$ and $\Sigma_{n+1}^{0}$, so the Borel hierarchy is increasing in $n$.

For a given set $C \subseteq X$, in order to find the exact Borel class of $C$, one must first calculate an upperbound for $C$, by showing, for example, that $C$ is $\Pi_{n}^{0}$ and then prove a lowerbound for $C$'s Borel class, for example, by showing that $C$ is $\Pi_{n}^{0}$-hard. Usually, finding the upperbound is fairly easy. However, it can be difficult to prove the hardness of $C$. Since the Borel classes $\Pi_{n}^{0}$ and $\Sigma_{n}^{0}$ are closed under continuous preimages, if $B$ is $\Gamma$-hard ($\Gamma$-complete) and $B = f^{-1}(C)$, where $f$ is a continuous function, then $C$ is $\Gamma$-hard ($\Gamma$-complete, if also $C \in \Gamma$). This remark is the basis of a common method for showing that a given set $B$ is $\Gamma$-hard: Choose an already known $\Gamma$-hard set $B$ and show that there is a continuous function $f$ such that $B = f^{-1}(C)$.

Now we define the $A, S, T$, and $U$ sets, from Mahler's classification. For convenience we use Koksma's notation, which is equivalent to that of Mahler. Given algebraic $\alpha \in \mathbb{C}$, let $p(x) \in \mathbb{Z}[x]$ be its minimal polynomial. Fix $d, h \in \mathbb{N}$. Let $X_{d,h}$ be the finite collection of polynomials with degree $\leq d$ whose largest coefficient has absolute value $\leq h$. Let the height of a polynomial, $ht(p)$, be the maximum of the absolute values of the coefficients. Let $A_{d,h}$ be the finite collection of algebraic numbers $\alpha$ such that for some $p \in X_{d,h}$, $p(\alpha)$ is zero (recall that $0 \notin \mathbb{N}$). Thus, $A_{d,h}$ is the finite collection of algebraic (complex) numbers whose minimal polynomial has degree $\leq d$ and $ht \leq h$. Let $\xi$ be any complex number and let $\alpha$ belong to $A_{d,h}$ such that $|\xi - \alpha|$ takes
the smallest positive value; define $\omega^*_d(\xi, h)$ by

$$|\xi - \alpha| = \frac{1}{h^d \omega^*_d(\xi, h + 1)}.$$ 

Set

$$\omega^*_d(\xi) = \limsup_{h \to \infty} \omega^*_d(\xi, h)$$ and $\omega^*(\xi) = \limsup_{d \to \infty} \omega^*_d(\xi).$

So the values of $\omega^*_d(\xi)$ and $\omega^*(\xi)$ measure how fast $\xi$ is approximated by algebraic numbers. We define, according to the values of $\omega^*_d(\xi)$ and $\omega^*(\xi)$, the $A$, $S$, $T$, and $U$ sets as follows:

$$A = \{\xi \in \mathbb{C} : \omega^*(\xi) = 0\},$$

$$S = \{\xi \in \mathbb{C} : 0 < \omega^*(\xi) < \infty\},$$

$$T = \{\xi \in \mathbb{C} : \omega^*(\xi) = \infty \text{ and } \forall d \in \mathbb{N} \ (\omega^*_d(\xi) < \infty)\},$$

$$U = \{\xi \in \mathbb{C} : \omega^*(\xi) = \infty \text{ and } \exists d \in \mathbb{N} \ (\omega^*_d(\xi) = \infty)\}.$$

Thus, the $A$ numbers are slowly approximated by algebraic numbers. The $S$ numbers are approximated a bit more quickly than $A$ numbers. On the other hand, the $T$ numbers and the $U$ numbers are very rapidly approximated, i.e., the value of $\omega^*(\xi)$ is infinite. In particular, the approximation of the $U$ numbers is so quick that for some $d \in \mathbb{N}$, $\omega^*_d(\xi)$ diverges. For these reasons, we claim that the set of complex numbers is naturally partitioned by the $A$, $S$, $T$, and $U$ numbers.

**Results**

**Lemma 1.** $\xi \in A \iff \xi$ is an algebraic number.

**Proof.** See [1, pp. 85–94].

**Proposition 2.** (i) The $A$ numbers are $\Sigma^0_1$-complete, and the $U$ numbers are $\Sigma^0_1$.

(ii) The $S$ numbers are $\Sigma^0_2$, while the collection of $T$ numbers are $\Pi^0_2$.

**Proof of Proposition 2.** (i) For each $d \in \mathbb{N}$, let $U_d$ be the collection of $\xi \in \mathbb{C}$ such that $\omega^*_d(\xi) = \infty$. Then $U_d$ is $\Pi^0_2$, since

$$\xi \in U_d \iff \omega^*_d(\xi) = \infty$$

$$\iff \forall a \in \mathbb{N} \forall b \in \mathbb{N} \exists c \in \mathbb{N} \ (\omega^*_d(\xi, b + c) > a)$$

$$\iff \forall a \in \mathbb{N} \forall b \in \mathbb{N} \exists c \in \mathbb{N} \exists \alpha \in A_{d,b+c} \left(0 < |\xi - \alpha| < \frac{1}{(b + c)^{ad+1}}\right)$$

$$\iff \xi \in \bigcap_{a \in \mathbb{N}} \bigcap_{b \in \mathbb{N}} \bigcap_{c \in \mathbb{N}} \bigcup_{\alpha \in A_{d,b+c}} V(a, b, c, \alpha),$$

where $V(a, b, c, \alpha)$ is the collection of $\xi \in \mathbb{C}$ such that $0 < |\xi - \alpha| < 1/(b + c)^{ad+1}$, which is open. Since it is easy to see that for each $d$, $\omega^*_d(\xi) = \infty$ implies $\omega^*_{d+1}(\xi) = \infty$, we have $U = \bigcup_{d=1}^{\infty} U_d$ and $U$ is $\Sigma^0_2$. It is well known that if $D$ is a countable dense set in a perfect Polish space, then $D$ is $\Sigma^0_2$-complete. Thus, by Lemma 1, $A$ is $\Sigma^0_2$-complete.

(ii) By definition, $T$ is the collection of $\xi \in \mathbb{C}$ such that $\omega^*(\xi) = \infty$ and $\forall a \in \mathbb{N} \ (\omega^*_a(\xi) < \infty)$. Thus, $T = M \cap N$, where $M = \{\xi \in \mathbb{C} : \omega^*(\xi) = \infty\}$
and $N = \{ \xi \in \mathbb{C} : \forall \alpha \in \mathbb{N} \ (\omega^*_\alpha(\xi) < \infty) \}$. Now $M$ is $\Pi^0_4$, since

$$\xi \in M \iff \forall a \in \mathbb{N} \ \forall b \in \mathbb{N} \ \exists c \in \mathbb{N} \ (\omega^*_b(\xi) > a)$$

$$\iff \forall a \in \mathbb{N} \ \forall b \in \mathbb{N} \ \exists c \in \mathbb{N} \ \exists d \in \mathbb{N} \ \forall e \in \mathbb{N} \ \exists f \in \mathbb{N}$$

$$\left( \omega^*_{b+c}(\xi, e+f) > a + \frac{1}{d+1} \right)$$

$$\iff \xi \in \bigcap_{a \in \mathbb{N}} \bigcup_{b \in \mathbb{N}} \bigcup_{d \in \mathbb{N}} \bigcup_{e \in \mathbb{N}} \bigcup_{f \in \mathbb{N}} W(a, b, c, d, e, f),$$

where $W(a, b, c, d, e, f)$ is the collection of $\xi \in \mathbb{C}$ such that $\omega^*_{b+c}(\xi, e+f) > a + 1/(d+1)$, which is open by the argument above. So $N$ is $\Pi^0_3$, since by (i) $U$ is $\Sigma^0_3$ and

$$\xi \in N \iff \forall a \in \mathbb{N} \ (\omega^*_a(\xi) < \infty)$$

$$\iff \xi \in \mathbb{C} - U.$$ 

Hence $T$ is $\Pi^0_4$, being the intersection of two $\Pi^0_4$ sets. Since $\xi \in S \iff \xi \notin T$, $\xi \notin U$, and $\xi \notin A$, $S$ is $\Sigma^0_3$. $\square$

In $2^\mathbb{N}$, $Q$ is the collection of sequences which end in zeros.

**Lemma 3.** There exists a continuous function $\nu$ from $2^\mathbb{N}$ to $\mathbb{N}^\mathbb{N}$ such that

(i) for each $\alpha \in \mathbb{N}$, $\nu(\alpha)(d) \leq \nu(\alpha)(d+1)$;

(ii) $\alpha \in Q \iff \lim_{d \to \infty} \nu(\alpha)(d) < \infty$.

**Proof of Lemma 3.** Let $\alpha \in 2^\mathbb{N}$. We produce $\beta = \nu(\alpha)$ recursively. First $\beta(1) = \alpha(1)$. Suppose that we have defined $\beta(i)$ for all $i \leq k$. Put $\beta(k+1) = \beta(k)$ if $\alpha(k+1) = 0$ and $\beta(k+1) = \beta(k)+1$ otherwise. It is easy to see that the function $\nu$ satisfies (i). As long as $\alpha$ ends in zeros, so does $\nu(\alpha)$ in constants. Otherwise, $\nu(\alpha)(d)$ goes to infinity as $d \to \infty$, because for infinitely many $d$'s, $\nu(\alpha)(d+1) = \nu(\alpha)(d) + 1$. So (ii) is valid. For given $d \in \mathbb{N}$, $\alpha_1, \alpha_2 \in 2^\mathbb{N}$, such that $\alpha_1(i) = \alpha_2(i)$ for all $i \leq d$, $\nu(\alpha_1)(i) = \nu(\alpha_2)(i)$ for all $i \leq d$. So $\nu$ is continuous. This completes Lemma 3. $\square$

From Lemma 3, $\alpha \notin Q \iff \lim_{d \to \infty} \nu(\alpha)(d) = \infty$. To prove our main theorem, we need a standard example of the $\Pi^0_3$-complete set.

**Lemma 4.** The set $P_3 = \{ \alpha = (\alpha_d) \in (2^\mathbb{N})^\mathbb{N} : \forall d \in \mathbb{N} \ (\alpha_d \notin Q) \}$ is $\Pi^0_3$-complete.

**Proof.** See [2].

The following theorem is the main result of the paper.

**Theorem 5.** There is a continuous function $f$ from $(2^\mathbb{N})^\mathbb{N}$ to $\mathbb{C}$ such that

$$\alpha \in P_3 \iff f(\alpha) \in T \quad \text{and} \quad \alpha \notin P_3 \iff f(\alpha) \in U.$$ 

In particular, $T$ is $\Pi^0_3$-hard and $U$ is $\Sigma^0_3$-complete.

Roughly speaking, the original statement of a theorem of Schmidt is the following: Let $\alpha_1, \alpha_2, \ldots$ be any nonzero algebraic numbers and let $\nu_1, \nu_2, \ldots$ be any real numbers exceeding 1. Then we may find $\xi \in \mathbb{C}$ such that according to $\alpha_1, \alpha_2, \ldots$ and $\nu_1, \nu_2, \ldots$, $\xi$ is a $U$ number or $T$ number.

By using $\nu$, which is constructed in Lemma 3, we shall effectively control $\nu_i$'s so that we are able to prove Theorem 5. In order to make it work, we need to state the reformulated version of a theorem of Schmidt which will play a crucial role in the proof of Theorem 5.
Theorem S (Schmidt). There exists a sequence \( S_n \) such that for each \( n \in \mathbb{N} \),
(i) \( S_n \) is a function from \( \mathbb{A}^n \times \mathbb{P}^n \) to \( \mathbb{A}^n \times (0, 1)^n \) and \( S_{n+1}|n = S_n \).
(ii) Suppose that
\[
S_n((\theta_1, \ldots, \theta_n), (\nu_1, \ldots, \nu_n)) = ((\gamma_1, \ldots, \gamma_n), (\lambda_1, \ldots, \lambda_n)).
\]
Then for each \( j < n \), \( \gamma_j/\theta_j \) is rational, \( H_j+1 > 2H_j \) and \( \frac{1}{4}H_j^{-1} < \gamma_{j+1} - \gamma_j < \frac{1}{2}H_j^{-1} \), where \( H_j = h_j^{(n)} \) and \( h_j = h(t(\gamma_j)) \), and furthermore, we have \( |\gamma_j - \beta| > B^{-1} \) for all algebraic numbers \( \beta \) with degree \( d \leq j \) distinct from \( \gamma_1, \ldots, \gamma_j \), where \( B = \lambda_d^{-1}b^{(3d)^4} \) and \( b \) denotes the height of \( \beta \).

Proof. See [1, pp. 85-94].

Using Theorem S we define the function \( S^* \) from \( \mathbb{A}^N \times \mathbb{P}^N \) to \( \mathbb{A}^N \times (0, 1)^N \) as follows: \( S^*((\theta_1, \theta_2, \ldots), (\nu_1, \nu_2, \ldots)) = ((\gamma_1, \gamma_2, \ldots), (\lambda_1, \lambda_2, \ldots)) \), where for each \( n \), \( S_n((\theta_1, \ldots, \theta_n), (\nu_1, \ldots, \nu_n)) = ((\gamma_1, \ldots, \gamma_n), (\lambda_1, \ldots, \lambda_n)) \). \( S^* \) is well defined by Theorem S(i).

Proof of Theorem 5. Let \( \alpha \in (2\mathbb{N})^N \). Fix a bijection \( (, ) \) from \( \mathbb{N} \times \mathbb{N} \) to \( \mathbb{N} \).
For each \( d, k \in \mathbb{N} \), define
\[
\nu((d,k)) = (\nu_1\nu_2\ldots) = \nu(\alpha_d)(k+1)(3d)^5 \quad \text{and} \quad \theta((d,k)) = \theta_d,k,
\]
where the function \( \nu \) is constructed in Lemma 3. Put \( A = \{\theta_d,k\} \) and \( \deg(\theta_d,k) = d \). Say
\[
S^*((\theta_1, \theta_2, \ldots), (\nu_1, \nu_2, \ldots)) = ((\gamma_1, \gamma_2, \ldots), (\lambda_1, \lambda_2, \ldots)).
\]
Then by Theorem S(ii), \( \gamma_1, \gamma_2, \ldots \) tends to a limit \( \xi \) which is a real number and satisfies
(1) \( |\xi - \beta| > B^{-1} \) for all algebraic numbers \( \beta \) distinct from \( \gamma_1, \gamma_2, \ldots \),
and also
(2) \( \frac{1}{4}H_j^{-1} \leq \xi - \gamma_j \leq H_j^{-1} \) for all \( j \).
Define
\[
f(\alpha) = \lim_{j \to \infty} \gamma_j = \xi.
\]
Claim. \( f \) is continuous from \( (2\mathbb{N})^N \) to \( \mathbb{C} \).

Proof of the claim. Suppose \( (\alpha_d^{(m)}) \to (\alpha_d) \) as \( m \to \infty \), where for each \( m \), \( (\alpha_d^{(m)}) \in (2\mathbb{N})^N \) and \( (\alpha_d) \in (2\mathbb{N})^N \). Say for each \( m \),
\[
f((\alpha_d^{(m)})) = \xi_m = \lim_k \gamma_k^{(m)} \quad \text{and} \quad f((\alpha_d)) = \xi = \lim_k \gamma_k,
\]
where for each \( k \in \mathbb{N} \), \( \gamma_k^{(m)} \) and \( \gamma_k \) are defined by \( S^* \), according to \( (\alpha_d^{(m)}) \) and \( (\alpha_d) \). Let \( \varepsilon > 0 \). Choose \( a_0 \) such that \( 1/2^{a_0^{-2}} < \varepsilon \). Since \( (\alpha_d^{(m)}) \) goes to \( (\alpha_d) \) as \( m \to \infty \), by the definition of \( \gamma_k^{(m)} \) and \( \gamma_k \) we may find \( N_0 \in \mathbb{N} \) such that \( |\gamma_k^{(m)} - \gamma_k| = 0 \) for all \( m \geq N_0 \). Then for all \( m \geq N_0 \), we have the following inequality:
\[
|\xi_m - \xi| \leq |\xi_m - \gamma_k^{(m)}| + |\gamma_k^{(m)} - \gamma_k| + |\gamma_k - \xi| < \frac{1}{2^{a_0^{-2}}} < \varepsilon,
\]
since from (2) and Theorem S(ii),
\[
|\xi_m - \gamma_a^{(m)}| \leq (H_a^{(m)})^{-1} < \frac{1}{2^{a-1}} (H_1^{(m)})^{-1} \leq \frac{1}{2^{a-1}}
\]
and
\[
|\xi - \gamma_a| \leq H_a^{-1} < \frac{1}{2^{a-1}} H_1^{-1} \leq \frac{1}{2^{a-1}}
\]
for all \( a \geq 1 \). So \( f \) is a continuous function. □

Now we show the main part of the theorem. Depending on the properties of \( \nu \), Theorem S guarantees that we produce a \( T \) number or \( U \) number. So we divide the following two cases so that one can have more intuitive ideas.

Case 1. \( \alpha = (\alpha_d) \notin P_3 \), i.e., \( \exists d \in \mathbb{N} \ (\alpha_d \notin \mathbb{Q}) \).

Fix such \( d \), i.e., \( \alpha_d \notin \mathbb{Q} \). Then by Lemma 3, we have
\[
\lim_{k \to \infty} (\nu(\alpha_d)(k) + 1) = \infty .
\]
It is clear that for all \( k, h = h(d, k) \),
\[
h^{-d} \omega_d^*(\xi, h) \leq |\xi - \gamma(d, k)| \leq h^{-\nu(d, k)} \quad \text{from (2) and the definition of } \omega_d^*(\xi, h),
\]
where \( f(\alpha) = \xi \). So \( d \omega_d^*(\xi, h(d, h)) \geq \nu(d, k) - 1 \), i.e.,
\[
(3) \quad \omega_d^*(\xi, h(d, k)) \geq \frac{\nu(d, k) - 1}{d} \geq (\nu(\alpha_d)(k)) + 1)3^5 d^4 - \frac{1}{d} \quad \text{for all } k .
\]
It is easy to see that \( \lim sup_{k \to \infty} h(d, k) = \infty \), since the right side of (3) goes to infinity as \( k \to \infty \). This shows that we may choose \( \{k_m\} \) such that \( k_m \to \infty \) and \( h(d, k_m) \to \infty \) as \( m \to \infty \). From (3) we get the following inequality:
\[
\omega_d^*(\xi) = \lim_{h \to \infty} \lim sup_{m \to \infty} \omega_d^*(\xi, h(d, k_m)) \geq \lim_{m \to \infty} (\nu(\alpha_d)(k_m) + 1)3^5 d^4 - \frac{1}{d} = \infty .
\]
Therefore, \( \omega_d^*(\xi) = \infty \) and \( f(\alpha) = \xi \in U \). So we derive \( \alpha \notin P_3 \Rightarrow f(\alpha) = \xi \in U \).

Case 2. \( \alpha = (\alpha_d) \in P_3 \), i.e., \( \forall d \in \mathbb{N} \ (\alpha_d \in \mathbb{Q}) \).

Fix \( d \in \mathbb{N} \). Then for all \( h, k, m \), we have
\[
\xi - \gamma(m, k) \geq \frac{1}{4} h^{-\nu(\alpha_d)(k)(3m)^5},
\]
\[
|\xi - \beta| \geq \lambda \deg(\beta)(ht(\beta))^{-(3 \deg(\beta))^4}
\]
for all algebraic numbers \( \beta \) distinct from \( \gamma_1, \gamma_2, \ldots \) from (1) and (2), where \( f(\alpha) = \xi \). In fact, all nonzero algebraic numbers appear in these two inequalities. Let \( h \) be a given natural number. Then from (4) and the definition of \( \omega_d^*(\xi, h) \), we have the following inequality:
\[
(5) \quad h^{-d} \omega_d^*(\xi, h) \geq \min\{\frac{1}{4} h^{-M_0 d^3}, \lambda(d)h^{-(3d)^4}\},
\]
where \( M_0 = \sup\{\nu(\alpha_d)(k) + 1 : s \leq d \text{ and } k < \infty\} \) and \( \lambda(d) = \min\{\lambda_s : s \leq d\} \).
Even if for \( s \leq d \), there is no \( k \) such that \( h(s, k) = h \), this inequality can be applied. The value \( \lambda(d) \) is positive and \( 1 \leq M_0 < \infty \), since \( \{\lambda_s : s \leq d\} \)
is the finite set of positive values and by assumption and Lemma 3, \( \forall d \in \mathbb{N} \) \( \lim_{k \to \infty} \nu(\alpha_d)(k) < \infty \). So from (5) we get

\[
\omega^*_d(\xi, h) \leq \max \left\{ \frac{\log 4}{\log h} + 3^5 M_0 d^4, \frac{\log 1/\lambda(d)}{d \log h} + 3^5 d^4 \right\} < \infty
\]

and

\[
\omega^*_d(\xi) = \limsup_{h \to \infty} \omega^*_d(\xi, h) \leq \max \{3^5 M_0 d^4, 3^5 d^4\} = 3^5 M_0 d^4 < \infty.
\]

Hence we can see that the inequality

\[
(6) \quad \omega^*_d(\xi) = \limsup_{h \to \infty} \omega^*_d(\xi, h) < \infty
\]

holds for all \( d \). But for all \( d, k \), we obtain

\[
\omega^*_d(\xi, h_{(d, k)}) = \frac{\nu(d, k) - 1}{d} \geq (\nu(\alpha_d)(k) + 1)3^5 d^4 - \frac{1}{d}.
\]

As in Case 1, \( \omega^*_d(\xi) \geq 3^5 d^4 M_1 - \frac{1}{d} \), where \( M_1 = \lim_{k \to \infty} \nu(\alpha_d)(k) + 1 \geq 1 \). Therefore,

\[
(7) \quad \omega^*_d(\xi) \geq (3d)^4 \quad \text{and} \quad \omega^*(\xi) = \limsup_{d \to \infty} \omega^*_d(\xi) = \infty.
\]

From (6) and (7), for all \( d \in \mathbb{N} \), \( \omega^*_d(\xi) < \infty \) and \( \omega^*(\xi) = \infty \), i.e., \( f(\alpha) = \xi \in T \). So we derive \( \alpha \in P_3 \Rightarrow f(\alpha) = \xi \in T \).

By Case 1 and Case 2, we obtain \( \alpha \in P_3 \Rightarrow f(\alpha) \in T \) and \( \alpha \notin P_3 \Rightarrow f(\alpha) \in U \). By definition of \( T, U \), it is easy to see that they are disjoint. So the continuous function \( f \) satisfies \( P_3 = f^{-1}(T) \) and \( C - P_3 = f^{-1}(U) \). This fact implies that \( T, U \) are \( \Pi^0_3 \)-hard, \( \Sigma^0_3 \)-complete respectively, since by Lemma 4, \( P_3 \) is \( \Pi^0_3 \)-complete. We complete the proof of Theorem 5. \( \square \)

Remark. We conjecture that \( S, T \) are \( \Sigma^0_4 \)-complete, \( \Pi^0_4 \)-complete, respectively.

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