A REMARK ON THE HAUSDORFF-YOUNG INEQUALITY

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ABSTRACT. We shall prove a sharp Hausdorff-Young inequality of Beckner type for functions on \( T \) with small support.

Let \( L^p(\mathbb{R}) \) denote the space of complex-valued \( L^p \) functions on \( \mathbb{R} \). For \( 1 < p < \infty \) and \( f \in L^p(\mathbb{R}) \) we set
\[
\|f\|_p = \left( \int_{\mathbb{R}} |f(x)|^p \, dx \right)^{1/p}.
\]

For \( f \in L^1(\mathbb{R}) \) we define the Fourier transform by setting
\[
\hat{f}(x) = \int_{\mathbb{R}} e^{-ixt} f(t) \, dt, \quad x \in \mathbb{R}.
\]

The sharp Hausdorff-Young inequality of Babenko [2] and Beckner [3] states that
\[
\|\hat{f}\|_{p'} \leq (2\pi)^{1/p'} B_p \|f\|_p, \quad 1 \leq p \leq 2,
\]
where \( 1/p + 1/p' = 1 \) and \( B_p = (p^{1/p} / p^{1/p'})^{1/2} \).

We shall also consider the corresponding inequality on \( T \). For \( g \in L^p(\mathbb{T}) = L^p(-\pi, \pi) \) we set
\[
\|g\|_{L^p(\mathbb{T})} = \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} |g(x)|^p \, dx \right)^{1/p} , \quad 1 \leq p < \infty,
\]
and define Fourier coefficients by setting
\[
c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-inx} g(x) \, dx, \quad n \in \mathbb{Z}.
\]

Also let \( c = (c_n)_{n=\infty}^{\infty} \) and define norms
\[
\|c\|_q = \left( \sum_n |c_n|^q \right)^{1/q} , \quad 1 \leq q < \infty,
\]
and
\[
\|c\|_{\infty} = \sup_n |c_n|.
\]
We then have the Hausdorff-Young inequality
\[ \|c\|_{p'} \leq \|g\|_{L^p(T)}, \quad 1 \leq p \leq 2, \]
and this inequality is sharp. The purpose of this note is to prove a Beckner type Hausdorff-Young inequality for \( T \). We shall do this by considering functions on \( T \) with small support. We therefore set
\[ H_p = \limsup_{a \to 0} \left\{ \frac{\|c\|_{p'}}{\|g\|_{L^p(T)}} ; g \in L^p(T), \operatorname{supp} g \subset [-a, a], \|g\|_{L^p(T)} \neq 0 \right\}, \]
where \( 1 \leq p \leq 2 \). M. E. Andersson [1] has proved that \( H_p \geq B_p \), \( 1 \leq p \leq 2 \), and that \( H_p = B_p \) if \( p' \) is an even integer. We shall here prove the following result.

**Theorem.** \( H_p = B_p \) for \( 1 \leq p \leq 2 \).

We have to prove the inequality \( H_p \leq B_p \) and to do this we shall first introduce an auxiliary function \( \varphi \). We shall use the Féjer kernel
\[ K(x) = \frac{1}{2\pi} \left( \frac{\sin x/2}{x/2} \right)^2, \quad x \in \mathbb{R}, \]
which has Fourier transform
\[ \hat{K}(\xi) = (1 - |\xi|)\chi(\xi), \]
where \( \chi \) denotes the characteristic function for the interval \([-1, 1]\). We also set \( K_\delta(x) = \delta K(\delta x), \delta > 0 \). Then
\[ \hat{K}_\delta(\xi) = \left( 1 - \frac{|\xi|}{\delta} \right)\chi_\delta(\xi), \]
where \( \chi_\delta \) is the characteristic function for the interval \([-\delta, \delta]\). For \( 0 < \delta < 1 \) it then follows that
\[ \hat{K}(\xi) - \delta \hat{K}_\delta(\xi) = 1 - \delta, \quad |\xi| \leq \delta, \]
and that the continuous function \( \hat{K} - \delta \hat{K}_\delta \) vanishes for \( |\xi| > 1 \) and is linear in the intervals \([\delta, 1]\) and \([-1, -\delta]\). We then set
\[ \psi(\xi) = \frac{1}{1-\delta} (\hat{K}(\xi) - \delta \hat{K}_\delta(\xi)) \]
so that \( \psi(\xi) = 1 \) for \( |\xi| \leq \delta \), \( \psi(\xi) \) vanishes for \( |\xi| > 1 \) and is linear in the above two intervals. We also have
\[ \hat{\psi}(x) = \frac{1}{1-\delta} (\hat{K}(x) - \delta \hat{K}_\delta(x)) = \frac{2\pi}{1-\delta} (K(x) - \delta K_\delta(x)) \]
and
\[ \int |\hat{\psi}| dx \leq \frac{2\pi}{1-\delta} \left( \int |K| dx + \delta \int |K_\delta| dx \right) = 2\pi \frac{1+\delta}{1-\delta}. \]

We then define the function \( \varphi \) by setting \( \varphi(x) = \psi(\delta x) \) so that \( \varphi(x) = 1 \) for \( |x| \leq 1 \). Then
\[ \varphi(x) = \frac{1}{\delta} \hat{\psi} \left( \frac{x}{\delta} \right) \]
and
\[ \int |\phi| \, dx = \int |\psi| \, dx \leq 2\pi \frac{1 + \delta}{1 - \delta} = 2\pi c_\delta, \]
where \( c_\delta = (1 + \delta)/(1 - \delta). \)

We then fix \( \delta. \)

We also set \( \phi_a(x) = \phi(x/a), \) \( 0 < a < 1, \) so that \( \phi_a(x) = 1 \) for \( |x| \leq a. \)
Then \( \phi_a(x) = a\phi(ax) \) and
\[ \int |\phi_a| \, dx \leq 2\pi c_\delta. \]

(2) \[ \int |\phi_a| \, dx \leq 2\pi c_\delta. \]

Now, assume that \( f \in L^p(\mathbb{R}), \) \( 1 < p \leq 2, \) and that \( f(x) = 0 \) for \( |x| > a. \)
Then \( f = \phi_a f \) and
\[ c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-inx} f(x) \, dx = \frac{1}{2\pi} \hat{f}(n), \quad n \in \mathbb{Z}. \]
It follows that
\[ c_n = (2\pi)^{-2} \phi_a * \hat{f}(n) = (2\pi)^{-2} \int \hat{f}(n-t) \phi_a(t) \, dt, \]
and invoking Hölder's inequality one obtains
\[ |c_n|^{p'} \leq (2\pi)^{-2p'} \left( \int |\hat{f}(n-t)||\phi_a(t)|^{1/p'}|\phi_a(t)|^{1/p} \, dt \right)^{p'/p} \]
\[ \leq (2\pi)^{-2p'} \left( \int |\hat{f}(n-t)|^{p'}|\phi_a(t)| \, dt \right)^{p'/p} \left( \int |\phi_a(t)| \, dt \right)^{p'/p} \]
\[ = (2\pi)^{-2p'} \left( \int |\hat{f}(n-t)| \phi_a(n-t) \, dt \right)^{p'/p} \left( \int |\phi| \, dt \right)^{p'/p}. \]

Now choose \( \epsilon > 0. \) If \( a \) is small, we then get
\[ \sum_n |c_n|^{p'} \leq (2\pi)^{-2p'} \left( \int |\hat{f}(t)|^{p'} \left( \sum_n a|\phi(an - at)| \right) \, dt \right)^{p'/p} \left( \int |\phi| \, dt \right)^{p'/p} \]
\[ \leq (2\pi)^{-2p'} \left( \int |\phi| \, dt + \epsilon \right)^{1+p'/p} \left( \int |\hat{f}(t)|^{p'} \, dt \right). \]
Invoking (1) and (2), we then obtain
\[ \|c\|_{p'} \leq (2\pi)^{-2} \left( \int |\phi| \, dt + \epsilon \right) \|\hat{f}\|_{p'} \]
\[ \leq (2\pi)^{-2} 2\pi(c_\delta + \epsilon)(2\pi)^{1/p'} B_p \|f\|_p \]
\[ = (2\pi)^{-1+1/p'}(c_\delta + \epsilon)B_p \left( \int |f|^p \, dx \right)^{1/p} \]
\[ = (c_\delta + \epsilon)B_p \left( \frac{1}{2\pi} \int |f|^p \, dx \right)^{1/p}. \]
Choosing \( \delta \) small we can get the value of \( c_\delta \) close to 1 and therefore we have
\[ \|c\|_{p'} \leq (1 + 2\epsilon)B_p \|f\|_{L^p(T)} \]
if \( a \) is small. It follows that \( H_p \leq B_p, \) and the proof of the theorem is complete.
References


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