PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 123, Number 10, October 1995

STABLE RANK OF H^∞ IN MULTIPLY CONNECTED DOMAINS

V. TOLOKONNIKOV

(Communicated by Theodore W. Gamelin)

ABSTRACT. The stable rank of the algebra $H^\infty(G)$ of bounded analytic functions in every finitely connected open Riemann surface is equal to one. The same is true for some infinitely connected plain domains (Behrens domains). The proof is based on the Treil theorem, which considered the case of H^∞ in the unit disk.

Let A be an associative ring with unit element being denoted 1. For natural n let $U_n(A)$ denote the set of (left)-unimodular elements, i.e.

$$\begin{aligned} U_n(A) &= \{a \in A^n : Aa_1 + \cdots + Aa_n = A\} \\ &= \{a \in A^n : \exists b \in A^n : b_1a_1 + \cdots + b_na_n = 1\}. \end{aligned}$$

This definition was generalized in [Va]. If A is a (associative) ring without unit, then $A^{(1)} \supset A$ is the result of adding the unit to the ring A, $e_1 = (1, 0, \dots, 0)$ is the element of $(A^{(1)})^n$ and $U_n(A) = \{a \in U_n(A^{(1)}) : a - e_1 \in A^n\}$. A unimodular element $a = c \oplus a_n$, $c \in (A^{(1)})^{n-1}$, $a_n \in A$, is called *reducible* if there exists $d \in A^{n-1}$ such that $c + d \cdot a_n \in U_{n-1}(A)$. The (*Bass*) *stable rank* $sr(A)$ is the least n such that every element a from $U_{n+1}(A)$ is reducible. In particular, $sr(A) = 1$ if and only if for every $(a, b) \in U_2(A)$ there exists $d \in A$ such that $a + bd \in U_1(a)$. The last set we denote by $A^{(-1)}$.

For the commutative complex banach algebra with unit the stable rank can be computed with the help of the maximal ideal space $M(A)$: $sr(A) \le n$ if and only if for every $a \in A$ the restriction induces a surjective mapping

$$[M(A), S^{2n-1}] \to [Z(a), S^{2n-1}] \tag{1}$$

where the brackets denote the homotopy class of the maps, $Z(a)$ is the zero set of the element a considered as a continuous function on $M(A)$ and S^{2n-1} is a unit sphere in R^{2n} [CoSu]. Let us note that this condition is equivalent to surjectivity of the map of the spaces of continuous functions

$$C(M(A), S^{2n-1}) \to C(Z(a), S^{2n-1}) \tag{2}$$

Received by the editors December 27, 1993 and, in revised form, March 30, 1994.

1991 *Mathematics Subject Classification*. Primary 46J15, 30H05; Secondary 13C10.

Key words and phrases. Stable rank, algebra H^∞, Riemann surfaces, homotopy, extension to invertible matrix.

for every $a \in A$, but that it is more convenient to work with homotopy classes [CoSu].

For an open Riemann surface G let $H^\infty(G)$ be the space of bounded analytic functions in G. We'll write simply H^∞ instead of $H^\infty(D)$ where D is the open unit disk. In 1992 Treil proved the following result.

Theorem A [Tr]. *Let* $f, g \in H^\infty$ *and*

$$1 \geq |f(z)| + |g(z)| \geq \delta > 0 \quad \textit{for all } z \textit{ in } D. \tag{3}$$

Then there exists $h \in H^\infty$, $\|h\|_\infty \leq C$, *such that* $f + h \cdot g \in (H^\infty)^{(-1)}$ *and* $\|(f + h \cdot g)^{(-1)}\|_\infty \leq C$, *where* C *depends only on* δ.

By Carleson's corona theorem condition (3) is satisfied if and only if $(f, g) \in U_2(H^\infty)$, so we have

Theorem B [Tr]. $sr(H^\infty) = 1$.

The main purpose of this paper is to prove the theorem generalizing Theorem B to the case of more common Riemann surfaces G including all finitely connected surfaces. The proof is based on Theorem A. The proofs of the corona theorem for finitely connected surfaces are based on Carleson's theorem. But there is one difference. The corona theorem is equivalent to the statement that the unit disk is dense in $M(H^\infty)$; that's why it has a local character. More exact formulation of this statement has been given by Gamelin in the following localization theorem.

Theorem C [Ga2]. *If for every point in the boundary of plain domain* G *there exists a neighbourhood* V *such that the corona theorem is true for* $H^\infty(G \cap V)$, *then the corona theorem is true for* $H^\infty(G)$.

But the problem of finding stable rank of a Banach algebra has a global character, as can be seen from properties (1) or (2), and the author is not aware of the theorem similar to Theorem C for stable rank of $H^\infty(G)$. So our reduction to the case of simply connected domains will be based on the following factorisation lemma.

Let G be a circular domain, i.e. an open disk with finitely many closed disjoint disks removed. Then G is the intersection of simply connected domains, $G = G_1 \cap G_2 \cap \cdots \cap G_n$, $G_i = \overline{\mathbf{C}} \setminus \operatorname{clos}(\mathbf{D}_i)$, $\mathbf{D}_i$-open disks in the extended complex plane $\overline{\mathbf{C}}$. Let us fix the points a_i in $\mathbf{D}_i$, $i = 1, 2, \ldots, n$; $a_1 = \infty$.

Lemma 1. *Under the conditions above every function* $f \in H^\infty(G)$ *can be decomposed as follows:* $f = f_1 \cdot f_2 \cdot \cdots \cdot f_n \cdot r$, $f_i \in H^\infty(G_i) \cap H^\infty(\bigcup_{j \neq i} \mathbf{D}_j)^{(-1)}$, r *is the rational function, all zeros and poles of which are in* $\{a_i, i = 1, 2, \ldots, n\}$.

Proof. First let $n = 2$. We can suppose that $G_1 = \mathbf{D}$, perhaps after linear changing of the variable. Remember that $Z(f)$ is the zero set of f in $M(H^\infty(G))$, and divide $Z(f) \cap G$ on two sets Z_1 and Z_2, where all cluster points of Z_1 lie in $\{z : |z| = 1\}$ and the ones of Z_2 in $\mathbf{D}$. Then Z_1 satisfies the Blaschke condition and for the corresponding Blaschke product b we have $f = b \cdot g$, $g \in H^\infty(G)$, and for some s, $0 < s < 1$, we have $g \neq 0$ in the ring $K = \{z : s < |z| < 1\}$. Let us take a point a in K and a branch of $\log(g)$ in a neighbourhood of a. After analytic continuation of $\log(g)$ along the circle $\{z : |z| = |a|\}$ we will have in point a the value $\log g(a) + 2k\pi \cdot i$, $k \in Z$.

So $g = r \cdot \exp h$, where $r = (z - a_2)^k$ and h is an analytic function with its real part bounded from above in the ring K. Let us write the Laurent series for the function h in the ring K, and define h_1 as the sum of the part of this series with positive degrees and h_2 as $h - h_1$. Then h_2 is bounded in the ring $K_1 = \{z : 1 > |z| > (1+s)/2\}$, so h_1 is analytic with upper bounded real part in D. Now we can take $f_1 = b \cdot \exp h_1$, $f_2 = f/(r \cdot f_1)$. So the existence of decomposition in the case $n = 2$ is proved.

We consider the general case by induction. Let $K = \operatorname{clos}(\mathbf{D}_2) \cup \cdots \cup \operatorname{clos}(\mathbf{D}_n)$, $G = \overline{\mathbf{C}} \backslash (\operatorname{clos}(\mathbf{D}_1) \cup K)$, and let us choose the open disk $\mathbf{D}_{n+1}$ such that G contains the boundary of $\mathbf{D}_{n+1}$ and $\mathbf{D}_{n+1} \supset K$. Then $G \backslash \operatorname{clos}(\mathbf{D}_{n+1})$ is the 2-connected circular domain, and for $f \in H^\infty(G) \subset H^\infty(G \backslash \operatorname{clos}(\mathbf{D}_{n+1}))$ with the help of the proved case we can write decomposition $f = f_1 \cdot f_2 \cdot r$, where f_2, as a result of division f and $f_1 \cdot r$, is in $H^\infty(G) \cap H^\infty(\overline{\mathbf{C}} \backslash \operatorname{clos}(\mathbf{D}_{n+1})) = H^\infty(\overline{\mathbf{C}} \backslash K)$. Then we can apply the induction assumption to the function f_2 in the $(n-1)$-connected domain $\overline{\mathbf{C}} \backslash K$. □

Theorem 1. *For every n-connected open Riemann surface G we have*

$$sr(H^\infty(G)) = 1.$$

Proof. If G is hyperbolic, then G is conformally equivalent to some circular domain [Go, p. 237], and in the other case $H^\infty(G) = C$. So let G be a circular domain, $f \in H^\infty(G)$, $g \in C(Z(f), S^1)$. Let us apply to f Lemma 1. Then $Z(f) = Z(f_1) \cup \cdots \cup Z(f_n)$, because r is invertible in $H^\infty(G)$. Let z be the coordinate function on G, and for $m \in M(H^\infty(G))$ define $t(m) = m(z)$. Then t is the continuous map from $M(H^\infty(G))$ to $\operatorname{clos}(G)$. For $i = 1, 2, \ldots, n$ let $\pi_i : M(H^\infty(G)) \to M(H^\infty(G_i))$ be the maps, induced by inclusions $G \to G_i$. As $t(Z(f_i)) \cap \partial G_j = 0$ for $i \neq j$, so the restriction of π_i on $Z(f_i)$ is injection and we can identify $Z(f_i)$ with the subset of $M(H^\infty(G_i))$. The region G_i is simply connected, so by Theorem B and (1) there exist $h_i \in C(M(H^\infty(G_i)), S^1)$ such that $h_i|_{Z(f_i)} \sim g|_{Z(f_i)}$ where " $\sim$ " is the sign of homotopy. For $i \neq j$ all cluster points of $t(Z(f_j))$ lie in $\partial \mathbf{D}_j$, so there exists a simply connected neighbourhood U of $\operatorname{clos}(\mathbf{D}_j)$ such that $G_i \supset \operatorname{clos}(U) \supset U \supset t(Z(f_j))$. Then every continuous function on $\operatorname{clos}(U)$ is homotopy to constant and $h_i|_{Z(f_i)} \sim 1$, $i \neq j$. So for $h = h_1 \circ \pi_1 \cdot h_2 \circ \pi_2 \cdots h_n \circ \pi_n$ we have $h|_{Z(f)} \sim g|_{Z(f)}$, and by property (1) we have $sr(H^\infty(G)) = 1$. □

In the proof of Theorem 1 we used only the corollary of Treil's theorem, i.e. Theorem B. The next theorem is based on estimates of solutions of the stable rank problem, i.e. Theorem A. It is easy to extend this theorem to every Banach algebra, where estimates of such type are known.

Theorem 2. *For the algebra $B = l^\infty(H^\infty) = \{f = (f_1, f_2, \ldots) : f_i \in H^\infty$, $\sup \|f_i\|_\infty < \infty\}$ with coordinatewise multiplication we have $sr(B) = 1$.*

Proof. If $(f, g) \in U_2(B)$, then there exists $(a, b) \in B^2$ such that $f_i \cdot a_i + g_i \cdot b_i = 1$ for all i, so

$$U_2(B) \subset \{(f, g) \in B^2 : \exists \delta > 0\ \forall i\ \forall z \in \mathbf{D}\ |f_i(z)| + |g_i(z)| \geq \delta\}.$$

In fact there is equality, but we don't need this simple result. So we can apply Theorem A to every component (f_i, g_i) of (f, g), and there is $h \in B$ such that $((f_i + h_i \cdot g_i)^{(-1)}) \in B$. □

For algebra A with the unit let $GL(n, A)$ be the group of all invertible n-dimensional matrices. For algebra A without the unit let $A^{[1]}$ be the result of adding the unit to the algebra A (this is not the same as $A^{(1)}$).

Lemma 2. *For every two-sided ideal J in algebra A we have*

$$\max(sr(J), sr(A/J)) \leq sr(A) \tag{4}$$

and

$$sr(J^{[1]}) = sr(J). \tag{5}$$

Proof. Inequality (4) was proved in [Va]; here Vaserstein proved also that if

$$\forall a \in U_n(A/J),\ n = sr(A/J),\ \exists b \in GL(n, A)\colon ba = e_1 \bmod J, \tag{6}$$

then inequality (4) became equality. In the case of the algebra $A = J^{[1]}$ we have that $J^{[1]}/J$ is a field, so condition (6) is fulfilled. □

Notice that for $J^{(1)}$ in [Va] it is proved that $sr(J^{(1)}) = \max(2, sr(J))$. The differences with (5) are connected with the following formulas: $J^{(1)}/J = Z$, $sr(Z) = 2$, while $sr(J^{[1]}/J) = 1$.

Let G be a domain obtained from the open unit disk $\mathbf{D}$ by deleting a sequence of disjoint closed disks $\mathbf{D}_i$ with the centers c_i and radii r_i, such that

$$\forall n\ |c_{n+1}|/|c_n| \leq a < 1,$$
$$\sum_{n=1}^{\infty} r_n/|c_n| < \infty.$$

The domains of such type were studied by Behrens [Be]. Let $\mathcal{M} = M(H^\infty(G))$ and let $\mathcal{M}_0 = t^{-1}(0)$ be the fiber over 0 for $\mathcal{M}$.

Lemma 3. $sr(H^\infty(G)|_{\mathcal{M}_0}) = 1$.

Proof. Let $I = c_0(H^\infty) = \{f = (f_1, f_2, \dots)\colon f_i \in H^\infty, \lim \|f_i\|_\infty = 0\}$ be the closed ideal in B, $C = B/I$, Z be the image of the element $(z, z, \dots)$ from B in the factor algebra C and $J = Z \cdot C$ be the closed ideal in C. Then in [Be, Lemma 7], it is proved that the algebra $H^\infty(G)|_{\mathcal{M}_0}$ is isometrically isomorphic to algebra $J^{[1]}$ (with norm from C). In fact in [Be] the definition of the algebra $J^{[1]}$ was given in terms of the maximal ideal spaces. Its equivalence to our definition follows from the known description of the maximal ideal spaces of factor algebras and algebras, generated by ideals [Gam1]. To finish the proof of Lemma 3 it is enough to apply Theorem 2 and Lemma 2. □

Theorem 3. *For every Behrens domain G we have*

$$sr(H^\infty(G)) = 1.$$

Proof. We'll apply again properties (1) and (2). If $f \in H^\infty(G)$ and $g \in C(Z(f), S^1)$, then by Lemma 3 and property (2) there exists a function $H_1 \in C(\mathcal{M}_0, S^1)$ such that $H_1|_{\mathcal{M}_0 \cap Z(f)} = g|_{\mathcal{M}_0 \cap Z(f)}$. Let $H_2 \in C(\mathcal{M}, \mathbf{C})$ be the continuation of H_1. Then $|H_2| > 1/2$ in some neighbourhood U of $\mathcal{M}_0$. We can take U in the form $U = \{m \in \mathcal{M}\colon |m(z)| < \varepsilon\}$. Let us take the smaller positive ε such that $T_\varepsilon = \{z \in C\colon |z| = \varepsilon\} \subset G$ and $\|(g - H_2)|_{U \cap Z(f)}\|_\infty < 1/2$. Dividing H_2 in U on $|H_2|$ and continuing this function from T_ε to the ring $\{z : \varepsilon < |z| < 1\}$ we construct the function $H_3 \in C(\mathcal{M}, S^1)$, such that $H_3|_{Z(f)\cap \operatorname{clos}(U)} \sim g|_{Z(f)\cap\operatorname{clos}(U)}$.

Now we apply Lemma 1 to f and the circular domain $G \cap \{z : |z| > \varepsilon\}$, so $f = f_1 \cdot f_2 \cdot \cdots \cdot f_n \cdot F$, $f_i \in H^\infty(\overline{\mathbf{C}} \setminus \operatorname{clos}(\mathbf{D}_i))$, $t(Z_F) \subset \{z : |z| \le \varepsilon\}$ and the function $F = f_{n+1} \cdot r$ can be analytically continued to the function from $H^\infty(G)$ (as a result of the division function f). Now acting as in the proof of Theorem 1 we'll find functions $h_1, \dots, h_n \in C(\mathcal{M}, S^1)$ such that $h_i|_{Z(f_i)} \sim g \cdot (H_3)^{(-1)}|_{Z(f_i)}$; $h_i|_{Z(f_j)} \sim 1$, $i \ne j$; $h_i|_{\{m\in\mathcal{M}\,:\, |t(m)|\le\varepsilon\}} \sim 1$. Then for $H_4 = H_3 \cdot h_1 \cdot \cdots \cdot h_n$ we have $H_4|_{Z(f)} \sim g|_{Z(f)}$, and by (1) $sr(H^\infty(G)) = 1$. □

Let A be a ring with the unit. Then the group $GL(m, A)$ acts through right multiplication on $U_m(A)$, and we'll define after [Ri] the *generalized stable rank* $gsr(A)$ of the ring A by the formula

(7) $$gsr(A) = \min\{n : \forall m \ge n \ GL(m, A) \text{ acts transitively on } U_m(A)\}.$$

If $B \in GL(m, A)$ and $a \cdot B = e_1$, then $a = e_1 \cdot B^{-1}$, and we see that the problem in the definition of the generalized stable rank is equivalent to the problem of extension every row from $U_m(A)$ to invertible matrix. In [Lam, Corollary 4.5, p. 26 and discussion there] it is proven that in the case of $gsr(A) = 1$ every rectangular left-invertible matrix can be extended to invertible matrix. Such rings are called here *Hermite rings* after the Hermite theorem: $gsr(\mathbf{Z}) = 1$. See also [CoSu, To2]. Bass proved that always $gsr(A) \le sr(A) + 1$ [Ri]. From definition (7) it is easy to see that $gsr(A)$ is never equal to 2, so from Theorem 3 we have

Corollary. *For every Behrens domain G algebra $H^\infty(G)$ is Hermite.*

For the case of the unit disk this result was formerly proved by the author in [To1] and for finitely connected plane domains in [To2] without Treil's theorem.

This result was mainly proved during the author's visit to the Mathematical Institute in Karlsruhe, Germany. The author is grateful to R. Mortini, M. Renteln and R. Rupp for useful discussions and hospitality.

References

[Be] M. Behrens, *The Corona conjecture for a class of infinitely connected domains*, Bull. Amer. Math. Soc. **76** (1970), 387–391.

[CoSu] G. Corach and F. D. Suarez, *Extension problems and stable rank in commutative Banach algebras*, Topology Appl. **21** (1985), 1–8.

[Gam1] T. Gamelin, *Uniform algebras*, Prentice-Hall, New York, 1969.

[Gam2] ——, *Localization of corona problem*, Pacific J. Math. **34** (1970), 73–81.

[Go] G. M. Golusin, *Geometric theory of function of a complex variable*, Amer. Math. Soc., Providence, RI, 1969.

[Lam] T. Y. Lam, *Serre's conjecture*, Lecture Notes in Math., vol. 635, Springer-Verlag, Berlin and New York, 1978.

[Ri] M. A. Rieffel, *Dimension and stable rank in the K-theory of C^*-algebras*, Proc. London Math. Soc. (3) **46** (1983), 301–333.

[To1] V. Tolokonnikov, *Estimates in the Carleson Corona theorem, ideals of the algebra H^∞, a problem of S.-Nagy*, J. Soviet. Math. **22** (1983), 1814–1828.

[To2] ——, *Extension problem to an invertible matrix*, Proc. Amer. Math. Soc. **117** (1993), 1023–1030.

[Tr] S. Treil, *Stable rank of H^∞ is equal to one*, J. Funct. Anal. **109** (1992), 130–154.

[Va] L. N. Vaserstein, *Stable rank of rings and dimensionality of topological spaces*, Functional Anal. Appl. **5** (1971), 102–110.

BLANSINGER STR. 15, RHEINWEILER, 79415 GERMANY