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A CHARACTERIZATION OF CLIFFORD MINIMAL HYPERSURFACES IN S^4

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ABSTRACT. In this note we give a characterization of Clifford minimal hypersurfaces in S^4 by the Ricci curvature condition.

1. INTRODUCTION AND THEOREM

Let S^4 be a 4-dimensional unit sphere space. It is well known (see Chern, do Carmo, and Kobayashi [1]) that the closed Clifford hypersurface $C_{1,2} = S^1(\sqrt{1/3}) \times S^2(\sqrt{2/3})$ in S^4 has two different principal curvatures $\lambda_1 = \sqrt{2}$, $\lambda_2 = \lambda_3 = -\sqrt{1/2}$. From the Gauss equation we easily know that the Ricci curvature of $C_{1,2} = S^1(\sqrt{1/3}) \times S^2(\sqrt{2/3})$ satisfies

$$0 \leq \mathrm{Ric}(C_{1,2}) \leq \tfrac{3}{2}. \tag{1}$$

A natural problem is that if M is a closed minimal hypersurface in S^4 and the Ricci curvature of M satisfies $0 \leq \mathrm{Ric}(M) \leq 3/2$, then is it the Clifford minimal hypersurface $C_{1,2} = S^1(\sqrt{1/3}) \times S^2(\sqrt{2/3})$?

In this note we give an affirmative answer for the above problem, that is, we prove the following

Theorem. *Let M be a closed minimal hypersurface in S^4 which satisfies the Ricci curvature condition*

$$0 \leq \mathrm{Ric}(M) \leq \tfrac{3}{2}. \tag{2}$$

Then $M = C_{1,2} = S^1(\sqrt{1/3}) \times S^2(\sqrt{2/3})$.

2. PRELIMINARIES

Let M be an n-dimensional closed minimal hypersurface in an $(n+1)$-dimensional unit sphere space S^{n+1}, and $e_1, \dots, e_n$ a local orthonormal frame field on M, $\omega_1, \dots, \omega_n$ its dual coframe field. Here for the sake of simplicity, we keep the notation of Peng and Terng [3]. We have the following formulas

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(see [3]):

$$I = \sum_i \omega_i^2, \tag{3}$$

$$II = h = \sum_{i,j} h_{ij}\omega_i\omega_j, \qquad h_{ij} = h_{ji}, \tag{4}$$

$$\nabla h = \sum_{i,j,k} h_{ijk}\omega_i\omega_j\omega_k, \qquad h_{ijk} = h_{ikj}, \tag{5}$$

$$\nabla^2 h = \sum_{i,j,k,l} h_{ijkl}\omega_i\omega_j\omega_k\omega_l, \tag{6}$$

where

$$h_{ijkl} = h_{ijlk} + \sum_m h_{mj}R_{mikl} + \sum_m h_{im}R_{mjkl}, \tag{7}$$

$$\Delta h_{ij} = (n-S)h_{ij}, \qquad \tfrac{1}{2}\Delta S = |\nabla h|^2 + S(n-S), \tag{8}$$

where $S = |h|^2$.

We introduce the following symmetric functions:

$$f_3 = \sum_{i,j,k} h_{ij}h_{jk}h_{ki}, \tag{9}$$

$$f_4 = \sum_{i,j,k,l} h_{ij}h_{jk}h_{kl}h_{li}. \tag{10}$$

By using (7) and (8), we can compute the Laplacian of the function f_4 as follows:

$$\Delta f_4 = 4[(n-S)f_4 + 2A + B], \tag{11}$$

where

$$A = \sum_{i,j,k,l,m} h_{ijk}h_{ijl}h_{km}h_{ml}, \qquad B = \sum_{i,j,k,l,m} h_{ijk}h_{klm}h_{jl}h_{im}. \tag{12}$$

Near any given point $p \in M$, we can choose a local frame field $e_1, \dots, e_n$ so that at p, we have

$$h_{ij} = \lambda_i\delta_{ij}.$$

Then, from the symmetricity of h_{ijk} about indices, we have

$$A + 2B = \frac{1}{3}\sum_{i,j,k} h_{ijk}^2(\lambda_i + \lambda_j + \lambda_k)^2 \geq 0. \tag{13}$$

We need the following lemma to prove our theorem

Lemma 1 (see Theorem 4 of Peng and Terng [3]). *Let M be a closed minimal hypersurface in S^{n+1}. Then*

$$\int_M Sf_4 - f_3^2 - S^2 - \frac{|\nabla S|^2}{4} = \int_M A - 2B, \tag{14}$$

where A and B are defined by (12).

Integrating both sides of (11), we have

$$\int_M (S-n)f_4 = \int_M 2A + B. \tag{15}$$

By (13), $-3\times(14)+4\times(15)$ implies the following lemma

Lemma 2. *Let* M *be a closed minimal hypersurface in* S^{n+1}. *Then*

$$\int_M [(S-4n)f_4 + 3S^2 + 3f_3^2 + \tfrac{3}{4}|\nabla S|^2] \geq 0. \tag{16}$$

Lemma 3. *Let* M *be a closed minimal hypersurface in* S^{n+1}. *Then*

$$\int_M |\nabla S|^2 \leq \frac{4n}{3n+2}\int_M S^2(S-n). \tag{17}$$

Proof. By using the Schwartz inequality, the minimality condition, and the Schwarz inequality again, we obtain (cf. Schoen, Simon, and Yau [4])

$$\begin{aligned}
|\nabla S|^2 &= 4\sum_k\left(\sum_{i,j} h_{ij}h_{ijk}\right)^2 \\
&= 4\sum_k\left(\sum_i \lambda_i h_{iik}\right)^2 \leq 4S\sum_{i,k} h_{iik}^2 \\
&= 4S\sum_{i\neq k} h_{iik}^2 + 4S\sum_i h_{iii}^2 \\
&= 4S\sum_{i\neq k} h_{iik}^2 + 4S\sum_i\left(\sum_{j\neq i} h_{jji}\right)^2 \\
&\leq 4S\sum_{i\neq k} h_{iik}^2 + 4S(n-1)\sum_{j\neq i} h_{jji}^2 \\
&= 4nS\sum_{i\neq j} h_{jji}^2.
\end{aligned} \tag{18}$$

On the other hand, by (5)

$$\begin{aligned}
|\nabla h|^2 &= \sum_{i,j,k} h_{ijk}^2 \\
&= 3\sum_{i\neq k} h_{iik}^2 + \sum_i h_{iii}^2 + \sum_{i,j,k\neq} h_{ijk}^2 \\
&\geq 3\sum_{i\neq k} h_{iik}^2 + \sum h_{iii}^2 \\
&= 2\sum_{i\neq k} h_{iik}^2 + \sum_i h_{iii}^2 + \sum_{i\neq k} h_{iik}^2.
\end{aligned} \tag{19}$$

From (18) and (19) we get

$$|\nabla S|^2 \leq \frac{4n}{n+2} S|\nabla h|^2. \tag{20}$$

By the second formula of (8),

$$S|\nabla h|^2 = \frac{1}{4}\Delta S^2 - \frac{1}{2}|\nabla S|^2 - (n-S)S^2. \tag{21}$$

Thus (17) comes from (20) and (21).

Combining Lemma 2 with Lemma 3, we have

Proposition 1. *Let M be a closed minimal hypersurface in S^{n+1}. Then we have*

$$\int_M \left[(S-4n)f_4 + 3S^2 + 3f_3^2 + \frac{3n}{3n+2}(S-n)S^2\right] \geq 0. \tag{22}$$

In the case $n=3$, $f_4 = S^2/2$, $f_3 = 3\det(h)$. We have

Corollary 1. *Let M be a closed minimal hypersurface in S^4. Then we have*

$$\int_M [\tfrac{1}{2}S^2(S-6) + 27(\det(h))^2 + \tfrac{9}{11}(S-3)S^2] \geq 0. \tag{23}$$

3. Proof of the Theorem

Let M be a closed minimal hypersurface in S^4. Near any given point p, we can choose a local frame field $e_1, \ldots, e_3$ so that at p, $h_{ij} = \lambda_i\delta_{ij}$. By Gauss equations

$$R_{ii} = 2 - \lambda_i^2, \qquad 1 \leq i \leq 3. \tag{24}$$

From assumption condition (2), we have

$$\lambda_i^2 \leq 2, \qquad i = 1, 2, 3.$$

Therefore, we obtain

$$\begin{aligned} 0 &\geq \prod_{i=1}^{3}(\lambda_i^2 - 2) \\ &= (\det(h))^2 + f_4 - S^2 + 4S - 8 \\ &= (\det(h))^2 - \tfrac{1}{2}S^2 + 4S - 8. \end{aligned} \tag{25}$$

Combining (25) with (23), we obtain

$$\int_M \tfrac{1}{2}(S-3)[\tfrac{29}{11}S^2 + 24S - 144] \geq 0, \tag{26}$$

that is

$$\int_M \tfrac{29}{11}(S-3)[(S + \tfrac{12}{19}(11+\sqrt{440}))(S - \tfrac{12}{19}(-11+\sqrt{440}))] \geq 0. \tag{27}$$

Since we assume $\mathrm{Ric}(M) \geq 0$, i.e.,

$$-\sqrt{2} \leq \lambda_i \leq \sqrt{2}, \qquad i = 1, 2, 3, \tag{28}$$

the minimality condition of M is

$$\lambda_1 + \lambda_2 + \lambda_3 = 0. \tag{29}$$

It is easily seen that the convex function $S = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$ of three variables $\lambda_1, \lambda_2, \lambda_3$ subject to linear constraint conditions (28) and (29) attains its maximum when (after renumbering e_1, e_2, e_3 if necessary)

$$\lambda_1 = -\lambda_2 = \sqrt{2}, \qquad \lambda_3 = 0. \tag{30}$$

Therefore, we have

$$S \leq 4. \tag{31}$$

On the other hand, since we also assume $\mathrm{Ric}(M) \leq 3/2$, i.e., $\lambda_i^2 \geq 1/2$, $i = 1, 2, 3$, we have $S \geq 3$ from (29). In view of $3 \leq S \leq 4$, we obtain from (27), $S \equiv 3$, i.e., $M = C_{1,2} = S^1(\sqrt{1/3}) \times S^2(\sqrt{2/3})$ (see Lawson [2] or Chern, do Carmo, and Kobayashi [1]). We complete the proof of the Theorem.

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